Introduction to Quiver varieties

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Arithmetic harmonic analysis on character and quiver varieties

June 4 - 8, 2007 at American Mathematical Institute

§0. Very brief introduction to representations theory of quivers

quiver = oriented graph (we assume finite)

I = set of vertices , Ω = set of oriented edges

$$\underset{o(h)}{\circ} \xrightarrow{\ h\ } \underset{i(h)}{\circ}$$

<u>Def.</u> a representation of a quiver (over $\mathbb{C}$) is a pair

(V, B) $V = \bigoplus V_i$: I-graded vector space $/\mathbb{C}$

$B = \bigoplus_{h \in \Omega} B_h \in \mathrm{End}(V)$: Ω-graded endomorphism of V

$B_h : V_{o(h)} \to V_{i(h)}$

Given a quiver (I, Ω), its representation form an abelian category. (In fact, it is a category of representations of the path algebra.)

- $\mathrm{Hom}((V,B),(V',B'))$

$$= \{ \xi = \bigoplus \xi_i \in \mathrm{Hom}(V, V') \mid I\text{- graded hom's respecting } B \text{ and } B' \}$$

$$\begin{tikzcd}
V_{o(h)} \arrow[r, "B_h"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & V_{i(h)} \arrow[d] \\
V'_{o(h)} \arrow[r, "B'_h"] & V'_{i(h)}
\end{tikzcd}$$

- $\dim(V,B)$ (or $\dim V$) $:= (\dim V_i)_{i \in I} \in \mathbb{Z}_{\geq 0}^{I}$

Th. (Kac)

Suppose there is no $\circlearrowleft$. We define a Cartan matrix by

$$C_{ij} = 2\delta_{ij} - \#\text{ edges joining } i \text{ and } j\text{, regardless to the orientation}$$

Then:

$\exists$ indecomposable representation with $\dim = v \in \mathbb{Z}_{\geq 0}^{I}$

$\iff$ v is a root of the Kac-Moody Lie algebra corresponding to C.

Example $\circ \rightrightarrows \circ$ Kronecker quiver

Fact. $D^b((I,\Omega)\text{-rep.}) \cong D^b(\mathrm{coh}\,\mathbb{P}^1)$

$v = (n, n+1)$

$$\mathbb{C}^n \underset{\begin{bmatrix} 0 & & \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}}{\overset{\begin{bmatrix} 1 & & \\ & \ddots & \\ & & 1 \\ 0 & & \end{bmatrix}}{\rightrightarrows}} \mathbb{C}^{n+1}$$

$V = (n+1, n)$

$$\mathbb{C}^{n+1} \underset{\begin{bmatrix} 0 & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}}{\overset{\begin{bmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & 0 \end{bmatrix}}{\rightrightarrows}} \mathbb{C}^n$$

$V = (n, n)$

$$\mathbb{C}^n \underset{\begin{bmatrix} z & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & z \end{bmatrix}}{\overset{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}{\rightrightarrows}} \mathbb{C}^n \quad \text{or} \quad \mathbb{C}^n \underset{\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix}}{\overset{\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ 0 & & & 0 \end{bmatrix}}{\rightrightarrows}} \mathbb{C}^n$$

$z \in \mathbb{C}$ $\qquad$ $(z = \infty)$

§1. Definition of quiver variety

$\boxed{\text{quiver variety} \doteqdot \text{cotangent bundle of moduli space of representations of a quiver}}$

$\overline{\Omega}$: opposite oriented arrows

$$\overline{} : \Omega \to \overline{\Omega}, \quad h \mapsto \overline{h}$$

$H = \Omega \sqcup \overline{\Omega}$

$$\mathbb{M}_{\Omega}(V) := \bigoplus_{h \in \Omega} \mathrm{Hom}(V_{o(h)}, V_{i(h)}) \curvearrowleft G_V := \prod_i GL(V_i)$$

$$PG_V := G_V / \mathbb{C}^*$$

$\mathbb{M}_{\Omega}(V)/PG_V$ = isom. classes of representations of given dimension.

$$\mathbb{M}(V) = \mathbb{M}_{\Omega}(V) \oplus \mathbb{M}_{\overline{\Omega}}(V) \cong T^*\mathbb{M}_{\Omega}(V)$$

1st approx:

$$\boxed{\text{quiver variety} \doteqdot T^*\left(\mathbb{M}_{\Omega}(V)/PG_V\right)}$$

recipe from the symplectic geometry (Marsden-Weinstein quotient)

cotangent space of a quotient ≑ quotient of the moment map $=0$
in the cotangent

$$\mu : \mathbb{M}(V) \longrightarrow \operatorname{Lie} G_V = \bigoplus_{i\in I} \mathfrak{gl}(V_i) \qquad \text{moment map}$$

$$\begin{array}{ccc} \mathbb{M}(V) & & \mathfrak{gl}(V_i) \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \bigoplus_{h\in H} B_h & \longmapsto & \left(\sum_{h:\, i(h)=i} \varepsilon(h) B_h B_{\bar h}\right)_{i\in I} \end{array}$$

$$\boxed{\begin{array}{l} 2^{\text{nd}} \text{ approx}: \\ \qquad \text{quiver variety} \doteqdot \mu^{-1}(0)/PG_V \quad \text{or} \quad \overset{\text{more generally}}{\mu^{-1}(\zeta_{\mathbb{C}})/PG_V} \end{array}}$$

twisted cotangent

$$\zeta_{\mathbb{C}} = \bigoplus_i \zeta_{\mathbb{C}}^i \operatorname{id}_{V_i} \in (\operatorname{Lie} PG_V)^* \text{ s.t. } \sum \zeta_{\mathbb{C}}^i \dim V_i = 0$$

<u>Remark</u>. $\mu_{\mathbb{C}}(B) = \zeta_{\mathbb{C}}$ is the defining relation of the deformed preprojective algebra of Crawley-Boevey + Holland

But the set theoretical quotient behaves badly in algebraic geometry.

recipe2 : quotient in algebraic geometry = geometric invariant theory

We consider two types of quotients:

① $\mu^{-1}(S_{\mathbb{C}}) /\!/ PG_V = \mathrm{Spec}\left(\mathbb{C}[\mu^{-1}(S_{\mathbb{C}})]^{PG_V}\right)$

$=$ the set of closed PG_V-orbits in $\mu^{-1}(S_{\mathbb{C}})$

$=$ the set of semisimple representations of the deformed preprojective algebra

$=: \mathfrak{M}_{0,S_{\mathbb{C}}}(v)$

This is an affine algebraic variety. (possibly not irreducible)

② $\zeta_{\mathbb{R}} = (\zeta_{\mathbb{R}}^i)_{i\in I} \in \mathbb{R}^I$ s.t. $\sum \zeta_{\mathbb{R}}^i \dim V_i = 0$ (parameter)

$B \in \mu^{-1}(\zeta_{\mathbb{C}})$ is $\zeta_{\mathbb{R}}$-semistable

$\overset{\text{def.}}{\iff}$ $\forall (S = \bigoplus_{i\in I} S_i, B|_S)$: subrepresentation of (V, B), we must have

$$\frac{\zeta_{\mathbb{R}}\cdot \dim S := \sum \zeta_{\mathbb{R}}^i \dim S_i}{\sum \dim S_i} \leqq 0 = \frac{\zeta_{\mathbb{R}}\cdot \dim V}{\sum \dim V_i}$$

$<$ unless $S = 0$ or V

$\zeta_{\mathbb{R}}$-semistable representations form an abelian subcategory closed under extensions

Def. $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}}) \in (\mathbb{R}\oplus\mathbb{C})^I$

$$\mathfrak{M}^s_\zeta(V) \overset{\text{def.}}{=} \{ B \in \mu^{-1}(\zeta_{\mathbb{C}}) \mid \zeta_{\mathbb{R}}\text{-stable} \} / PG_V$$

$$\mathfrak{M}_\zeta(V) \overset{\text{def.}}{=} \{ B \in \mu^{-1}(\zeta_{\mathbb{C}}) \mid \zeta_{\mathbb{R}}\text{-semistable} \} / S\text{-equiv.}$$

semisimplification is isomorphic

These are quasiprojective varieties.

If $\zeta_{\mathbb{R}}=0$, we get the quotient in ①.

Practically we only consider $\zeta_{\mathbb{R}}=0$ or generic $\zeta_{\mathbb{R}}$.

$\exists$ projective morphism $\pi : \mathfrak{M}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}(v) \longrightarrow \mathfrak{M}_{0,\zeta_{\mathbb{C}}}(v)$

(resolution of singularities in many cases)

$\mathfrak{M}^{s}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}} \subset \mathfrak{M}_{\zeta_{\mathbb{R}},\zeta_{\mathbb{C}}}$ open

($=$ in many cases)

§2. Examples

(1) cotangent bundle of Grassmann manifold

$W = \mathbb{C}^r$: fix

$$\begin{tikzcd} 1 \cdots \circ \arrow[d, "r \text{ arrows}"] \\ \infty \cdots \circ \end{tikzcd}$$

We think
$$\begin{tikzcd} \mathbb{C}^k \arrow[d, shift left] \\ \mathbb{C} \arrow[u, shift left] \end{tikzcd}$$
as
$$\begin{tikzcd} \mathbb{C}^k \arrow[d, shift left, "b"] & PG_V \cong GL(k) \arrow[l] \\ \mathbb{C}^r = W \arrow[u, shift left, "a"] \end{tikzcd}$$

$$\mu = \zeta_{\mathbb{C}} \iff ab = \zeta^1_{\mathbb{C}}$$

$$\zeta_{\mathbb{R}}\text{-stable} \iff \begin{cases} b : \text{injective} & \text{if } \zeta^1_{\mathbb{R}} > 0 \\ a : \text{surjective} & \text{if } \zeta^1_{\mathbb{R}} < 0 \end{cases}$$

If $\zeta_{\mathbb{C}} = 0$, $\zeta^1_{\mathbb{R}} > 0$, then $\left(\operatorname{Im} b \subset W,\ ba : W/\operatorname{Im} b \longrightarrow \operatorname{Im} b\right)$ gives a point in $T^* Gr(k, r)$.

$$\therefore\ \mathfrak{M}_{\zeta}(v) \cong T^* Gr(k, r)$$

On the other hand $\mathfrak{M}_{0,\zeta_{\mathbb{C}}}(v) \cong \left\{ X = ba \;\middle|\; \begin{array}{l} X^2 = \zeta_{\mathbb{C}} X \\ \operatorname{rank} X \leq k \end{array} \right\}$ X has eigenvalues 0 & $\zeta_{\mathbb{C}}$.

② More generally, partial flag variety and closures of conjugacy class can be realised as quiver varieties.

$$\begin{array}{c} 1 \leftarrow 2 \leftarrow 3 \leftarrow \cdots \leftarrow n-1 \leftarrow n \\ \Downarrow \\ \infty \end{array} \quad \rightsquigarrow \quad \begin{array}{l} V_1 \rightleftarrows V_2 \rightleftarrows \cdots \rightleftarrows V_n \\ \downarrow\uparrow \\ W = \mathbb{C}^r \end{array} \quad \circlearrowleft PG_V = \prod GL(V_i)$$

r arrows

Suppose $\zeta_{\mathbb{R}}^i > 0$ $(i \neq \infty)$ for simplicity.

$\Leftrightarrow$ all $\leftarrow$ are injective

$$\mathfrak{M}_{0,\zeta_{\mathbb{R}}}(v) \cong \{(\mathbb{C}^r = V_0 \supset V_1 \supset \cdots \supset V_n, X) \mid X(V_i) \subset V_{i+1}\}$$
$$\cong T^*(\text{n-step partial flag variety})$$

$$\mathfrak{M}_{\zeta_{\mathbb{C}},0}(v) \cong \{X \in \mathrm{End}(W) \mid X(X-\zeta_{\mathbb{R}}^1)\cdots(X-\zeta_{\mathbb{R}}^n) = 0 \}$$
$+$ rank condition

③ star-shaped quiver $\rightsquigarrow$ additive version of the Deligne-Simpson problem, explained in the multi. version.

§3. Properties of quiver varieties

Th. Suppose V is primitive. (not a multiple $V \overset{\exists}{=} m \overset{\exists}{V'}$ $m \in \mathbb{Z}_{>1}$)

$\Longrightarrow$ For generic $\zeta = (\zeta_{\mathbb{R}}, \zeta_{\mathbb{C}})$ (complement of $\underbrace{\bigcup_{\theta}}_{\text{finite union}} (\mathbb{R} \oplus \mathbb{C}) \otimes \underbrace{D_\theta}_{\text{hyperplane}}$)

(1) $\mathfrak{M}_\zeta(V)$ is nonsingular of dimension $= 2 - (V, CV)$ $C =$ Cartan matrix if it is non-empty.

(2) $\mathfrak{M}_\zeta(V) \cong \mathfrak{M}_{\zeta'}(V)$ diffeomorphic, has the same Hodge numbers.

(3) $H_*(\mathfrak{M}_\zeta(V), \mathbb{Z})$: torsion free, odd deg. $= 0$, only (p,p) classes

(4) $H_k(\mathfrak{M}_\zeta(V), \mathbb{Z}) = 0$ if $k > \dim_{\mathbb{C}} \mathfrak{M}_\zeta(V) = \frac{1}{2} \dim_{\mathbb{R}} \mathfrak{M}_\zeta(V)$.

(5) [Crawley-Boevey] $\mathfrak{M}_\zeta(V)$ is connected.

Suppose $\zeta_{\mathbb{C}} = 0$. (Most complicated, most interesting case)

$$\mathbb{C}^* \curvearrowright \mathfrak{M}_\zeta(V) \text{ by } B_h \bmod PG_V \longmapsto t\cdot B_h = \begin{cases} tB_h & h\in\Omega \\ B_h & h\in\overline{\Omega} \end{cases} \bmod PG_V.$$

$$\mathfrak{L}_\zeta(V) \overset{\text{def.}}{=} \{[B]\in\mathfrak{M}_\zeta(V) \mid \lim_{t\to\infty}[t\cdot B_h] \text{ exists}\}$$

<u>Th.</u> (Lusztig, N)

Suppose V: primitive and $\zeta = (\zeta_{\mathbb{R}}, 0)$ is generic (hence $\underset{\text{smooth}}{\mathfrak{M}_\zeta(V)}$)

(1) $\mathfrak{L}_\zeta(V)$ is of pure dimension of $\frac{1}{2}\dim\mathfrak{M}_\zeta(V)$

(2) $\mathfrak{L}_\zeta(V) \simeq \mathfrak{M}_\zeta(V)$ deformation retract.

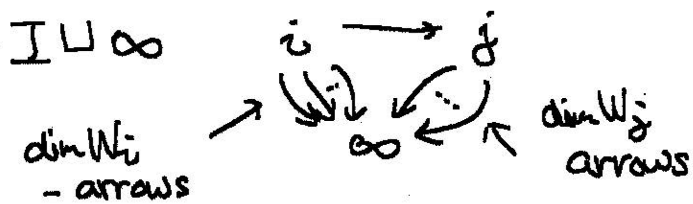
hence $H_*(\mathfrak{L}_\zeta(V)) \cong H_*(\mathfrak{M}_\zeta(V))$

§4. Representations of Kac–Moody Lie algebra

(I,Ω) : quiver

$W=\bigoplus_{i\in I} W_i$: another I-graded vector space $/\mathbb{C}$

$\rightsquigarrow$ a new quiver $\hat{I} = I \sqcup \infty$, $\hat{\Omega}$



$\hat{V}$: $\hat{I}$-graded vector space with $\hat{V}_\infty = \mathbb{C}$ (1-dim.)

$\quad = \bigoplus_{i\in I} V_i \oplus \mathbb{C}_{\text{at } \infty}$

We denote $\mathfrak{M}_\zeta(\hat{V})$ by $\mathfrak{M}_\zeta(V,W)$. (Original definition in [N, 1994])

Suppose (I,Ω) has no $\circlearrowleft$. We have the corresponding Kac-Moody Lie algebra $\mathfrak{g}$.

Th [N. 1994, 1998]

$\bigoplus_V H_{mid}(\mathfrak{M}_\zeta(V,W))$ has a structure of the integrable highest weight representation of $\mathfrak{g}$ with highest weight $= \sum_{i\in I} \dim W_i \Lambda_i$

$mid = \dim_{\mathbb{C}} \mathfrak{M}_\zeta(V)$
$= \frac{1}{2}\dim_{\mathbb{R}} \mathfrak{M}_\zeta(V)$

such that (1) $H_{mid}(\mathfrak{M}_\zeta(V,W))$ is the weight space with weight $= \sum \dim W_i \Lambda_i - \dim V_i \alpha_i$

(2) $[\mathfrak{M}_\zeta(0,W) = \text{point}]$ is the highest weight vector.

(Rem. We work with $\zeta = (\zeta_{\mathbb{R}}, 0)$ s.t. $\zeta_{\mathbb{R}}^i > 0$ $\forall i \in I = \hat{I} \setminus \infty$
and $\mathfrak{L}_\zeta(V,W)$ (not $\mathfrak{M}_\zeta(V,W)$))

Therefore $\dim H_{mid}(\mathfrak{M}_\zeta(V,W))$ is given by the Weyl-Kac character formula.

(About the proof)

Define $h \in \mathfrak{h}$ by $h = \langle h, \sum \dim W_i \Lambda_i - \dim V_i \alpha_i \rangle$ on $H_{mid}(\mathfrak{L}(v,w))$.

Define operators e_i, f_i by correspondences and check the defining relations of $\mathfrak{g}$. $[e_i, f_j] = \delta_{ij} h_i$ etc

e_i :

(f_i : similar)

$$\begin{tikzcd} & \mathfrak{P}_i(v,w) \subset \mathfrak{M}_\zeta(v,w) \times \mathfrak{M}_\zeta(v-\alpha_i, w) \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathfrak{M}_\zeta(v,w) & & \mathfrak{M}_\zeta(v-\alpha_i, w) \end{tikzcd}$$

$v-\alpha_i$: $\dim V_i$ is decreased by 1, other $\dim V_j$ are unchanged.

$$\begin{tikzcd} H_*(\mathfrak{L}_\zeta(v,w)) \arrow[r, "p_{2*}(p_1^*(-) \cap [\mathfrak{P}_i(v,w)])"] & H_*(\mathfrak{L}_\zeta(v-\alpha_i, w)) \\ H_{mid}(\mathfrak{L}_\zeta(v,w)) \arrow[u, phantom, "\cup"] \arrow[r] & H_{mid}(\mathfrak{L}_\zeta(v-\alpha_i, w)) \arrow[u, phantom, "\cup"] \end{tikzcd}$$

where

$\mathfrak{P}_i(v,w) =$ Hecke correspondence

$\overset{\text{def.}}{=} \{ (B^1, B^2) \in \mathfrak{M}_\zeta(v,w) \times \mathfrak{M}_\zeta(v',w) \mid B^2$ is a subrepresentation of $B^1 \}$.

Rem. A representation of the quantum loop algebra $U_q(L\mathfrak{g}_{KM})$ can be constructed on $\bigoplus_V K^{\mathbb{C}^* \times G_W}(\mathfrak{L}(V,W))$

$$G_W = \prod_i GL(W_i)$$

$\Longrightarrow$ Applications to the representation theory of $U_q(L\mathfrak{g}_{KM})$, e.g. character formula for irreducible representations

so far proved only by this method.

Lower degree homology groups have representation theoretic meanings.

§5. Lower degree homology groups

For a fixed $s \in \mathbb{Z}_{\geq 0}$

$H_{mid-s}(\mathfrak{M}_\zeta(v,w))$ is also an integrable representation of $\mathfrak{g}$,

not irreducible in general.

$\exists$ representation theoretic meaning of its irreducible decompostion via quantum loop algebra

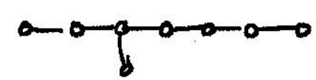
But so far purely representation theoretic approach is not known.

So we reverse the logic : understand H_{mid-s} by geometry

$\Rightarrow$ application to rep. theory.

<u>Th.</u> $\exists$ combinatorial algorithm to compute Betti numbers of $\mathfrak{M}_\zeta(v,w)$, in fact more generally each component of $\mathfrak{M}_\zeta(v,w)^{\mathbb{C}^*}$ (fixed point set).

— $\exists$ computer program http://www.math.kyoto-u.ac.jp/~nakajima/Qchar/Qchar.html

For E_8, fund. rep. corr. to  (most complicated fund. rep.)

weight = 0

$$1357104 + 2232771 t^2 + 2002423 t^4 + 1317308 t^6 + 716312 t^8 + 342421 t^{10} + 148512 t^{12} + 59490 t^{14} + 22162 t^{16} + 7687 t^{18} + 2463 t^{20} + 726 t^{22} + 192 t^{24} + 44 t^{26} + 8 t^{28} + t^{30}$$

computation in the super-computer. (120 G byte of memory, 300 hours

<u>Rem</u> ① $\exists$ conjectural algorithm in the representation theory
(Kirillov - Reshetikhin)
$\Longrightarrow$ conjectural algorithm for Betti numbers (Lusztig)
(drastically simpler than the above)

② Hausel has also an algorithm. math.AG/0511163

- It produces an algorithm which looks like ① (Mozgovoy),
  but in fact very different by a different definition
  of the q-binomial coefficients

- It is not clear how Hausel's algorithm
  is related to mine.

§6. Multiplicative quiver variety after Yamakawa

$\mathbb{M}(V) = \mathbb{M}_\Omega(V) \oplus \mathbb{M}_{\overline{\Omega}}(V)$ as before.

Define $\Delta : \mathbb{M}(V) \longrightarrow \mathbb{C}$ by $\prod_{h \in H} \det(1 + B_h B_{\bar{h}})$

$$\mathbb{M}(V)^\circ = \{\Delta \neq 0\} \subset \mathbb{M}(V) \quad \text{open}$$

Φ : multiplicative moment map

$$\Phi(B) = \left(\prod_{i(h)=i} (1 + B_h B_{\bar{h}})^{\varepsilon(h)} \right)_i$$

fix a total ordering on H

$\mathfrak{z} = (\mathfrak{z}_i) \in (\mathbb{C}^*)^I$ s.t. $\prod \mathfrak{z}_i^{\dim V_i} = 1$

<u>Def.</u>
$$\mathcal{M}^s_{\mathfrak{z}, \zeta_{\mathbb{R}}}(V) = \Phi^{-1}(\mathfrak{z})^{\zeta_{\mathbb{R}}\text{-stable}} / PG_V$$
$$\cap \text{ open}$$
$$\mathcal{M}_{\mathfrak{z}, \zeta_{\mathbb{R}}}(V) = \Phi^{-1}(\mathfrak{z})^{\zeta_{\mathbb{R}}\text{-semistable}} / S\text{-equiv.}$$

<u>Rem</u> (1) $\zeta_{\mathbb{R}}$ is the same as the additive case.

(2) $\zeta_{\mathbb{R}} = 0$ case $\mathcal{M}_{\mathfrak{z},0}(V)$: affine variety appears in the work of Crawley-Boevey (and Shaw).

$\exists \pi : \mathcal{M}_{\mathfrak{z}, \zeta_{\mathbb{R}}}(V) \longrightarrow \mathcal{M}_{\mathfrak{z},0}(V)$ projective morphism, which is resolution of singularities in many cases.

Examples ① Type A

$$\begin{tikzcd} 1 \arrow[d, shift left] & 2 \arrow[l] & 3 \arrow[l] & \cdots \arrow[l] \\ \infty \arrow[u, shift left] & & & \end{tikzcd}$$

r arrows

as in the additive case.

$$\Rightarrow \mathcal{M}_{q,\zeta_{\mathbb{R}}}(v) \cong \mathfrak{M}_{\zeta_{\mathbb{C}},\zeta_{\mathbb{R}}}(v) \cong \begin{cases} T^* \text{ partial flag variety} \\ \text{closure of conjugacy class} \end{cases}$$

where $\zeta_{\mathbb{C}}^{(i)} = q_1 \cdots q_{i-1}(1-q_i)$

Key point

$$\begin{tikzcd} {} \arrow[r, shift left, "A"] & V_i \arrow[l, shift left, "B"] \arrow[r, shift left, "C"] & {} \arrow[l, shift left, "D"] \end{tikzcd}$$

$$(1+AB)(1+DC)^{-1} = q_i$$

$$\Rightarrow 1+AB = q_i(1+DC)$$

essentially additive equation

$$A'B' - D'C' = \zeta_{\mathbb{C}}$$

② star-shaped quiver (comet-shaped quiver ?)

$$\begin{tikzcd}
 & {[1,1]} \arrow[dl] & {[1,2]} \arrow[l] & \cdots \arrow[l] & {[1,k_1]} \arrow[l] \\
* & {[2,1]} \arrow[l] & {[2,2]} \arrow[l] & \cdots \arrow[l] & {[2,k_2]} \arrow[l] \\
 & & & \vdots & \\
 & {[n,1]} \arrow[uul] & {[n,2]} \arrow[l] & \cdots \arrow[l] & {[n,k_n]} \arrow[l]
\end{tikzcd}$$

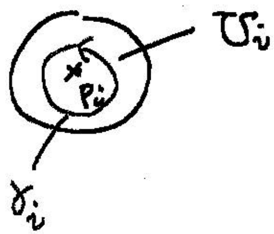
$\Longrightarrow$ $\mathcal{M}_{\xi,\zeta_{\mathbb{R}}}(v)$: moduli space of $\zeta_{\mathbb{R}}$-semistable filtered local system on $X = \mathbb{P}^1 \setminus \{p_1, \cdots, p_n\}$

e.g. $\zeta_{\mathbb{R}} = 0$

- each tail gives $A_i \in$ the closure of a conjugacy class
- the equation at $*$: $\prod_{i=1}^{n} A_i = 1$ $\rightsquigarrow$ representation of $\pi_1(X)$

This is due to Crawley-Boevey. (additive case : also by Hausel.)

But a general $\zeta_{\mathbb{R}}$-case is new.



$E|_{\sigma_i}$ has a filtration

$$E|_{\sigma_i} = E_{i0} \supset E_{i1} \supset \cdots\cdots \supset E_{ik_i}$$

The monodromy of γ_i $\sim$ upper triangular and the eigenvalues gives ξ.

More precisely, if ζ_{ij} $(0 \leqq j \leqq k_i)$ are eigenvalues,

$$\Longrightarrow \xi_{ij} \overset{\text{def.}}{=} \zeta_{ij} / \zeta_{ij-1} \quad , \xi_* \overset{\text{def.}}{=} \prod_i \zeta_{i0} .$$

And $\xi_{\mathbb{R}}$ defines the stability condition of the filtered loc. system.

Some results for additive quiver varieties seem to have analogs for multiplicative ones, but many are still conjectural.

Ok — the theory of reflection functors $\left(\begin{array}{l}\text{CB for } \zeta_{\mathbb{R}} = 0 \\ \text{Yamakawa for general } \zeta_{\mathbb{R}}\end{array}\right.$

? — The Hodge number independent of the choice of generic $(\zeta_{\mathbb{R}}, \xi)$?

$\mathcal{M}_{\xi, \zeta_{\mathbb{R}}}$ is noncompact, so the deformation invariance is not clear.

In the additive case, the hyper-Kähler structure is used.

OK —
$$
\begin{tikzcd}
\mathcal{M}_{1,\zeta_{\mathbb{R}}}(V) \arrow[r, "\pi"] & \mathcal{M}_{1,0}(V) \\
\mathcal{L}_{\zeta_{\mathbb{R}}}(V) \arrow[u, hook] \arrow[r] & {[B=0]} \arrow[u, hook]
\end{tikzcd}
$$

V : $V_i = \mathbb{C}$ for some i

i.e., corresponding to $\overbrace{\text{my}}^{\text{original}}$ quiver var.

$\# \operatorname{Irr} \mathcal{L}_\zeta(V)$ is the same as the additive case.

? — This seems to be compatible with Hausel + Letellier + Rodriguez-Villegas

pure part of $H_*(\mathcal{M}_{1,\zeta_{\mathbb{R}}}(V))$ is the same as the additive one.

$\mathcal{L}_{\zeta_{\mathbb{R}}}(V)$ isomorphic to the additive one?