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Middle convolution

$$F(a,b,c;\lambda) := \int \underbrace{x^{a-c}(1-x)^{c-b-1}}_{f(x)} (\lambda - x)^{-a}\, dx$$

$$\chi(x) := x^{-a}$$

$$F(\lambda) := \int f(x)\, \chi(\lambda - x)\, dx$$

convolution

$\mathbb{G}_a := \mathbb{A}^1$ under $+$, $\quad + : \mathbb{G}_a \times \mathbb{G}_a \to \mathbb{G}_a$

$$\mathbb{G}_a(\mathbb{F}_q) \times \mathbb{G}_a(\mathbb{F}_q) \to \mathbb{G}_a(\mathbb{F}_q)$$

$f \searrow \quad g \searrow \quad \downarrow$

$$f * g(\lambda) := \sum_{x+y=\lambda} f(x) g(y)$$

want to lift this construction to cplx of sheaves

K, L on $\mathbb{G}_a$

$\rightsquigarrow K * L$ on $\mathbb{G}_a$ s.t.

$$tr_{K*L} | G(\mathbb{F}_q) = tr_K | G(\mathbb{F}_q) * tr_L | G(\mathbb{F}_q)$$

External tensor product

$$pr_1^* K \otimes pr_2^* L =: K \boxtimes L$$

$$K *_! L := R\, Add_! (K \boxtimes L)$$

works by Lefschetz trace fmla

perverse sheaf on $\mathbb{A}^1$ (any curve really)

cplx K only $\mathcal{H}^0$, $\mathcal{H}^{-1}$ possibly $\neq 0$

$\mathcal{H}^0$ = punctual

$\mathcal{H}^{1}$ = sheaf with no nonzero punctual sections lisse on open $U \overset{j}{\hookrightarrow} \mathbb{A}^1$

i.e. $\mathcal{F} \hookrightarrow j_* j^* \mathcal{F}$

"best" sort

$j_*\left(\begin{array}{l}\text{lisse } \mathcal{F} \text{ on open } U \\ \text{which is irred as} \\ \text{rep}^n \text{ of } \pi_1(U)\end{array}\right)[1]$

puts it on $\mathcal{H}^{-1}$

"simplest" sort

$(\bar{\mathbb{Q}}_\ell)_\alpha$ skyscraper sheaf at point α

(or δ_α)

would be nice if L, K perverse so is ~~[illegible]~~

$K *_! L$ but it's false!

false also with $K *_* L$!

Is there are any perverse K s.t. for all perverse L s.t. K $*_!$ L is perverse?

ok if $K = \delta_\alpha$ ($*_! \delta_\alpha$ = translation by α)

not ok if $K = \bar{\mathbb{Q}}_\ell[1]$ output is constant

$\mathcal{H}^0$ = constant if L is $\mathcal{G}[1]$
sheaf $H^2_c(\mathbb{A}^1, \mathcal{G})$

Say K has $\mathcal{P}_!$ if K $*_!$(perv) = perv

(ditto $\mathcal{P}_*$ ... $*_*$)

on $\mathbb{A}^1/\mathbb{C}$ K has both $\mathcal{P}_!$ and $\mathcal{P}_*$
$\Leftrightarrow$ K has perv sheaf has no sub or quotient object $\mathbb{Q}_\ell[1]$

on $\mathbb{A}^1/\bar{\mathbb{F}}_p$ ... $\mathcal{L}_\psi(\lambda x)$ $\lambda \in \mathbb{F}_p$
<u>Artin-Schreier</u> sheaf

$*_!$, $*_*$ are assoc. operations on cplxs.

So if K & L have $\mathcal{P}_*$ so does K $*_*$ L
.... $\mathcal{P}_!$ .... K $*_!$ L

K $*_!$ L $\xrightarrow{\text{forget supports}}$ K $*_*$ L

both are perverse

In category of perverse sheafs take

$$\mathrm{Im}\,(K *_! L \longrightarrow K *_* L)$$

as defn of $K *_{mid} L$.

$\mathcal{F}$ lisse on open curve $U \overset{\subsetneq}{j} C / \mathbb{F}_q$

pure of wt w

Frobenius eigenvalues are alg numbers of abs $q^{w/2}$.

$$\mathrm{Im}\,(H^1_c(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}) \to H^1(U \otimes \overline{\mathbb{F}}_q, \mathcal{F}))$$

wts $\geq w+1$

wts $\leq w+1$

$$= H^1(C \otimes \overline{\mathbb{F}}_q, j_* \mathcal{F})$$

If K, L have P_* & $P_!$ (say just P)

so does $K *_{mid} L$ and $*_{mid}$ is assoc.

on $\{$ have $P\}$

2 if $0 \to A \to B \to C \to 0$ exact seq. of perv sheaves each with P

then $0 \to A *_{mid} K \to B *_{mid} K \to C *_{mid} K \to 0$

need not be exact

were

Gabber-Loeser (in multiplicative setting)

Think of $\mathcal{P}$ as perv / neglibeble
Neg

Neg: perverse sheaf which is a successive extension of $\mathcal{L}_\psi$'s [1]

Neg $\leftrightarrow$ Fourier transform is punctual

different structure of abelian category (quotient).

$\pi_1(\mathbb{G}_m) \twoheadrightarrow \mathbb{Z}(1)_{\text{not } p}$

$\mathbb{A}^1 \supseteq \mathbb{G}_m$

Kummer sheaf $\quad \mathcal{L}_\chi$ on $\mathbb{G}_m$ lisse rk 1

$\chi \neq \mathbb{1} \qquad j_* \mathcal{L}_\chi = j_! \mathcal{L}_\chi$ write as $\mathcal{L}_\chi$

$\mathcal{L}_\chi[1]$ on $\mathbb{A}^1$ perverse in $\mathcal{P}$.

$\underset{\text{in } \mathcal{P}}{(\text{perverse})} *_{\text{mid}} \mathcal{L}_\chi[1]$

$\chi \neq \mathbb{1}$

$$\mathcal{L}_\chi[1] *_{\text{mid}} \mathcal{L}_\rho[1] = \begin{cases} \mathcal{L}_{\chi\rho}[1] & \chi\rho \neq \mathbb{1} \\ \delta_0 & \chi\rho = \mathbb{1} \end{cases}$$

$\pi_1(\mathbb{G}_m) \twoheadrightarrow \varprojlim_m \mathbb{G}_m(\mathbb{F}_{p^m})$

Abhyankar 50's

$$\begin{array}{ccc} \mathbb{G}_m & & x \\ \downarrow & & \downarrow \\ \mathbb{A}^1 & & x^p + \frac{1}{x} \end{array}$$

finite étale

$$\begin{array}{c} \mathbb{G}_m \setminus \mu_{p+1} \\ \downarrow \\ \mathbb{G}_m \end{array}$$

$K \longmapsto K *_{mid} \mathcal{L}_\chi[1]$

is an invertible operation (inverse

$K \longmapsto K *_{mid} \mathcal{L}_{\bar{\chi}}[1]$)

on the Fourier transform side this is ~~to~~ twisting by χ.

K, L perverse

$$K \otimes_{mid} L := j_*(\underset{\text{on } U}{\mathcal{F} \otimes \mathcal{G}})[1]$$

$K|_U = \mathcal{F}[1]$, $L|_U = \mathcal{G}[1]$

Could have defined $*_{mid}$ this way in char p but want to do it / $\mathbb{C}$.

$$\pi_1(\mathbb{P}^1_{\mathbb{C}} \setminus \{s_1, \dots, s_k\}) \xrightarrow{\rho} Gl_n(\mathbb{C})$$

$A_1 \cdots A_k = 1$

irred $\leftrightarrow$ $\langle A_1, \dots A_k \rangle$ acts irred on $\mathbb{C}^n$

rigid $\leftrightarrow$ $B_1 \cdots B_k = 1$ B's in $Gl_n(\mathbb{C})$

A_i conjugate to B_i $i = 1, 2, \dots, k$

then $\exists g \in Gl_n(\mathbb{C})$ s.t. $g A_i g^{-1} = B_i$ ".

Cohom. criterion for $\mathcal{F}$ irred

rigid iff $\chi(\mathbb{P}^1, j_* \mathcal{E}nd(\mathcal{F})) = 2$

$\quad\quad\quad\quad \|$

index of regularity $:= \underset{\substack{\uparrow \\ \text{irreducibility}}}{2} - h^1(\mathbb{P}^1, j_* \mathcal{E}nd(\mathcal{F}))$ $\quad \underset{=0}{}$

$H^1(\mathbb{P}^1, j_* \mathcal{E}nd(\mathcal{F}))$ symplectically self-dual

so in partic. even dim'l

Suppose $\chi = 2$ $\quad$ $\mathcal{G}$ same loc monodr.

$\chi(j_* \mathcal{E}nd\, \mathcal{F}) = \chi(j_* \mathcal{H}om(\mathcal{F}, \mathcal{G}))$

$H^0 \quad : \quad \mathrm{Hom}_U(\mathcal{F}, \mathcal{G})$

$\mathrm{hom}_U(\mathcal{F}, \mathcal{G}) - h^1 +$ dual to $H^0(j_* \mathcal{H}om(\mathcal{G}, \mathcal{F}$

one of them is nonzero

since $\mathcal{F}$ is irred $\quad \mathcal{F} \simeq \mathcal{G}$ in either case.

Rigid + irred loc. systems on $\mathbb{P}^1 \setminus S$?

In Gauss fmla: $\Big[\underbrace{\mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{\rho(1-x)}}_{\text{rk } 1}\Big] \ast_{mid} \mathcal{L}_{\Lambda}$

rigid of rk 2

Suggests

$$j_*\begin{pmatrix}\text{irred}\\ +\text{rigid}\\ \text{on } \mathbb{A}^1\setminus S_1\end{pmatrix} *_{\text{mid}} \mathcal{L}_\Lambda \overset{?}{=} j_*(\text{rigid}+\text{irred})[1]$$

$\mathbb{P}^1 \setminus S = \mathbb{A}^1 \setminus S_1$ $\quad S_1 \cup \{\infty\} = S.$

assume $\infty \in S$ always

Almost true (recall $\mathcal{L}_\Lambda *_{\text{mid}} \mathcal{L}_\Lambda = \delta_0[1]$)

$$\text{rig}\begin{pmatrix}\text{local}\\ \text{syst}\end{pmatrix} = \text{rig}\begin{pmatrix}\text{local}\\ \text{syst}\end{pmatrix} *_{\text{mid}} \mathcal{L}_\Lambda)$$

Prove in char p by proving

$$\text{rig}(K) = \text{rig}(FT(K))$$

$$FT(K *_{\text{mid}} \mathcal{L}_\Lambda) \sim FT(K) \otimes \mathcal{L}_\Lambda$$

E.g. $\mathcal{L}_{\chi_2(x(x-1))} *_{\text{mid}} \mathcal{L}_{\chi_2}$

$= \text{Legendre}(\lambda)$ $\quad y^2 = x(x-1)(x-\lambda)$

loc. monodromy

$0 \quad \begin{pmatrix}1&1\\0&1\end{pmatrix}$

$1 \quad \begin{pmatrix}1&1\\0&1\end{pmatrix}$

$\infty \quad -\begin{pmatrix}1&1\\0&1\end{pmatrix}$

rig $\otimes$ (well chosen rk 1) $*_{mid}$ $\mathcal{L}_\Lambda$

rk ≥ 2

eigenv. with max multiplicity

multiplicity at ∞?

= rig of lower rk

Keep doing this until you get rk 1.

Simplest non-rigid Huma equation ← check Wittaker Watson

rk 2, 4 punctures. on $\mathbb{P}^1$.

character variety = affine cubic surface

D_4 case

$\underline{\mathcal{F}}$ $*_{mid}$ $\mathcal{L}_\Lambda = \mathcal{G}$, what happens to rank?

$$\mathcal{G}(\infty) := M \otimes \chi / (\cdots)^{I(\infty)}$$

$$\mathrm{rk}\, M = \sum_{s_i} \dim (\mathcal{F}(s_i) \cancel{\otimes \chi}) / (\cdots)^{I}$$