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Deligne Mixed Hodge Theory

U "open" smooth / $\mathbb{C}$

$= X \setminus \bigcup_i D_i$ ← smooth divisor in X

normal crossings

$j : U \hookrightarrow X$

$H_c(U, \mathbb{Q}) := H(X, j_! \mathbb{Q}_U)$

$j : \mathbb{Q}_U \to \mathbb{Q}_X \to \bigoplus_i \mathbb{Q}_{D_i} \to \bigoplus_{i<j} \mathbb{Q}_{D_i \cap D_j} \to \cdots$

$j_! \mathbb{Q}_U = [\mathbb{Q}_U \to \mathbb{Q}_X \to \bigoplus_i \mathbb{Q}_{D_i} \to \cdots]$

simplest case

E.g. $U = X \setminus D$

$j_! U : \mathbb{Q}_X \to \mathbb{Q}_D$

$\cdots \to H^i_c(U) \to H^i_c(X) \to H^i_c(D) \to H^{i+1}_c(U)$

$E_1^{a,b} = \bigoplus_{i_1 < \cdots < i_a} H^b\left(\begin{matrix}\text{a-fold} \cap \\ D_{i_1} \cap \cdots \cap D_{i_a}\end{matrix}\right) \Rightarrow H_c(U)$

0-fold $\cap$ is X by defn

Simplest case example

$$0 \to H^{i-1}(D)/H^{i-1}(X) \to H^i_c(U) \to$$
$$\mathrm{Ker}(H^i X \to H^i D)$$
$$\to 0$$

smooth normal crossing divisors

$\Rightarrow$ a-fold $\cap$

$D_{i_1} \cap \cdots \cap D_{i_a}$

is smooth projective

Spectral seq. degenerates $d_r = 0$ for $r \geq 2$

$H^i_c(U, \mathbb{Q})$ has an increasing filtration

$W_{-1} = 0 \subseteq W_0 \subseteq W_1 \subseteq \cdots \subseteq W_i$

$W_j / W_{j-1} \otimes \mathbb{C}$ is a pure Hodge structure weight j

Miracle:

(i) Structure is indep. of the auxiliary choices of X, D_i used to define it.

(ii) Functoriality

(Assume we define MHS for <u>all</u> varieties already)

$V \subseteq Y$ open $\quad Z := Y \setminus V$

$$\cdots \to H^i_c(V) \to H^i_c(Y) \to H^i_c(Z) \to H^{i+1}_c(V) \to \cdots$$

respects W filtration and on $gr^j_W(\cdots)$ is an exact sequence of pure Hodge structures of weight j

<u>ordinary cohom</u>

$U = X \setminus \bigcup_i D_i$

$U \xrightarrow{j} X$ Leray spectral seq.

$$E_2^{a,b} = H^a(X, R^b j_* \mathbb{Q}_U) \Rightarrow H^{a+b}(U, \mathbb{Q})$$

$\rightsquigarrow$ gives filtration

$W_{-1} = 0 \subseteq W_1 \subseteq \cdots$ on $H^{a+b}(U, \mathbb{Q})$

$\Omega_X(\log D)$ filter this according to # $\frac{dx_i}{x_i}$ allowed

$\mathbb{H}(\cdots) = H(U, \mathbb{C})$

E - polynomial

$$E(U; x, y) \in \mathbb{Z}[x, y]$$

$$= \sum_i (-1)^i \left\{ \sum_{a,b} x^a y^b h^{a,b} \, gr_W^{a+b} \left(H_c^i(U) \right) \right\}$$

Excision property $\Rightarrow$

V open in Y, $Z := Y \setminus V$

$$E(V; x, y) = E(Y; x, y) - E(Z; x, y)$$

$K(sch/\mathbb{C})$

free abelian group on $[T]$

T separated $\mathbb{C}$-scheme of finite type

- $[T] \sim [T^{red}]$
- $[V] \sim [Y] - [Z]$

N.B. In $K(sch/\mathbb{C})$ every element is

$[\text{proj nonsing } X] - [\text{proj nonsing } Y]$

(not necessarily connected)

<u>Pf</u> Reduce to affine

$Z \subset \mathbb{A}^n$

$[\mathbb{A}^n] - [\mathbb{A}^n \setminus Z] = [Z]$

inclusion-exclusion

$U = [X] - \sum_i [D_i] + \sum_{i<j} [D_i \cap D_j] - \cdots$

Deligne MHS $\Rightarrow$ E is well defined on $K(sch/\mathbb{C})$ (write it as $E(X; x,y) - E(Y; x,y)$ for $[X] - [Y]$)

<u>Polynomial count varieties</u>

Arbitrary $X/\mathbb{F}_q$ is <u>polynomial-count</u> if there exists $P(t) \in \mathbb{C}[t]$ s.t. for all finite fields $k/\mathbb{F}_q$

$\# X(k) = P(\# k)$

e.g. $X = \mathbb{A}^n$, $P(t) = t^n$

$X = \mathbb{P}^n$, $P(t) = 1 + t + \dots + t^n$

Ex. X polynomial count P has $\mathbb{Z}$-coeffs.
(Note P is uniquely determined).
use $Z(X,T)$ is a rational fctns
can we do it without this?

$X/\mathbb{C}$ is polynomial count if there exists a spreading out $\mathcal{X}/R \subseteq \mathbb{C}$ finitely generated $\mathbb{Z}$-algebra
and $P \in \mathbb{C}[t]$ s.t.

$R \xrightarrow{\varphi} k$ = finite field

$\mathcal{X} \otimes_{R \nearrow \varphi} k$ is polynomial with polynomial P.

Defn In K group say:

$$[X] \underset{zeta}{\sim} [Y]$$

if there exist spreadings out $\mathcal{X}$ and $\mathcal{Y}$ over some R s.t. for all

$R \xrightarrow{\varphi} k$

$\# \mathcal{X}_k(k) = \# \mathcal{Y}_k(k)$
(equiv. they have the same zeta fctn)

$$K(\text{schemes}/R) \longrightarrow K(\text{schemes}/\mathbb{C})$$

In this language: polynomial count means zeta equivalent to $\sum_n a_n [\mathbb{A}^n]$

$$\left(P(t) = \sum_n a_n t^n \right)$$

OR zeta equivalent to $\sum_n b_n [\mathbb{P}^n]$

<u>THM</u> If $X, Y \in K(sch/\mathbb{C})$ are zeta equiv then

$$E(X; x, y) = E(Y; x, y)$$

<u>Pf</u> $[X] = [S] - [T]$ $\quad S, T, S_1, T_1$

$[Y] = [S_1] - [T_1]$ $\quad$ proj. smooth

$$[S] + [T_1] \underset{\text{zeta}}{\sim} [S_1] + [T]$$

Fontaine-Messing Faltings $\Rightarrow$ $E(S \amalg T_1; x, y)$

$= E(S_1 \amalg T; x, y)$

(same Hodge numbers)

For polynomial count X

$$E(X; x, y) = \sum_n b_n E(\mathbb{P}^n; x, y)$$

<u>Cor</u> $E(X; x, y) = P(xy)$

Example

E_1, E_2 two isogenous elliptic curves

$$\mathbb{P}^2 \setminus E_1 \amalg E_2 \underset{\text{zeta}}{\sim} \mathbb{P}^2$$

$$\begin{array}{c} \mathbb{P}^2 \setminus E_1 \amalg E_2 \\ j \big\downarrow \\ \mathbb{P}^N \end{array}$$

Let $X = \mathbb{P}^N \setminus \text{image } j$

polynomial count not paved by affines.

$$\# E(\mathbb{F}_q) = q + 1 - a_q$$