June 8, 2007 A. Ram

Life begins w/

W_0 finite group acting on a $\mathbb{Z}$-lattice P generated by reflections. A <u>reflection</u> is $s \in W_0 \subseteq Gl(P)$ s.t. $V^s = \{\lambda \in V \mid s\lambda = \lambda\}$
$\subseteq Gl(V)$ is codim 1

$V := \mathbb{R} \otimes_{\mathbb{Z}} P$

Chevalley $(W_0, P) \leftrightarrow (G, T)$

Examples

1. $G = Gl_n$, $W_0 = S_n$

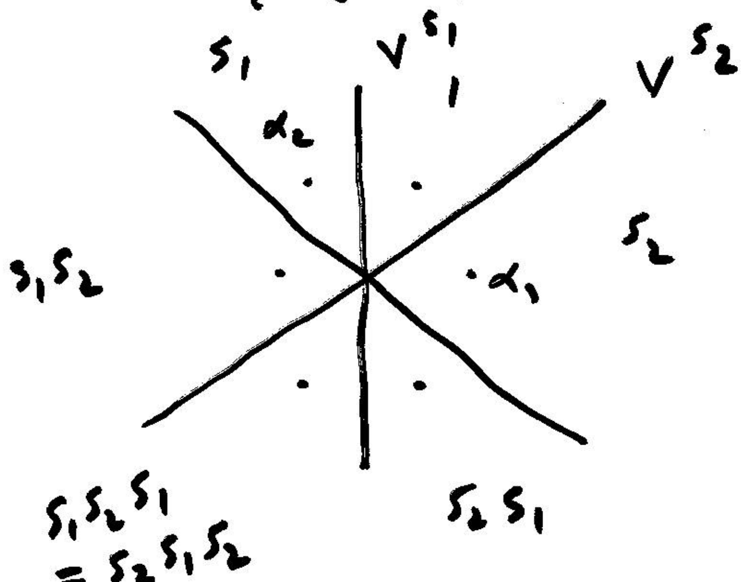
$\cup\!|$

$T = \left\{ \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix} \right\}$ $P = \mathrm{span}\{\varepsilon_1, \ldots, \varepsilon_n\}$

2. $G = Sl_3$ $W_0 =$ dihedral order 6

$\cup\!|$

$T = \left\{ \begin{pmatrix} * & & 0 \\ & * & \\ 0 & & * \end{pmatrix} \right\}$ $P = \mathrm{span}\{\alpha_1, \alpha_2\}$



DAHA and Macdonald polyns

The double affine Hecke algebra H is generated by

$$\mathbb{C}[X] = \mathrm{span}\{X^\lambda \mid \lambda \in P\}$$

$$X^\lambda X^\mu = X^{\lambda+\mu}$$

$$H_0 = \mathrm{span}\{T_w \mid w \in W_0\}$$

$$T_{s_i}^2 = (t^{1/2} - t^{-1/2})\, T_{s_i} + 1$$

$\mathbb{C}[Y]$ replace Y's in $\mathbb{C}[X]$

w/relations to move X^λ left, Y^μ right

s.t.

$$H = \mathbb{C}[X] \otimes H_0 \otimes \mathbb{C}[Y]$$ (triang. decomp.)

The <u>affine Hecke algebra</u>

$\hat{H} = H_0 \otimes \mathbb{C}[Y]$ has a 1-diml module $\mathbb{1}$

$$T_w \mathbb{1} = t^{\frac{1}{2}\ell(w)} \mathbb{1}, \quad Y^\mu \mathbb{1} = q^{\langle \rho, \mu \rangle} \mathbb{1}$$

$\rho = \frac{1}{2}\sum$ positive roots

and <u>polynomial representation</u> is

$$\mathrm{Ind}_{\hat{H}}^{H}(\mathbb{1}) = \mathbb{C}[X]\,\mathbb{1}$$

The Macdonald polynomials are the simultaneous eigenvectors of Y^{μ} on $\mathbb{C}[X]\cdot \mathbb{1}$

Affine Hecke algebra and Satake and $\mathcal{F}_0$

Let $\tilde{H} = \mathbb{C}[X] \otimes H_0$ and $\mathbb{1}_0$ is the 1-diml H_0 module $\quad (T_w \mathbb{1} = t^{\frac{1}{2}\ell(w)} \mathbb{1}_0)$

$$\mathbb{1}_0 = \sum_{w \in W_0} (t^{\frac{1}{2}})^{\ell(w)} T_w$$

and $\mathrm{Ind}_{H_0}^{\tilde{H}}(\mathbb{1}_0) = \tilde{H}\,\mathbb{1}_0 = \mathbb{C}[X]\,\mathbb{1}_0$

$$\mathbb{C}[X]^{W_0} = Z(\tilde{H}) \xrightarrow[\sim]{\text{satake}} \mathbb{1}_0 \tilde{H} \mathbb{1}_0$$

spherical Hecke algebra

$\uparrow$

<u>Bernstein</u>

$$f \longmapsto f\,\mathbb{1}_0$$

<u>Hall-Littlewood</u> $\quad P_\lambda \longleftarrow \mathbb{1}_0 X^\lambda \mathbb{1}_0$

Weyl characters $\quad s_\lambda$

$$\varepsilon_0 = \sum_{w \in W_0} (-t^{1/2})^{\ell(w)} T_w$$

$\mathbb{C}[X]^{W_0}$ = Rep^{n} ring of $G^\vee$

$$\mathbb{1}_0 \tilde{H} \mathbb{1}_0 \longrightarrow \varepsilon_0 \tilde{H} \mathbb{1}_0$$

$$f\,\mathbb{1}_0 \longmapsto A_\rho f\,\mathbb{1}_0$$

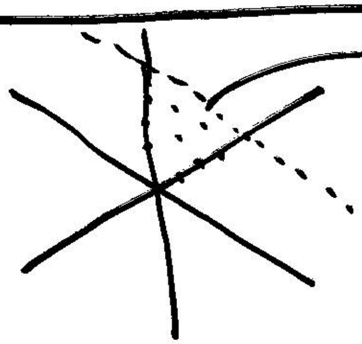
$$\mathbb{C}[W]^{W_0} \xrightarrow{\sim} \mathbb{1}_0 \tilde{H} \mathbb{1}_0 \longrightarrow \varepsilon_0 \tilde{H} \mathbb{1}_0 =: \mathcal{F}_0$$

$$s_\lambda \longmapsto C'_\lambda \longleftarrow A_{\lambda+\rho} = \varepsilon_0 X^{\lambda+\rho} \mathbb{1}_0$$

Kazdan-Lusztig basis

level 0 Fock space

Level ℓ Fock space



$$\mathcal{F}_\ell = \bigoplus_{\nu \in A_\ell} \varepsilon_0 \tilde{H} \mathbb{1}_\nu$$

where $\mathbb{1}_\nu = \sum_{w \in W_\nu} (t^{1/2})^{\ell(w)} T_w$

$|\lambda\rangle = \varepsilon_0 X^\beta T_w \mathbb{1}_\nu$ if $\lambda = \ell\beta + w\nu$ with

w min length in W/W_0

W_0 = stabilizer of ν

Then

$$\mathcal{F}_\ell\big|_{t=1} \xrightarrow{\sim} K(U_q\mathfrak{g}\text{-modules}), \quad q^\ell = 1$$

standard $|\lambda\rangle \longmapsto \Delta(\lambda)$ standard modules

KL $G^-_\lambda \longmapsto L(\lambda)$ simple modules

$$U_q\mathfrak{g}\text{-modules} = \bigoplus_{\nu \in A_\ell} (U_q\mathfrak{g}\text{-modules})^{[\nu]}$$

(missing Langlands dual ?)

<u>Translation</u> $\mathfrak{g} = \mathfrak{gl}_n$, $U_q\mathfrak{g}$

V = standard n-diml $U_q\mathfrak{g}$-module

$(\Delta(\mu) \otimes V)^{[\mu+\varepsilon_i]}$

has factors $\Delta(\mu+\varepsilon_j)$ in block $[\mu+\varepsilon_i]$

$$=: T_\mu^{\mu+\varepsilon_i}(\Delta(\mu)) = (\Delta(\mu) \otimes V)^{[\mu+\varepsilon_i]}$$

Keep track of factors according to their depth in a Jantzen filtration

Fix $\langle\, , \rangle_\Delta$ $U_q\mathfrak{g}$ contravariant form on $\Delta(\mu)$ $\langle\, , \rangle_V$ on V and set

$$\langle m_1 \otimes v_1, m_2 \otimes v_2 \rangle = \langle m_1, m_2 \rangle_\Delta \langle v_1, v_2 \rangle_V$$

on $\Delta(\mu) \otimes V$

$$N^+_{\mu+\varepsilon_j} = \left\{ \begin{array}{l} \text{highest wt vectors of} \\ \text{wt } \mu+\varepsilon_j \text{ in } \Delta(\mu_j) \otimes V \end{array} \right\}$$

$$V_{\varepsilon_j} = \left\{ \begin{array}{l} \text{wt vectors of wt} \\ \varepsilon_j \text{ in } V \end{array} \right\}$$

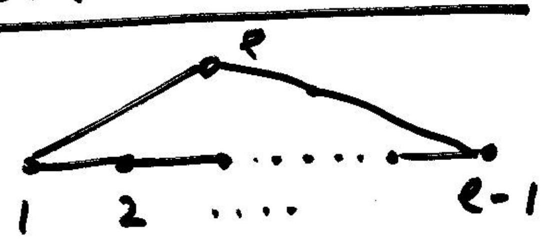
Let $a_{\mu+\varepsilon_j}(\mu)$ be given by

$$\det\left(\begin{array}{c} \langle , \rangle \text{ on} \\ N^+_{\mu+\varepsilon_j} \end{array} \right) = a_{\mu+\varepsilon_j}(\mu) \deg t(V_{\varepsilon_j})$$

For $1 \le i \le \ell$ define

$$(*) \qquad f_i |\mu\rangle = \sum_{\substack{\text{factors } \Delta(\mu+\varepsilon_{j'}) \text{ in} \\ \text{block } \mu+\varepsilon_i}} t^{\#([\ell] \text{ in } a_{\mu+\varepsilon_{j'}}(\mu))} |\mu+\varepsilon_{j'}\rangle$$

Hall algebra action on $\mathcal{F}_\ell$

$Q :=$ 

A representation of Q is a functor from Q to vector spaces over $\mathbb{F}_t$

The Hall algebra of Q is

$$U_t^- = \text{span}\left\{ \begin{matrix} \text{isom. classes } F_\lambda \text{ of} \\ \text{nilpotent repn of } Q \end{matrix} \right\}$$

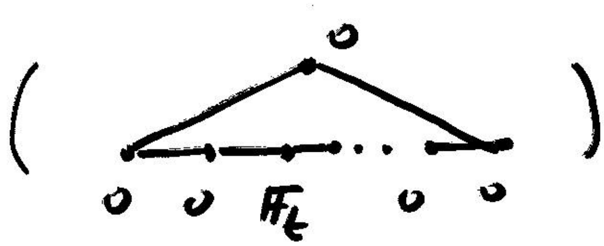
with product

$$F_\mu F_\nu = \sum_\lambda t^{-x_{\mu\nu}} g_{\mu\nu}^{\lambda}(t^{-2}) F_\lambda$$

where

$$g_{\mu\nu}^{\lambda} = \#\left\{ \gamma \subseteq \lambda \;\middle|\; \begin{matrix} \gamma \simeq \mu \\ \lambda/\gamma \simeq \nu \end{matrix} \right\}$$

$= \#\{$subrepns of λ of type μ and cotype $\nu\}$

$F_i =$ 

$(*)$ makes $\mathcal{F}_\ell$ into a U_t^- module