June 7, 2007 F. Bergeron

$$\tilde{H}_\mu(z;q,t) = \sum_{|\lambda|=n} \tilde{K}_{\lambda\mu}(q,t)\, s_{\lambda}(z)$$

Schur functions

$z = z_1, z_2, \ldots$

$\tilde{K}_{\lambda\mu}(q,t)$ non-negative integer coeff.

(q,t)-Kotska polynom

Macdonald polynomial

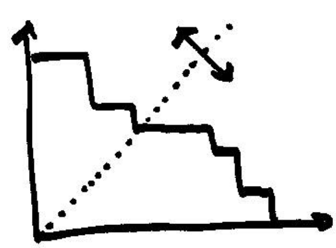
$|\mu| = 4$

$$\tilde{H}_4 = s_4 + (q^3+q^2+q)\, s_{31} + (q^4+q^2)\, s_{22} + (q^5+q^4+q^3)\, s_{211} + q^6\, s_{1111}$$

$$\tilde{H}_{31} = s_4 + (q^2+q+t)\, s_{31} + (q^2+qt)\, s_{22} + (q^3+q^2t+qt)\, s_{211} + q^3t\, s_{1111}$$

$$\tilde{H}_{22} = s_4 + (qt+q+t)\, s_{31} + (q^2+\ \ )\, s_{22} + \cdots$$

conjugate partition



Symmetry

$$\tilde{H}_{\mu'}(z;q,t) = \tilde{H}_\mu(z;t,q)$$

Another symmetry

$$\tilde{H}_\mu(z; q, t) = q^{n(\mu')} t^{n(\mu)} \, \omega \, \tilde{H}_\mu(z; q^{-1}, t^{-1})$$

$$n(\mu) = \sum_i (i-1)\mu_i$$

ω involution on symmetric fctns

$$\omega s_\lambda = s_{\lambda'}$$

$$\left[\begin{array}{l} \tilde{K}_{\lambda\mu}(q,t) = \tilde{K}_{\lambda\mu'}(t,q) \\ \tilde{K}_{\lambda\mu}(q,t) = q^{n(\mu')} t^{n(\mu)} \tilde{K}_{\lambda'\mu}(q^{-1}, t^{-1}) \end{array}\right.$$

$$\tilde{H}_\mu(z; 1, 1) = s_4 + 3s_{31} + 2s_{22} + 3s_{211} + s_{1111}$$

$|\mu| = 4$

$$= s_1^{\,4}$$

$$\tilde{K}_{\lambda\mu}(1,1) = \chi^\lambda(1) \quad \text{any } \mu$$

$\underline{t=1}$, $\quad \mu = \mu_1 \geq \mu_2 \geq \cdots \geq \mu_k$

$$\tilde{H}_\mu(z; q, 1) = \tilde{H}_{\mu_1}(z; q, 1)\, \tilde{H}_{\mu_2}(z; q, 1) \cdots \tilde{H}_{\mu_k}(z; q$$

$\uparrow$
does not actually depend on t

$$\tilde{H}_n(z; q, 1) = (1-q)\cdots(1-q^n)\, h_n\left[\frac{z}{1-q}\right]$$

$$h_n(z) = \sum_{1d_1 + 2d_2 + \cdots + nd_n = n} \frac{\left(\frac{p_1}{1}\right)^{d_1}}{d_1!} \frac{\left(\frac{p_2}{2}\right)^{d_2}}{d_2!} \cdots$$

$$p_k = z_1^k + z_2^k + \cdots$$

$$h_n(z) = \sum_{|\lambda|=n} \frac{p_\lambda}{z_\lambda} = \frac{1}{n!} \sum_{\sigma\in S_n} p_{\lambda(\sigma)}$$

$$p_\lambda = p_{\lambda_1} \cdots p_{\lambda_k}$$

$$h_n\left[\frac{z}{1-q}\right] = \sum_{|\lambda|=n} \frac{1}{z_\lambda} \, p_{\lambda_1}\left[\frac{z}{1-q}\right] p_{\lambda_2}\left[\frac{z}{1-q}\right] \cdots$$

$$p_m\left[\frac{z}{1-q}\right] = \frac{p_m(z)}{1-q^m}$$

<u>EX.</u> Expand $h_n\left[\frac{z}{1-q}\right]$ in terms of Schur functions

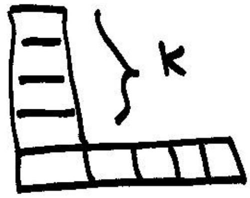
$$\lim_{q\to 1} \tilde{H}_n(z;q,1) = (s_1)^n$$

<u>$q=0$</u> $\quad \tilde{H}_\mu(z;0,t)$ <u>Hall-Littlewood</u>

$$\tilde{H}_\mu(z;q,q^{-1}) = q^{-n(\mu)} \prod_{c\in\mu} (1-q^{h(c)}) \, s_\mu\left[\frac{z}{1-q}\right]$$

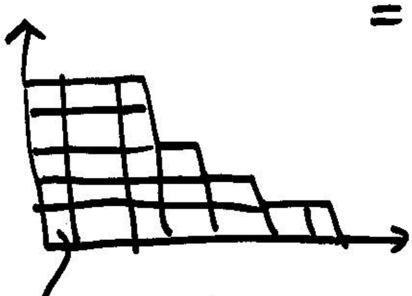
$$s_\mu(z) = \sum_{|\lambda|=n} \chi^\mu(\lambda) \frac{p_\lambda}{z_\lambda}$$

(Proofs in original paper by Macdonald)

$\lambda =$ 

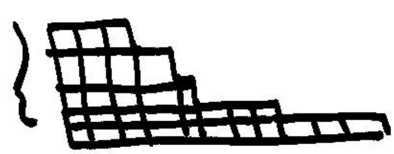
$$\tilde{K}_{\lambda,\mu}(q,t) = e_k\Big[\sum_{(a,b)\in\mu} q^a t^b - 1\Big]$$

$$= e_k(q, t, qt, \dots)$$



In general: There exist k_γ symmetric functions s.t.

$$\tilde{K}_{\lambda\mu}(q,t) = k_\gamma\Big[\sum_{(a,b)\in\mu} q^a t^b\Big]$$



$\gamma :=$ remove first row of λ

e.g. $k_{(2)} = q^{-1}t^{-1}\big((1+q+t)e_2 - e_1^2 + (1-qt)e_1\big)$

Rule for computing k_γ using an operator ∇

$$\sum_{|\lambda|=n} \frac{\tilde{H}_\lambda(z_1)\,\tilde{H}_\lambda(z_2)\cdots\tilde{H}_\lambda(z_k)}{\prod(q^a - t^{l+1})(q^{a+1} - t^l)}$$

$\underline{k=2}$ $$e_n\left[\frac{xy}{(1-t)(1-q)}\right] = \sum_{|\lambda|=n} \frac{\tilde{H}_\lambda(x)\,\tilde{H}_\lambda(y)}{\cdots}$$

If $y=(1-t)(1-q)$ then

$$e_n(x) = \sum_{|\lambda|=n} \frac{\tilde{H}_\lambda[(1-t)(1-q)]}{\cdots\cdot}\tilde{H}_\lambda(x)$$

expansion of e_n in basis $\tilde{H}_\lambda$.

∇ linear operator on symmetric fctns defined by

$$\nabla(\tilde{H}_\lambda) := q^{n(\lambda')}\, t^{n(\lambda)}\,\tilde{H}_\lambda$$

(∇ is also multiplicative for $t=1$)

Apply ∇ to e_n

$$\nabla e_n(x) = \sum_{|\lambda|=n} \frac{\tilde{H}_\lambda[(1-t)(1-q)]}{\cdots}\, q^{n(\lambda')}\, t^{n(\lambda)}\, \tilde{H}_\lambda(x)$$

$$\langle \tilde{H}_\lambda, \underset{\underset{e_n}{\shortparallel}}{S_{1^n}} \rangle = q^{n(\lambda')}\, t^{n(\lambda)}$$

Space of diagonal coinvariants

$X = x_1 \cdots x_n$

$Y = y_1 \cdots y_n$

$$\mathbb{C}[X,Y] \Big/ \left(\sum_{i=1}^{n} x_i^p y_i^q \,,\; p+q>0 \right)$$

dim $(n+1)^{n-1}$

information on this encoded by $\nabla(e_n)$

in Schur basis.