DEGENERATIONS OF SCHUBERT VARIETIES: A SURVEY

ALLEN KNUTSON

CONTENTS

In this survey I list the degenerations of Schubert (and similar) varieties I know about. It is not historically exhaustive; in particular my degeneration in §7 subsumes a number of previously known special cases. There are a few conjectures and problems included. References are given, but no proofs, and combinatorial details are mostly omitted.

1. HODGE'S DEGENERATION OF THE GRASSMANNIAN TO A STANLEY-REISNER SCHEME

The map

$$\begin{array}{rcl} M_{k\times n} & \to & \mathrm{Alt}^k\,\mathbb{C}^n \\ (\vec{v}_1,\ldots,\vec{v}_k) & \mapsto & \vec{v}_1\wedge\cdots\wedge\vec{v}_k \end{array}$$

is SL_k-invariant, so descends to a map

$$SL_k\backslash\backslash M_{k\times n} =: \widehat{Gr_k(\mathbb{C}^n)} \hookrightarrow \mathrm{Alt}^k\,\mathbb{C}^n.$$

The $\binom{n}{k}$ determinants (p_λ) of $k\times k$ submatrices (called **Plücker coordinates**) generate this ring of SL_k-invariants (this is the **first fundamental theorem of invariant theory**). We will

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index $\binom{n}{k}$ by partitions inside a $k \times (n-k)$ box. The relations among the (p_λ) are generated by the **straightening laws**

$$p_\lambda p_\mu = p_{\lambda\cup\mu} p_{\lambda\cap\mu} + \dots$$

for λ, μ incomparable (the **second fundamental theorem of invariant theory**).

The projectivization of this map is called the **Plücker embedding**.

Theorem (Hodge, predating Gröbner basis theory)**.** *There is flat projective degeneration of subschemes of Plücker space $\mathbb{P}(\mathrm{Alt}^k \mathbb{C}^n)$, taking the straightening laws to their initial forms*

$$p_\lambda p_\mu = 0$$

for λ, μ incomparable.

The initial scheme is a union of coordinate subspaces, one for each standard Young tableau in the $k \times (n-k)$ box.

Inside $M_{k\times n}$ is the subset of reduced row-echelon forms with a fixed set μ of pivot columns. Its image $X^\mu_\circ$ in $\mathrm{Gr}_k(\mathbb{C}^n)$ is often called a **Bruhat cell**, with closure X^μ a **Schubert variety**. Similarly we define X_λ by asking that the left-right reversed matrix be in echelon form, and X^μ_λ as their (transverse) intersection, a **Richardson variety**.

Theorem (Hodge, basically)**.** *The equations defining X_λ are linear, $p_\mu = 0\ \forall \mu \not\geq \lambda$. The same holds for the initial scheme of X_λ (likewise X^μ_λ).*

In particular, the initial scheme of X^μ_λ is a union of coordinate spaces, one for each standard skew-tableau of shape μ/λ. The corresponding simplicial complex is homeomorphic to a ball [Björner-Wachs].

These are not the only subvarieties one can follow nicely into the degeneration; in §8 we'll do the same with *positroid varieties*.

2. The less drastic Gel'fand-Cetlin degeneration

Theorem. *There is a Gröbner degeneration of the Grassmannian to the* irreducible *toric variety defined by*

$$p_\lambda p_\mu = p_{\lambda\cup\mu} p_{\lambda\cap\mu}$$

The moment polytope can be identified with real *Young tableaux (neither set of inequalities being strict) with values in $[0,1]$. Its vertices are the Young tableaux with values $\{0,1\}$, corresponding to partitions.*

The flatness of this follows from the flatness in Hodge's more drastic degeneration. The fact that we can chain the degenerations in order shows that the Hodge simplicial complex (the order complex of Young's lattice) gives a triangulation of the Gel′fand-Cetlin polytope. E.g. for $\mathrm{Gr}_2(\mathbb{C}^4)$, the GC polytope is the solutions to

$$\begin{array}{ccccccccc} & & & & A & & & & \\ & & & \geq & & \geq & & & \\ & & B & & & & C & & \\ & \geq & & \geq & & \geq & & \geq & \\ 1 & & & & D & & & & 0 \end{array}$$

which breaks into two simplices along the plane $A = D$.

In particular, there's a map on face posets, taking a skew-tableaux (indexing a stratum in the Hodge degeneration) to a face of the Gel′fand-Cetlin polytope. These faces are indexed by which entries are 1 (by the West corner), which entries are 0 (by the East corner), and which are NE and SE equalities hold. From a skew-tableau (turned counterclockwise 45°), it is clear how to obtain this data.

E.g. for $Gr_2(\mathbb{C}^4)$, the tableaux $\begin{smallmatrix}1&2\\3&4\end{smallmatrix}$ and $\begin{smallmatrix}1&3\\2&4\end{smallmatrix}$ provide the two top simplices, glued along the one from $\begin{smallmatrix}1&2\\2&4\end{smallmatrix}$.

Question: In both this and the Hodge degeneration, one can follow the *conormal variety* of the Grassmannian through the degeneration. What indexes the components of the resulting scheme, and what are the multiplicities of the components?

3. The extension of that to GL_n/B by Gonciulea-Lakshmibai

When a subring (e.g. an invariant ring) of a domain degenerates to another domain, as in the last section, it's suggestive that it could degenerate *as a subring*, which will be how we pursue it further.

The fundamental theorems of invariant theory also apply to $N_- \backslash\backslash M_{n\times n}$, where N_- is lower unipotent matrices. Now the Plücker determinants are of all sizes k, using any k columns and the top k rows. We can correspond these 2^n generators with partitions in the upper triangle of a matrix, and from there to $\{0,1\}$ Young tableaux, such as

$$\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 1 \\
 & 0 & 0 & 0 & 1 & 1 \\
 & & 0 & 1 & 1 & 1 \\
 & & & 1 & 1 & 1 \\
 & & & & 1 & 1 \\
 & & & & & 1
\end{array}$$

The correspondence goes as follows: start in the NE corner, and go left/down n times, keeping the 1s to the SE and 0s to the NW. The number of 1s on the diagonal gives k.

Let p'_λ be the *diagonal term* of the determinant p_λ.

Theorem. [GL, MS] *There is a flat (SAGBI) degeneration of the N_--invariant subring to the monomial subring generated by the (p'_λ). Its monoid can be identified with* **Gel′fand-Cetlin patterns,** *which are Young tableaux in the NE triangle. The $\{0,1\}$ tableaux above are the Hilbert basis of that monoid (once we ditch the all-0s pattern, corresponding to the 0×0 Plücker determinant).*

A big difference between the Grassmannian and flag manifold Gel′fand-Cetlin degenerations is that the degenerate flag manifold acquires new T-fixed points. There has been much work by de Loera and others on the moment images of these new points (which need not be integral).

Every SAGBI degeneration has an associated Gröbner degeneration. It is then interesting to follow Richardson varieties under this degeneration. Here are some facts for which I don't have references:

Theorem 1. (1) init *of a Richardson variety is again reduced, i.e. is the union of toric varieties $TV(F)$ corresponding to a certain set $\{F\}$ of faces of the Gel′fand-Cetlin polytope.*

(2) *For each face* F *of the polytope, there is a unique smallest* X_w^v *with* $\mathtt{init}\ X_w^v \supseteq TV(F)$*. It is read this way: in each of the two sets of GC inequalities, the locations of equalities determines a* pipe dream *whose* Demazure product *is* v *or* w*, depending. (These definitions are in §7.)*
(3) *The union of those faces is homeomorphic to a ball.*

Some of that is in [Kogan-Miller], some [Björner-Wachs], some [K]. I believe more will be in a Speyer student thesis soon.

4. Feigin et al.'s degeneration of GL_n/B to a cominuscule Schubert variety

In [Fe] is given a SAGBI-like degeneration of GL_n/B to an irreducible variety, with many nice properties, one being that the open N-action degenerates to an open action of the vector space $\mathfrak{n}$, giving a sort of $\mathbb{G}_a$-version of a toric variety.

The resulting variety was recognized in [IL] as a Schubert variety. Many of Feigin et al.'s results on this "degenerate flag variety", such as it having a Frobenius splitting and a Bott-Samelson-"like" resolution, follow trivially from [IL].

Lanini and I showed that this degeneration can be seen as embedded:

Theorem 2. *Let* $\Delta_\pm : \mathbb{C}^n \to \mathbb{C}^n \oplus \mathbb{C}^n$ *take* $\vec{v} \mapsto \vec{v} \oplus \pm\vec{v}$*. Define* $Fl(1,2,3,\ldots,n) \to Fl(2,4,6,\ldots,2n)$ *by*

$$(0 < V_1 < V_2 < \ldots < V_n) \mapsto (\ldots < \Delta_+(V_i) + \Delta_-(\mathbb{C}^i) < \ldots)$$

Then the image has a Gröbner degeneration to a certain Schubert variety.

5. Chirivì's extension of Hodge to general G/P and line bundle

The Lakshmibai-Seshadri conjecture gave a type-independent way to compute the multiplicities in G-irreps, by counting lattice points in a union of stretched simplices. It inspired and was proven using Littelmann's path model. It was given geometric form by Chirivì:

Theorem 3. [Ch] *Let* ν *be a dominant weight of* G*, and* $G/P \hookrightarrow \mathbb{P}(V_\nu)$ *the orbit of the high weight vector of the irrep* V_ν *(e.g. the Plücker embedding, in the case* $G = GL_n$ *and* ν *fundamental).*

Then there is an embedded degeneration of G/P *to a seminormal union of toric varieties (each an equivariant cover of projective space), with one component for each maximal chain in* W/W_P*.*

Note that this does not create any new T-fixed points. In particular, in the flag manifold case the Chirivì degeneration is *not* more drastic than the Gel′fand-Cetlin degeneration (as was true on Grassmannians); they are incomparable degenerations.

Question: Is there a common degeneration of both?

Chirivì's proof is by analysis of Lusztig's canonical basis for V_ν. I believe that there should be a more geometric proof based off of

Lemma 1. *Let* $E \leq V_\nu$ *be the sub-*$N(T)$*-representation consisting of extremal weight spaces, and* $V_\nu \twoheadrightarrow E$ *the* T*-equivariant projection. Then the composite*

$$e : \ G/P \hookrightarrow \mathbb{P}(V_\nu) \dashrightarrow \mathbb{P}E$$

is globally defined (unlike the second map), and finite (making G/P *a "branchvariety" of* $\mathbb{P}E$ [AK]*).*

I suspect that there should be a straightforward Gröbner degeneration of the subvariety $e(G/P)$, with G/P itself degenerating as a branchvariety of $\mathbb{P}E$. More specifically, we know that G/P is defined by quadratic equations (the kernel of the map $Sym^2 V_\lambda \to V_{2\lambda}$) [D. Garfinkel's thesis, or findable in Brion-Kumar]. I conjecture that

- e is birational to its image, so they have the same degree
- $(\mathrm{init}\, e(G/P))_{\mathrm{red}} = SR(\mathrm{order}(W/W_P))$ w.r.t. lex order of linear extensions of W/W_P.

The first example is $\lambda = \rho$ for $G = SL(3)$, where this initial ideal is

$$(b{\otimes}x)(c{\otimes}y)(a{\otimes}z),\ (c{\otimes}y)^2(a{\otimes}z)^2,\ (b{\otimes}z)(c{\otimes}y)(a{\otimes}z)^2,\ (c{\otimes}x)(c{\otimes}y)^2(a{\otimes}z),$$

$$(c{\otimes}x)(b{\otimes}z)(c{\otimes}y)(a{\otimes}z),\ (c{\otimes}x)(b{\otimes}z)^2(a{\otimes}z),\ (c{\otimes}x)^2(b{\otimes}z)(c{\otimes}y),\ (c{\otimes}x)^2(b{\otimes}z)^2$$

with the correct radical $\langle (c{\otimes}x)(b{\otimes}z), (c{\otimes}y)(a{\otimes}z)\rangle$.

Questions:

(1) What is the ramification locus of e?
(2) Does the standard Frobenius splitting on G/P restrict to one on $e(G/P)$?
(3) These would explain why the degeneration of G/P is a seminormal union of seminormal toric varieties, each an equivariant cover of projective space. Why are the components of Chirivì's degeneration actually *normal*?

6. Caldero's extension of Gonciulea-Lakshmibai to general G/P

For those who prefer degenerations that stay irreducible, one has

Theorem 4. [Ca02]

(1) *For any G/P in any $\mathbb{P}(V_\lambda)$, there exist Gröbner degenerations of G/P to irreducible toric varieties. (Their polytopes were already described in* [BZ99]*.) There are many; they depend on the choice of a reduced word for w_0. Being Gröbner, they provide compatible degenerations of all subschemes, e.g. the Schubert varieties.*
(2) *For any single Schubert variety X_w in G/P, one can ensure that $\mathrm{init}\ X_w$ remains irreducible under this degeneration, by making sure that the chosen word is "adapted" to w.*
(3) *(O. Mathieu's observation) There is no degeneration of G/P that works for all X_w simultaneously, since some $X_w \cap X_v$ can themselves be reducible.*

The first is based on canonical basis techiques, the second a purely combinatorial fact.

Question: are there reduced words "adapted" to Richardson varieties (or even their projections, as in §8)?

7. My degeneration of Kazhdan-Lusztig varieties to Stanley-Reisner schemes

Lemma 2 (Kazhdan-Lusztig)**.** *The open set $(v \cdot N_- B/B) \cap X_w$ on X_w, centered at the T-fixed point vB/B, is isomorphic to*

$$X_w \cap X_\circ^v \qquad \textit{a } \textbf{Kazhdan-Lusztig variety}$$

times a (generally irrelevant) vector space X_v°.

To understand the singularities of X_w, it's enough to look near T-fixed points, and thanks to this lemma it's enough to look at K-L varieties.

Theorem 5. [K, §7.3] *Let* Q *be a reduced word for* v*, and consider the map*

$$\begin{array}{rcl} m: \mathbb{A}^{\ell(v)} & \to & G/B \\ (z_1, \ldots, z_{\ell(v)}) & \mapsto & \left(\prod e_{q_i}(z_i) r_{q_i}\right) B/B \end{array}$$

where e_q, r_q *are w.r.t. pinnings of the root* SL_2*s in* G*. Then*

(1) m *is an isomorphism with* $X^v_\circ$*.*
(2) $\text{lex init}\ m^{-1}(X^v_\circ \cap X_w)$ *is the Stanley-Reisner scheme of the* **subword complex** $\Delta(Q, w)$*, whose facets are the complements in* Q *of reduced subwords with product* w*.*
(3) *This* lex init *commutes with unions and intersections, applied to K-L varieties.*
(4) [KnM03] *Those subword complexes are homeomorphic to balls.*

We can cast this in another way [K]. Given a (not necessarily reduced) word Q, there exists a unique[1] Bruhat maximum in $\{\prod S \ :\ S \subseteq Q\}$, called the **Demazure product**. Then the geometrically defined map

$$\begin{array}{rcl} 2^Q & \to & [1, v] \\ F & \mapsto & \min\{w \ :\ \text{lex init}\ m^{-1}(X_w) \supseteq \mathbb{A}^F\} \end{array}$$

is defined (those mins are unique, which uses part (3) of the above) and matches the Demazure product. This gives a partition of the simplex with face lattice 2^Q into open balls, whose closures are closed balls (subword complexes), whose combinatorics matches that of the Bruhat order on $[1, v]$.

7.1. **Example: matrix Schubert varieties.** If $v = w_0 w_0^P$, then the composite $X^v_\circ \hookrightarrow G/B \twoheadrightarrow G/P$ is an isomorphism with the big cell. If G/P is cominuscule, e.g. a Grassmannian, then v has a unique reduced word Q up to commuting moves. (In type A this "fully commutative" condition on v is equivalent to v being 321-avoiding.)

Under the graph construction, this big cell is isomorphic to $k \times (n-k)$ matrices. In these matrix coordinates, the $n-1$ Schubert divisors pull back to

- $k-1$ many NW determinants,
- the $(n-k) - k + 1$ many $k \times k$ determinants that use k consecutive columns, and
- $k-1$ many SE determinants.

Subwords of Q correspond to *pipe dreams*, in which we either use a matrix entry to make two pipes cross or forego the opportunity.

Then theorem 5 recovers many of the results of [KnM05] (at least over a field).

8. Projected Richardson varieties, and their Chirivì degenerations

The projections of Schubert varieties (B_--orbit closures) in G/B to G/P are exactly the Schubert varieties in G/B. But the projections of Richardson varieties may not be Richardson: the intersection of projections is typically bigger then the projection of the intersection.

Theorem 6. [KLS1, KLS2]

[1] Most easily proven by observing that the image of a Bott-Samelson map must be an opposite Schubert variety, and following the T-fixed points.

(1) *Projected Richardson varieties are normal and Cohen-Macaulay, with rational singularities.*
(2) *The set of projected Richardson varieties in* G/P *forms a stratification (the intersection of two is a union of others), and its poset is ranked.*
(3) *The boundary of a projected Richardson variety (in this stratification) is an anticanonical divisor.*
(4) *The projected Richardson stratification is the coarsest stratification by irreducible varieties that includes the components of* G/P*'s boundary (i.e. it is "generated" by that anticanonical divisor).*
(5) *Under the standard Frobenius splitting on* G/P*, the projected Richardson varieties are exactly the compatibly split subvarieties.*
(6) *The projected Richardson stratification on* $\mathrm{Gr}_k(\mathbb{C}^n)$ *is the* **cyclic Bruhat decomposition**, *the common refinement of the* n *cyclic shifts of the Bruhat decomposition. Its strata are called* **positroid varieties**.

While (1) and (2) hold for the Richardson stratification of G/P, (3) and (4) hint that the finer projected Richardson stratification is better, and simpler!

As for degeneration:

(1) Theorem [KLS2]: The Hodge degeneration of a positroid variety $\pi(X_w^v) \subseteq \mathrm{Gr}_k(\mathbb{C}^n)$ is the Stanley-Reisner scheme of a ball, π(order complex of $[w, v]$).
(2) Theorem [KLS1]: For any G/P, π(order complex of $[w, v]$) is a shellable ball.
(3) Conjecture: The Chirivì degeneration of $\pi(X_w^v) \subseteq G/P$ is the Stanley-Reisner scheme of π(order complex of $[w, v]$).

We know a bit more about the positroid case. Consider the degenerations

$$\mathrm{Gr}_k(\mathbb{C}^n) \dashrightarrow TV(GC) \dashrightarrow SR(\mathrm{order}(W/W_P)),$$

inducing maps backwards on posets,

$$\text{positroids} \leftarrow \{\text{faces of GC}\} \leftarrow \{\text{skew-tableaux}\}$$

The map from skew-tableaux to faces was explained in §2. The map from faces to Richardsons was explained in theorem 1. From there we get a positroid variety as a projected Richardson variety.

9. Vakil's degeneration of Grassmannian Richardson varieties through interval positroid varieties

So far the degenerations have been of projected Richardson (e.g. Richardson (e.g. Schubert)) or K-L varieties

- in affine or projective space (Hodge, theorem 5)
- mapping finitely to projective space (Chirivì, conjecturally)
- inside a bigger flag manifold (our interpretation of Feigin's)
- or done as SAGBI degenerations (Gonciulea-Lakshmibai, Caldero).

and based on $\mathbb{G}_m$-actions (AKA Gröbner degenerations).

Vakil introduced an embedded degeneration-in-stages of Richardson varieties in Grassmannians to unions of Schubert varieties, with two new features: after each stage, one has to break into components (to avoid creating multiplicities), and it's based of $\mathbb{G}_a$-actions.

In particular it requires an extra tweak, if one wants to compute the T-*equivariant* cohomology class of a Richardson variety, not just the ordinary cohomology.

9.1. **Combinatorial shifting.** The "shift" operation $Ш_{i\to j}$ due to [Erdős-Ko-Rado] "turns i into j, unless something's in the way". We'll define the **combinatorial shift** for numbers k, sets S of numbers, and collections $\mathcal{C}$ of sets of numbers:

$$Ш_{i\to j}k \;:=\; \begin{cases} k & \text{unless } m = i \\ j & \text{if } m = i \qquad \text{(nothing can be in the way)} \end{cases}$$

$$Ш_{i\to j}S \;:=\; \left\{ \begin{cases} Ш_{i\to j}k & \text{unless } Ш_{i\to j}k \in S \\ k & \text{if } Ш_{i\to j}k \in S \end{cases} \;:\; k \in S \right\} = \begin{cases} S & \text{unless } j \notin S \ni i \\ S \setminus \{i\} \cup \{j\} & \text{if } j \notin S \ni i \end{cases}$$

$$Ш_{i\to j}\mathcal{C} \;:=\; \left\{ \begin{cases} Ш_{i\to j}S & \text{unless } Ш_{i\to j}S \in \mathcal{C} \\ S & \text{if } Ш_{i\to j}S \in \mathcal{C} \end{cases} \;:\; S \in \mathcal{C} \right\}$$

Note that $Ш_{i\to j}$ preserves cardinality. The linear analogues of $k, S, \mathcal{C}$ will be basis elements, linear subspaces, and subschemes of $\mathrm{Gr}_k(\mathbb{C}^n)$, for which we define[2] the **geometric shift** as

$$Ш_{i\to j}X := \lim_{t\to\infty} \exp(te_{ij}) \cdot X$$

where e_{ij} is a matrix with only an (i, j) entry.

The combinatorial and geometric shifts are related by two constructions:

$$\left\{ \text{collections } \mathcal{C} \subseteq \binom{n}{k} \right\} \qquad \sim \qquad \{\text{T-invariant subschemes } X \subseteq \mathrm{Gr}_k(\mathbb{C}^n)\}$$

$$X^T \quad \text{fixfix} \quad X$$

$$\mathcal{C} \quad \mapsto \quad X(\mathcal{C}) := \bigcap_{S \in \binom{n}{k} \setminus \mathcal{C}} \{p_S = 0\}$$

where the elements of X^T are coordinate k-planes, the set of whom we identify naturally with $\binom{n}{k}$.

Theorem 7. [K2]

(1) *For any* $\mathcal{C}$, $X(\mathcal{C})^T = \mathcal{C}$.
(2) *For any* reduced X, $X(X^T) \supseteq X$.
(3) *If* X *is a Schubert variety, then* $X(X^T) = X$ *(Hodge's theorem from before). This also holds for positroid varieties* [KLS2].
(4) *For any* X, $(Ш_{i\to j}X)^T \subseteq Ш_{i\to j}(X^T)$.
(5) *If* $X = X(\mathcal{C})$ *or is irreducible, then* $(Ш_{i\to j}X)^T = Ш_{i\to j}(X^T)$.

We will also need a **sweep** operation, which combinatorially is just $\Psi_{i\to j}X := X \cup Ш_{i\to j}X$. The **geometric sweep** is

$$\Psi_{i\to j}X := \overline{\bigcup_{t\in\mathbb{A}^1} \exp(te_{ij}) \cdot X}$$

[2]If such limits are unfamiliar, look ahead to the definition of "sweep" below.

which is obviously the projection of a flat family

$$\widetilde{\Psi_{i\to j}X} := \overline{\bigcup_{t\in\mathbb{A}^1}\{t\}\times \exp(te_{ij})\cdot X} \quad \subseteq \mathbb{P}^1\times \mathrm{Gr}_k(\mathbb{C}^n)$$

over $\mathbb{P}^1$. The shift is exactly defined as the fiber over ∞, added in taking the closure.

Theorem 8. (1) *In ordinary cohomology and* K*-theory,* $[X] = [Ш_{i\to j}X]$.
(2) *In* T*-equivariant cohomology, with base ring* $H^*_T = \mathbb{Z}[y_1,\ldots,y_n]$,

$$[X] = [Ш_{i\to j}X] + d(y_i - y_j)[\Psi_{i\to j}X]$$

where d *is the degree of the map* $\widetilde{\Psi_{i\to j}X} \twoheadrightarrow \Psi_{i\to j}X$ *(or* 0 *if* X *is* $Ш_{i\to j}$*-invariant).*
(3) *If* $d = 1$ *and* $\Psi_{i\to j}X$ *has rational singularities, then in* T*-equivariant* K*-theory, with base ring* $H^*_T = \mathbb{Z}[\exp(\pm y_1),\ldots,\exp(\pm y_n)]$,

$$[X] = \exp(y_j - y_i)[Ш_{i\to j}X] + (1-\exp(y_j - y_i))[\Psi_{i\to j}X]$$

9.2. **Reducedness properties of judiciously chosen shifts.** Vakil gives a particular **degeneration order**, an ordering $((i_k, j_k))$ of the pairs $\{(i,j)\ :\ i<j\}$ in which to do shifts.

We can state most of his results fairly quickly, in an abstract way. Let V_k be a set of subvarieties of $\mathrm{Gr}_k(\mathbb{C}^n)$, defined inductively as follows:

- $V_0 := \{X^\mu_\lambda\ :\ \lambda,\mu \in \binom{n}{k}\}$
- $V_{k>0} := \bigcup_{X\in V_{k-1}}\{\text{components of } Ш_{i_k j_k}X\}$

Vakil proved some important theorems about these varieties and their shifts:

Theorem 9. [V]

(1) *For* $X \in V_{k-1}$, *the shift* $Ш_{i_k j_k}X$ *is generically reduced, with one or two components (explicitly described, but not here).*
(2) $V_{\binom{n}{2}}$ *is the set of opposite Schubert varieties.*
(3) *Hence, the homology class* $[X^\nu_\mu]$ *is an explicit sum (with repetition but not explicit multiplicities) of classes* $[X^\lambda]$, *which computes the Littlewood-Richardson coefficients positively.*

I strengthened these in several ways:

Theorem 10. [K2]

(1) *If one expands* V_0 *to consist of* interval Schubert varieties *(defined below), then each* $V_k \subseteq V_0$. *In particular this gives equations for all of Vakil's varieties (which he defined as closures of certain locally closed sets).*
(2) *Each shift* $Ш_{i_k j_k}X$ *is reduced (not just generically), and the intersection of any set of* m *components is codimension* m *and an interval Schubert variety (in particular, reduced).*
(3) *Each sweep is an interval Schubert variety, and its* $\widetilde{\Psi_{i\to j}X} \to \Psi_{i\to j}X$ *has degree* 1.
(4) *Hence, the* K_T*-homology class* $[X^\nu_\mu]$ *is an explicit sum (with repetition but not explicit multiplicities) of classes* $[X^\lambda]$ *times factors* $\exp(y_j - y_i)$ *and* $1-\exp(y_j-y_i)$. *This computes several generalizations of Littlewood-Richardson coefficients appropriately positively.*

We now get around to defining interval positroid varieties. There is one for each upper triangular partial permutation matrix f of rank $n-k$, defined by

$$\Pi_f := \left\{ rowspan(M) \ : \ M \in M_{k\times n}^{rank=k},\ rank(\text{columns } [i,j]) \leq |[i,j]| - \sum_{i\leq a\leq b\leq j} f_{ab} \right\}$$

Lemma 3. [K2]

(1) *If f's 1s run NW/SE, then Π_f is a Richardson variety X_μ^ν, with μ determined by the rows of f's 1s and ν determined by the columns.*
(2) *In particular, if f's 1s run NW/SE and are in the top r rows of f (where $r = rank(f)$), then $\Pi_f = X^\nu$.*

To actually derive a formula from these results one needs to index the elements of each V_k, and determine the components of each shift (and their intersections for K-theory, and the sweep for equivariant cohomology). Vakil indexes V_k by 2-dim "checkerboards", so that any single term in the answer is given by a $(2+1)$-dim "checker game". I streamline the checkerboards to 1-dim objects, and the checker games to $(1+1)$-dim "IP pipe dreams".

It is unfortunate that these techniques apply only to *interval* positroid varieties, not arbitrary positroid varieties, whose classes solve the Schur-times-Schubert problem. (This problem has been solved combinatorially in [ABS].) Unfortunately, there are examples of positroid varieties for which no shift is a nontrivial union of positroid varieties [Fo].

Question: Are there other $\mathrm{Gr}_k(\mathbb{C}^n)$-embedded degenerations of positroid varieties, perhaps through some larger class, giving a geometric proof of the [ABS] formula?

One combinatorial motivation to do this is that geometric proofs typically extend to equivariant K-theory, not just cohomology.

10. A fact seeking a degenerative proof

Consider the T-space $\prod_{\beta\in\Delta_+} \mathbb{P}(\mathbb{C}_{\beta/2}\oplus\mathbb{C}_{-\beta/2})$, carrying the line bundle $\boxtimes_{\Delta_+}\mathcal{O}(1)$, with space of sections $\bigotimes_{\beta\in\Delta_+}(\mathbb{C}_{\beta/2}\oplus\mathbb{C}_{-\beta/2})$. This T-rep is isomorphic to V_ρ, as is straightforward to prove from WCF once you're told it. More generally,

Proposition 1 (Kostant, personal communication)**.** *The T-spaces $\prod_{\beta\in\Delta_+} \mathbb{P}(\mathbb{C}_{\beta/2}\oplus\mathbb{C}_{-\beta/2})$ with this line bundle, and $G/B \hookrightarrow \mathbb{P}(V_\rho)$, have the same (multigraded) Hilbert function.*

Consequently, Hartshorne's thesis says that they can be connected by a flat family of subschemes of $\mathbb{P}(V_\lambda)$. However, neither can degenerate to the other, as they are smooth with different topologies.

Questions:

- Is there a common degeneration of both?
- Is there a flat family of T-invariant subschemes connecting them?
- (Together) Is there a T-equivariant degeneration of both?

For example, one could G-L degenerate $\mathbb{P}(V_\lambda)$ to the GC toric variety, and then construct isomorphic tesselations of the GC polytope and the cube, the isomorphism compatible with their maps to $\mathfrak{t}^*$. For $GL(3)$, this is easy enough to do.

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E-mail address: allenk@math.cornell.edu