June 6, 2007 T. Hausel

Hyper kähler mflds

A Riemannian mfld M is hyperkähler if it is Kähler w.r.t. three cplx structures I, J, K, $IJK = -1$

In physics susy
$N=4$ supersymmetric algebra acts on $\Omega^*(M)$ via operators in hyperkähler Hodge theory.

Examples $N=4$ susy Yang-Mills theory on $X = \mathbb{R}^4$ or $\widetilde{\mathbb{C}^2/\Gamma}$ ALE space

$\rightsquigarrow$ moduli spaces $\mathcal{M}$ of Yang-Mills instantons on X $\rightsquigarrow$ certain quiver varieties.

$\Big\downarrow$ reduction to 2-dimensions (impose symmetry under translation in 2-dimensions)

Sigma model with target moduli $\mathcal{M}$ space of Hitchin pairs on a Riemann surface Σ

Physics S-duality: $SL_2(\mathbb{Z})$ acts on $N=4$ susy YM

<u>Nakajima</u> $H^*(M)$ is acted upon by an affine Kac-Moody algebra

$\rightsquigarrow$ modular forms via Kac

<u>Kapustin-Witten</u> $\rightsquigarrow$ relation between S-duality & Geometric Langlands program

moduli space of Hitchin pairs

$\mathcal{M}$

I

semistable

$\mathcal{M}_{Dol} = \left\{ \begin{array}{c} \text{moduli space of} \\ \text{Higgs bundles} \\ \text{on } C \end{array} \right\}$

$\uparrow$ alg curve structure on Σ

J, K

$\mathcal{M}_{DR} = \left\{ \begin{array}{c} \text{moduli space of flat} \\ Gl_n(\mathbb{C}) \text{ connections on } C \end{array} \right\}$

$\mathcal{M}_{Dol} \simeq \mathcal{M}_{DR}$ as differential mflds

Have only limited information on $H^*(\mathcal{M}_{Dol})$

$\mathcal{M}_{DR} \xrightarrow[\text{Riemann Hilbert map}]{}$ $\mathcal{M}_B = \left\{ \begin{array}{c} \text{moduli space of} \\ Hom(\pi_1(C)) \, Gl_n(\mathbb{C}) \end{array} \right\} / \text{conj}$

RH is an analytic isomorphism. but $\mathcal{M}_{DR}$, $\mathcal{M}_{B}$ are different as algebraic variety.

Σ genus g compact Riemman surface

$S = p_1, \dots, p_k \in \Sigma$ punctures, $k > 0$

$\mu = (\mu^1, \dots, \mu^k)$, $|\mu^i| = n$

$C_1, \dots, C_k$ conjugacy classes
semisimple with type $\mu^1, \dots, \mu^k$
(i.e. multiplicity of eigenvalues determined by μ^i)

$$\mathcal{M}_\mu := \{ \operatorname{Hom}_{C_1,\dots,C_k}(\pi_1(\Sigma \setminus S), Gl_n(\mathbb{C})) /\!/ Gl_n(\mathbb{C})$$

 $\longmapsto C_i$

$$= \left\{ [x_1, y_1] \cdots [x_g, y_g] z_1 \cdots z_k = 1 \;\; z_i \in C_i \right\} /\!/ Gl_n(\mathbb{C})$$

by conjugation

<u>THM</u> For each μ can choose generic choice of C_i s.t. M_μ is smooth.

Goal Compute Mixed Hodge polynomial of $\mathcal{M}_\mu$.

$$H_c^k(\mathcal{M}_\mu) \simeq \bigoplus H_{p,p}^k(\mathcal{M}_\mu)$$

(thm in some cases
conj. in general)

$k \geq 2p$ (pure if $k = 2p$ but not necessarily all is)

$$H_c(\mathcal{M}_\mu; q, t) = \sum h_{p,p}^{k} q^p t^k$$

$$P_c(\mathcal{M}_\mu, t) = H_c(\mathcal{M}_\mu; 1, t)$$

ie. refinement of Poincaré polynomial

$$E(\mathcal{M}_\mu; q) := H_c(\mathcal{M}_\mu; q, -1)$$

(another specialization)

Can compute this

If X is polynomial count

$$E(X, q) = \# X(\mathbb{F}_q)$$

Frobenius $E(\mathcal{M}_\mu, q) = \# \mathcal{M}_\mu(\mathbb{F}_q)$

$$= \sum_{\chi \in \mathrm{Irr}(Gl_n(\mathbb{F}_q))}$$

Thm 1.1.

Rewrite

$$H_c(\mathcal{M}_\mu, q, t) = \mathcal{H}_c(\mathcal{M}_\mu, qt^2, \tfrac{1}{q})$$

$$PH_c(\mathcal{M}_\mu, q) = \mathcal{H}_c(\mathcal{M}_\mu, q, 0)$$

(another specialization)

Pure part of MH polynomial

<u>Purity conjecture</u> $\quad PH_c^* = \bigoplus H^{2k}_{k,k}$

$Q\mu$ <u>quiver variety</u>

additive version of $\mathcal{M}_\mu$

$C_1, \dots, C_k$ adjoint orbits in $gl_n(\mathbb{C})$
semisimple

$$Q\mu := \left\{ (X_1Y_1 - Y_1X_1) + \dots + (X_gY_g - Y_gX_g) + Z_1 + \dots + Z_k = 0 \right\} /\!/ Gl_n(\mathbb{C})$$

$$Z_i \in C_i$$

μ indivisible
(i.e. gcd allparts $= 1$)
can choose C_i so $Q\mu$ is smooth.
generic

$\underline{g=0}$ $\quad Q\mu \ni$ point $\rightsquigarrow$ meromorphic flat connection

$$\sum_i Z_i \frac{dz}{z - p_i} \qquad Q\mu \subseteq \mathcal{M}_{DR} \xrightarrow{RH} \mathcal{M}_\mu$$

<u>Purity conjecture</u> 1st version

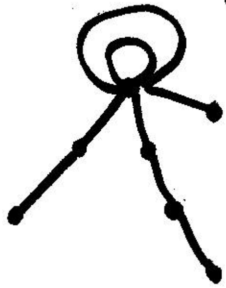
μ indivisible $g=0$

RH: $Q_\mu \longrightarrow M_\mu$

$$H_c^*(Q_\mu) \simeq PH_c^*(M_\mu)$$

The quiver here is

$\leftarrow g$ loops



$\leftarrow$ leg lengths given by μ^i

In fact conjecture in general $g \geq 0$

$$P_c(Q_\mu\mathcal{g}, t^2) = PH_c(M_\mu, t)$$

μ indivisible

Even in the divisible case can relate Kac A polynomial (counts abs. indec. representation of quiver over $\mathbb{F}_q$) to $PH_c(M_\mu, t)$

$$PH_c(M_\mu; q) = \mathcal{H}_c(M_\mu; q, 0)$$
$$= P_c(Q_\mu, \sqrt{q})$$
$$= E(Q_\mu, q) = \# Q_\mu(\mathbb{F}_q)$$
$$= \sum \ldots \text{(Fourier transform of char fctn of } C_i\text{)}$$

irred. chars of $gl_n(\mathbb{F}_q)$