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Characters of $Gl_n(\mathbb{F}_q)$ à la Green

$=: G_n$

50's

1. Conjugacy classes

$f \in \mathbb{F}_q[t] \rightsquigarrow U(f)$ companion matrix

size $= \deg f$

$$\begin{pmatrix} U(f_1) & 1 & & \\ & U(f_1) & 1 & \\ & & & \ddots \end{pmatrix}$$ Jordan form

$(f_1^{\lambda_1}, f_2^{\lambda_2}, \dots)$ λ_i partition

f_i distinct irred $\neq t$

Functions: $\lambda : \mathbb{F}_q^\times \longrightarrow \mathcal{P}$ (set of all partitions)

- λ is constant on Frob_q-orbits (i.e. map on closed points of $\mathbb{G}_m/\mathbb{F}_q$)
- $\displaystyle\sum_{f \in \mathbb{F}_q^\times/\mathrm{Frob}_q} \deg f \, |\lambda(f)| = n$

$\lambda = (\lambda_1 \geq \lambda_2 \geq \dots)$

$|\lambda| = \sum \lambda_i$

2. Harish Chandra Induction

(1969 Green proved $Gl_n(\mathbb{F}_q)$ case)

G_n contains "split Levi" subgroups

conjugate to

$$L_\lambda := \begin{pmatrix} G_{n_1} & & \\ & G_{n_2} & \\ & & \ddots \end{pmatrix} \subseteq \begin{pmatrix} & * \\ & \end{pmatrix} =: P$$

(contained in a parabolic / $\mathbb{F}_q$)

Feature of Gl_n all components of Levi are of same type (only happens for $Gl(n)$) Allows to define characters recursively.

Up to conjugacy: $L \sim L_\lambda$, $|\lambda| = n$

$$HC_L^G : Cl(L) \longrightarrow Cl(G)$$

$$\chi \longmapsto Ind_P^G(\tilde{\chi}_{L \subseteq P})$$

Independent of choice of P (even if these are not conjugate)

<u>Defn</u> A class function φ of G is <u>cuspidal</u> (belongs to a discrete series)

if $(\varphi, HC_L^G(\chi)) = 0$ for all χ, $L \neq G$.

<u>THM</u> (HC, 1969) 1. Every irred character φ of G $\in HC_L^G(\chi)$ some cuspidal $\chi \in \mathrm{Irr}(L)$

2. The cuspidal pair (L, χ) s.t. $\varphi \in HC_L^G(\chi)$ is uniquely determined up to conjugacy by G.

3. Suppose (L_1, δ_1), (L_2, δ_2) are cuspidal pairs then

$$(HC_{L_1}^G \delta_1, HC_{L_2}^G \delta_2) = |W(\delta_1, \delta_2)|$$

where

$$W = L_2 \backslash \{g \in G \mid {}^g\delta_1 = \delta_2\} / L_1$$

$\sim$ ($\subseteq$ Weyl group of G = permutation matrices)

<u>Cor</u>. Suppose $E(L, \delta) := \mathrm{End}_G(HG_L^G(\delta))$ for a cuspidal pair (L, δ)

$$\mathrm{Irred}(G) \leftrightarrow \{(L, \delta, \Delta) \mid (L, \delta) \text{ cuspidal}, \ \Delta \in \mathrm{Irr}(E(L, \delta))\} / G$$

(This description works for any reductive G / finite field. Particularly simple for Gl_n.)

<u>THM</u> (HL, 1980)

i) $E(L, \delta) \simeq H_{W(\delta)}(p^{k(\delta)})$

where

$W(\delta) := W(\delta, \delta)$ (as above)

= ramification group

ii) $W(\delta)$ it is (almost) a reflection group.

<u>Heckealgebra</u>: Deform relations $s_i^2 = 1$ in a Coxeter group to

$(s-q)(s+q^{-1}) = 1$

<u>Cor</u> $\mathrm{Irr}(G) \sim (\underbrace{L, \delta}_{\text{cuspidal pair}}, \nu) / G$ ← $\in \mathrm{Irr}(W(\delta))$

3. <u>Green's theory</u>

$d/e \quad N_{e,d} : \mathbb{F}_{q^e}^\times \longrightarrow \mathbb{F}_{q^d}^\times$ norm map surjective

induces

$$\hat{\mathbb{F}}_{q^d}^\times \hookrightarrow \hat{\mathbb{F}}_{q^e}^\times$$

$$\varinjlim_{e} \hat{\mathbb{F}}_{q^e}^\times =: \hat{\mathbb{F}}_q$$

Let $\psi : \mathbb{F}_{q^n}^\times \longrightarrow \mathbb{C}^\times$ be a character

Define

$$J_n^{\langle\psi\rangle} : G_n \longrightarrow \mathbb{C}$$

by

$$J_n^{\langle\psi\rangle}(\underbrace{\ldots f^{\lambda(f)} \ldots}_{c}) = \begin{cases} K(p(\lambda), q^{d(f)}) \\ \quad (\psi(\eta) + \cdots + \psi(\eta^{q^{d-}} & c \sim (f^\lambda) \\ 0 & \text{otherw.} \end{cases}$$

conjugacy class $\qquad f(\eta) = 0$

$p(\lambda) :=$ length of $\lambda \qquad d = \deg f$

$$K(m,t) = (t-1)(t^2-1)\ldots(t^{m-1}-1)$$

<u>THM</u> (i) $J_n^{\langle\psi\rangle}$ is a $\overbrace{\text{generalized}}^{\text{(virtual)}}$ char of G_n

(ii) $\langle J_n^{\langle\psi_1\rangle}, J_n^{\langle\psi_2\rangle}\rangle = (\psi_1, \psi_2) + (\psi_1, \psi_2^q) + \ldots + (\psi_1, \psi_2^{q^{n-1}})$

(iii) $J_n^{\langle\psi\rangle}$ $\deg\psi = n$ (n-regular)

are all the cuspidal irred char of G_n

<u>THM</u> If $\delta = \left(J_d^{\langle\psi\rangle}\right)^{n/d}$ $(\psi \in \hat{\mathbb{F}}_{q^d}^{\times})$

cuspidal on $\underset{L :=}{L_d^{n/d}} = \begin{pmatrix} d & & \\ & \boxed{d} & \\ & & \ddots \end{pmatrix}$

then

$W(\delta) \simeq \mathrm{Sym}_{n/d}$

<u>Cor</u> The irred. components $HC_L^{G_n}(\delta)$

$\leftrightarrow \{\lambda \vdash n/d\}$

We can hence write them as

$J^{\langle\psi\rangle}(\lambda)$, $\deg\psi = d$, $|\lambda| = n/d$

(defined as the χ_λ component of $HC_L^{G_n}(\delta)$)

<u>THM</u> (Green '55) The irred chars of G_n are parameterized by functions

$\lambda : \hat{\mathbb{F}} \to \mathcal{P}$ satisfying

(i) λ is constant on Frob_q orbits

(ii)
$$\sum_{\Theta \in \hat{\mathbb{F}}/\mathrm{Frob}_q} |\Theta|\,|\lambda(\Theta)| = n$$

The corresponding character is:

$$HC_L^{G_n}\left(J^{\langle\psi_1\rangle}(\lambda(\psi_1))\, J^{\langle\psi_2\rangle}(\lambda(\psi_2)) \dots \right)$$

$\langle\psi_i\rangle$ distinct Frob orbits

Source of $J_n^{\langle\psi\rangle}$

<u>THM</u> $\psi \in \hat{\mathbb{F}}^{\times}_{q^{n!}}$ for $g \in G_n$ let $\lambda_1(g), \dots, \lambda_n(g)$ be the eigenvalues in $\mathbb{F}^{\times}_{q^{n!}}$ then for any symmetric function σ, $g \mapsto \sigma(\psi(\lambda_1), \dots, \psi(\lambda_n))$ is a virtual character of G_n. $= \chi^{\psi}(\sigma)$

(Lusztig 1971)

<u>Explicit form</u>

$\theta : \mathbb{F}^{\times}_{q^{n!}} \longrightarrow \mathbb{C}^{\times}$ faithful

$$\chi^{\theta}_{\sigma} = \sum_{j=0}^{n} (-1)^j HC_{L_j}^{G_n}\left(J^{\langle\theta\rangle} \otimes 1_{n-j} \right)$$

$$\begin{array}{c} j \\ n-j \end{array} \begin{pmatrix} J_j^{\langle\theta\rangle} & \\ & 1 \end{pmatrix}$$

Rational tori of G_n

$\mathbb{F}_{q^{n_1}}^\times \times \cdots \times \mathbb{F}_{q^{n_r}}^\times \qquad n_1 + \cdots + n_r = n$

$\mathbb{F}_{q^{n_i}}^\times \sim \begin{pmatrix} \eta & & \\ & \ddots & \\ & & \eta^{q^{n_i-1}} \end{pmatrix}$

$\longleftrightarrow \quad T_\lambda \quad$ partition $\lambda = (n_1, n_2, \ldots n_r)$

$\psi \in \hat{T}_\lambda \longleftrightarrow \psi = (\psi_1, \psi_2, \ldots)$

$\psi_i \in \hat{\mathbb{F}}_{q^{n_i}}^\times$

$$R_{T_\lambda}(\psi) = HC_L^{G_n}(J^{\langle\psi_1\rangle} \otimes \cdots \otimes J^{\langle\psi_r\rangle})$$

- Green proved: all irred. char are $\mathbb{Q}$ - linear combination of the $R_{T_\lambda}(\psi)$

"uniform function"

- E.g. $\quad HC_T^{G_n}(1) = \bigoplus_{|\lambda|=n} \chi^\lambda(1)\, u_\lambda$

$T = \begin{pmatrix} \cdot & & \\ & \ddots & \\ & & \cdot \end{pmatrix} \qquad u_\lambda = \underset{n!^{-1}}{|W|^{-1}} \cdot \sum_{w \in W} \chi^\lambda(w) R_{T_{\lambda(w)}}(1)$

split torus

"almost character"

- All character values by a "degeneracy rule" reduces computation of value to

$R_{T_\lambda}(\psi)(u_\mu)$ ← unipotent element

Indep of $+ \rightsquigarrow$ Green polynomials

$Q_{\mu}^{\lambda}(q)$