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Reflection Functors

1. Quivers

Q quiver w/ vertices $1, \dots, n$.
The path algebra $\mathbb{C}Q$ is the associative algebra with basis the paths in Q

E.g.

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r, bend left, "a"] \arrow[r, "b"'] & \underset{2}{\bullet} \arrow[r, "c"] & \underset{3}{\bullet} \end{tikzcd}$$

$e_1, e_2, e_3, a, b, c, ca, cb \leftarrow$ basis

$c \cdot a = ca$

$c \cdot c = 0$

$e_i^2 = e_i$

$e_1 + e_2 + e_3 = 1$

$\{\text{modules} / \mathbb{C}Q\} \leftrightarrow \{\text{repn of } Q\}$

Fact: Any f.g. associative algebra A can be presented as a quotient $A = \mathbb{C}Q/J$ (J 2-sided ideal). Then A-modules $\simeq$ category repns satisfying the relations in J.

E.g. Free associative algebra on n quantities $\leftrightarrow$ Q:

Kac's theorem

There is an indec. repn of Q of dimension vector $\alpha \in \mathbb{Z}^n_{\geq 0}$ $\leftrightarrow$ α is a positive root

E.g.

$\bullet \rightrightarrows \bullet \to \bullet$ simple roots

$1 \rightrightarrows 0 \to 0$

$0 \rightrightarrows 1 \to 0$

$0 \rightrightarrows 0 \to 1$

real roots $1 \rightrightarrows 0 \to 0 \rightsquigarrow 1 \rightrightarrows 2 \to 0 \rightsquigarrow 1 \rightrightarrows 2 \to 2 \rightsquigarrow$

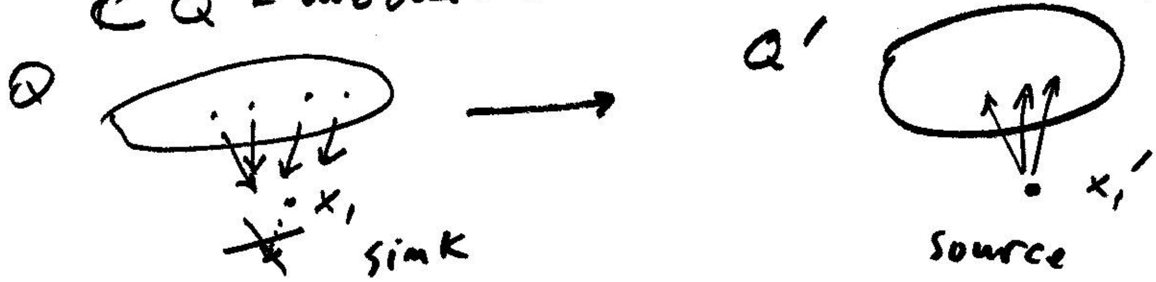
add dim of adjacent vertices

in fund. region

imaginary roots $2 \rightrightarrows 2 \to 0 \rightsquigarrow 2 \rightrightarrows 2 \to 2 \rightsquigarrow 2 \rightrightarrows 4 \to 2$

Reflection functor of Bernstein Gelfand & Ponomarev

$\mathbb{C}Q$-modules $\longrightarrow$ $\mathbb{C}Q'$-modules



$$X_1' = \mathrm{Ker}(X_2 \oplus X_3 \oplus \cdots \to X_1)$$

Bijection

$$\{\text{indec. for } Q\} \setminus \{S_1\} \to \{\text{indecomp for } Q'\} \setminus \{S_1\}$$

$\rightsquigarrow$ proof of Kac's theorem

S_1 $\quad 0 \to \mathbb{C} \leftarrow 0 \ \cdots \ 0 \to \mathbb{C}$

<u>Generalizations</u>

Tilting theory Derived equivalences

Cluster tilting theory

<u>(Deformed) preprojective algebras</u>

Q quiver $\quad \lambda \in \mathbb{C}^n$

$$\Pi^\lambda(Q) := \mathbb{C}\overline{Q} \Big/ \Big(\sum_{a \in Q} (aa^* - a^*a) - \sum_{i=1}^{n} \lambda_i e_i \Big)$$

$\overline{Q}$ = double of Q: add reverse arrow to each arrow in Q

Modules for $\Pi^\lambda(Q) \leftrightarrow$ representations of $\overline{Q}$ satisfying

$$\sum_{\substack{a \in Q \\ h(a)=i}} aa^* - \sum_{\substack{a \in Q \\ t(a)=i}} a^*a = \lambda_i \Big|_{X_i} \quad \text{for all } i$$

Additive Deligne-Simpson Pblm

$C_1, \dots, C_m$ $\quad gl_n(\mathbb{C})$ conj. classes

solution to $\quad$ There exist starshaped Q, λ, α s.t.

$$A_1 + \cdots + A_k = 0 \quad \longleftrightarrow \quad \text{"strict" repn of } \Pi^\lambda(Q) \text{ of dim vector } \alpha$$

~~same starshaped~~

$A_i \in C_i$

Say $A_i^{\in C_i}$ has minimal polynomial

$(t - \xi_{i1})(t - \xi_{i2}) \cdots$ $\quad$ Fix ordering of roots

$$\begin{array}{ccc} & \mathbb{C}^n & \\ {\scriptstyle A_1 - \xi_{i1} I} \swarrow\nearrow & & \searrow\nwarrow \\ \mathrm{Im}(A_1 - \xi_{i1} I) & & \ddots \\ \swarrow\nearrow & & \\ \mathrm{Im}(A_1 - \xi_{i1})(A_1 - \xi_{i2}) & & \\ \swarrow\nearrow & & \\ \vdots & & \end{array}$$

α dimension vector determined by $\mathrm{Im}(A_1 - \xi_{i1} I) \cdots$ etc.

"strict" means $\swarrow \nearrow$ in all cases

<u>THM</u> There exist a simple repn of $\Pi^\lambda(Q)$ of dim vector α

$$\Updownarrow$$

α is a positive root

$$\lambda\alpha := \sum_i \lambda_i \alpha_i = 0$$

There is no nontrivial decomposition

$\alpha = \beta + \gamma + \cdots$ $\quad \beta, \gamma$ positive roots

with $\lambda\cdot\beta = \lambda\cdot\gamma = \cdots = 0$

and $p(\alpha) \le p(\beta) + p(\gamma) + \cdots$

$\alpha \in \mathbb{N}^n$ $\qquad$ quadr. form on root lattice

$$p(\alpha) = 1 - q(\alpha) = \begin{cases} 0 & \text{real roots} \\ >0 & \text{imag. roots} \end{cases}$$

(Conditions are easy to check; there are other simple forms of it)

Also

... simple rigid ... $\Leftrightarrow$ ... positive real ...

3. Multiplicative case

$Q \quad 1,\dots,n \qquad q \in (\mathbb{C}^\times)^n$

$\Lambda^q(Q) = \mathbb{C}\bar{Q}$ universally localize $(1 + aa^*)$

invert $\quad a \in \bar{Q} \quad (a^*)^* = a$

$$\Big/ \Big(\prod_{a \in \bar{Q}} (1 + aa^*)^{\pm 1} - \sum_i q_i e_i \Big)$$

Deligne-Simpson pblm

THM analogous to the additive case.

$A_1 \cdots A_k = 1 \quad A_i \in C_i \quad$ in $Gl_n(\mathbb{C})$

MC <u>Middle convolution</u> (N. Katz)

Algebraic version by Dettweiler & Reiter

<u>First</u> Reflection functors of CB + Holland (Nakajima, Rump)

Given $\Pi^\lambda(Q)$ Suppose i is a vertex with no loop at i, $\lambda_i \neq 0$

$\Pi^\lambda(Q)$-modules $\longleftrightarrow$ $\Pi^{\lambda'}(Q)$-modules

which acts as the reflection at i on dimension vectors.

$$\begin{tikzcd} X_1 \arrow[r, bend left, "a"] & X_i \arrow[l, bend left, "a^*"] \arrow[r, bend right, "b^*"'] \arrow[d, squiggly] & X_2 \arrow[l, bend right, "b"'] \\ & X_i' & \end{tikzcd}$$

$$\begin{tikzcd} X_i \arrow[r, "p"] \arrow[rr, bend right, "\lambda_i 1_{X_i}"'] & X_1 \oplus X_2 \arrow[r, "q"] & X_i \end{tikzcd}$$

$$\underset{X_i}{\mathrm{Im}(p)} \oplus \underset{X_i'}{\mathrm{Ker}(q)} = X_1 \oplus X_2$$

<u>Back to $\Lambda^q(Q)$</u>

Can reformulate MC as reflection functor

$\Lambda^q(Q)$-modules $\longleftrightarrow$ $\Lambda^{q'}(Q)$-modules

$q_i \neq 1$

(subtle to define use matrices in Dettweiler & Reiter originally arising from studying Pochammer diff eqn.)

(Complicated conditions in DR correspond to restricting to simple rep^n supported on central vertex which are those related to the Deligne-Simpson pblm)

<u>THM</u> There exist simple rep^n of $\Lambda^q(Q)$ of dim α

$\Downarrow$ $\quad$ $\Uparrow$

not yet written up

α positive root

$q^\alpha := \prod q_i^{\alpha_i} = 1$

no decomp… ..

Constructive for rigid case