Open Questions in AIM Workshop on Brownian Motion and Random Matrices, Dec 14 - 18, 2009

April 30, 2010

1.(Forrester) The equivalence of the SDE description of the β-circular and β-Gaussian ensembles in the bulk limit.

2.(Forrester et al) Analyse recursions for other (compare 1 above) β-ensembles and derive the corresponding bulk / edge limits.

3.(Virag) Is there a tridiagonal (or recursion) type representation for the natural lattice version of the $sine_\beta$ process (for eg in the context of weight-modified Plancherel measure) ?

4.(Edelman) To go from the SDE for the β-Laguerre ensemble to the known formula for the hard edge; $\beta = 1, 2$ first, general later.

5.(Zeitouni) What can be done about the process given by $|\Delta(\lambda)|^\beta e^{-N\sum_{i=1}^N V(\lambda_i)}$ (in $\mathbb{C}^n$ or other geometries) ?

6.(Stoiciu) Transition from clock to Poisson behaviour as β changes for β-ensembles other than the β-circular one (more examples other than the 1-d Anderson model).

7.(Killip) Prove (or disprove) the conjecture that for the $sine_\beta$ process the Laplace transform is analytic in $\beta \in [0, \infty]$.

8.(Rains) Is it possible to extend analytically (if answer to 7 above is affirmative) to $\beta < 0$, and if it is possible what does that mean? Connection to negative Jack polynomials ?

9.(Rains) Is there an explicit formula for the differential equation associated with some higher order Selberg integrals ?

10.(Killip) Can we explain empirically the random matrix limiting behaviour of some interesting deterministic models using the technology of this workshop? If so, how? Is this relationship universal ?

11.(Forrester) Low rank perturbations of β-ensembles and their limiting behaviour (starting with rank one) beyond known special cases.

12.(Virag) Can we control the tri-diagonalisation of Wishart models coming from some general covariance matrices (having some known limiting behaviour)? What is the connection, if any, with free probability? (Zeitouni) Alternatively, start with GUE + (full rank) diagonal model.

13.(Killip / Virag) Make sense of the $N \to \infty$ limit of the β-ensemble analog of Dyson Brownian motion (beyond what is known) ? (Bloemendal) What are the correct ways to put time evolution on the limiting operator or point process? (Zeitouni) Relate and expand the McKean-Vlasov with singular drift ?

14.(Rider) Hard to soft edge transition or soft to bulk transition, observed at the limiting operator level.

15.(Valko) Connection between $sine_\beta$ and $sine_{4/\beta}$, same for Airy process? (Virag) Based on erasing points ?

16.(Corwin/Bloemendal/Forrester) $Airy_1$ relating to KPZ and interlacing (Warren-type) Brownian motion.

17.(Virag) Let $p_n(x)$ be characteristic polynomial for GUE (or GOE). Consider $\frac{p_n(\frac{x}{\sqrt{n}})}{p_n(0)}$, or similar normalisation. Does it converge in distribution to a random analytic function?

18.(Breuer) Suppose we have a compactly supported measure on $\mathbb{R}$. Consider the orthogonal polynomials. Prove a lower bound on the replusion of the zeroes in terms of the Hausdorff dimension of the measure.

19.(Killip) Describe the law of the limiting spectral (orthogonality) measures, for eg for

$$\begin{bmatrix} N(0,1) & \chi_\beta & 0 & \cdots \\ \chi_\beta & N(0,1) & \chi_{2\beta} & \cdots \\ 0 & \chi_{2\beta} & N(0,1) & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

Show that in the circular case, it is mutually absolutely continuous to the multiplicative cascade, ie weak* limit as $N \to \infty$ of

$$\frac{1}{Z_N} exp[c_\beta \sum_{j=1}^{N} \frac{Re(X_j e^{ij\theta})}{\sqrt{j}}]d\theta$$

where the X_j s are iid complex Gaussian.

20.(Killip) (a)Why is $lim_{L\to\infty} \sum f(\frac{x_i}{L})$ (with $\int f = 0$) Gaussian with variance proportional to $||f||^2_{\dot{H}^{1/2}}$?

(b)β-analogue in Ginibre case.

21.(Virag / Killip) Understand and define charactersitic polynomials of minors for general β as random analytic functions.

22.(Conrey) Tracy Widom law in the setting of the zeta function.

23.(Conrey) Lower order arithmetic terms (finite T) for neighbour spacing of ζ.

24.(Conrey) What is the radial distribution of zeroes of $\Lambda_X^{'}(s)$ on a scale of $1/N$. Here

$$\Lambda_X(s) = \prod_{j=1}^{N}(1 - se^{-i\theta_j})$$

and

$$Z_X(s) = s^{-N/2}\Lambda_X(s).$$

25.(Conrey)

$$\int_{U(N)} |\Lambda_X'(1)|^{2k} dX$$

$$|Z_X'(1)|^{2k} dX \approx b_k N^{k^2+2k} (asN \to \infty).$$

$k \in \mathbb{Z}$. What about for k real ? (interpolation ...)

$$b_k = g_k(\frac{d}{dx})^{2k}_{|x=0} e^{-x/2} x^{-k^2/2} exp[\frac{x}{2} - \int_0^x (\sigma(s) + k^2)\frac{ds}{s}]$$

26.(Conrey)

$$\int_{U(N)} |Z_X(1)Z_X'(1)| dX \approx \frac{e^2 - 5}{4\pi} N^2$$

Done in number theory assuming Riemann hypothesis.

27.(Conrey) Moments: Riemann Hypothesis implies $\zeta(1/2 + it) << t^{\varepsilon}$.

$$\int_1^T |\zeta(1/2 + it)|^2 dt \approx TlogT \ldots (1916)$$

$$\int_1^T |\zeta(1/2 + it)|^4 dt \approx \frac{T}{2\pi^2} log^4 T \ldots (1926)$$

$$\ldots$$

$$\int_1^T |\ zeta(1/2 + it)|^{2k} dt \approx g_k a_k \frac{T(logT)^4}{k!}$$

where

$$a_k = \prod (1 - \frac{1}{p})^{(k-1)^2} (1 + \frac{\binom{k-1}{1}}{p} + \frac{\binom{k-1}{2}^2}{p^2} + \ldots)$$

$$g_{2k} = \int_{U(N)} |det(I - X)|^{2k} dX$$

$$=_{Selbergintegral} \frac{(N+1)(N+2)^2 \ldots (N+k)^k (N+k+1)^{k-1} \ldots (N+2k-1)}{1.2^2 \ldots . k^k (k+1)^{k-1} \ldots (2k-1)}$$

28.(Conrey) Prove that

$$M = \int_{U(N)} \Lambda_X(e^{-\alpha_1}) \ldots \Lambda_X(e^{-\alpha_n}) \Lambda_{X^k}(e^{-\beta_1}) \ldots \Lambda_{X^k}(e^{-\beta_n}) dX$$

where

$$M = \sum_{S\subset A, T\subset B, |S|=|T|} e^{-N\sum_{\alpha\in S,\beta\in T}(\alpha+\beta)} Z((A-S)\cup T; (B-T)\cup S)$$

$$\int_{U(k)} Z(A;B) = \prod_{\alpha\in A,\beta\in B} z(\alpha+\beta)$$

$$z(x) = \frac{1}{1-e^{-x}} <--- > \zeta(1+x).$$