June 8, 2007 Schiffmann

Recall $\mathcal{M}_\mu$: character variety

Q_μ = quiver variety

Nakajima $\bigoplus_{n} H^{mid}(Q_{\mu n}) \circlearrowleft$ Kac-Moody algebra $= g_{KM}$

HISTORY

The point is: Q_μ "$\simeq$" $T^* M_\mu$

M_μ = "moduli space" of rep n of quiver $\vec{Q}$

one can recover g_{KM} (or $U_q(g_{KM})$) from $Rep\,\vec{Q}$. This is the Hall algebra construction.

I) Defn of a Hall Algebra

Machine (nice) abelian category $\rightsquigarrow$ assoc. algebra

$\mathcal{A}$ abelian category s.t. $\forall M, N \in \mathcal{A}$

$$\begin{cases} \# \mathrm{Hom}(M,N) < \infty \\ \# \mathrm{Ext}^1(M,N) < \infty \end{cases}$$

$\mathcal{I} = \mathrm{obj}\,\mathcal{A}/\sim$

$$H_\mathcal{A} := \bigoplus_{M \in \mathcal{I}} \mathbb{C}\, u_M$$

$$u_M \cdot u_N := \varepsilon_{MN} \sum_{L \in \mathcal{I}} P^L_{MN} \, u_L$$ well defined by finiteness conditions on $\mathcal{A}$

$$P^L_{MN} = \# \left\{ K \subseteq L \;\middle|\; \begin{array}{l} K \simeq N \\ L/K \simeq M \end{array} \right\}$$

structure constants $\rightsquigarrow$ assoc. algebra (easy to check)

$$0 \to N \to L \to M \to 0$$

i.e. "count extensions"

- $\varepsilon^2_{MN} \overset{\#}{=} \prod_{i \geq 0} \frac{\mathrm{Ext}^{2i}(M,N)}{\mathrm{Ext}^{2i+1}(M,N)}$
- $\varepsilon_{MN} > 0$

__Better__

$$H_{\mathcal{A}} = \{ f : \mathcal{I}_{\mathcal{A}} \to \mathbb{C} \mid \# \operatorname{supp}(f) < \infty \}$$

$$f * g(L) := \sum_{K \subseteq L} f(L/K)\, g(K)$$

convolution product.

__Examples__

1. "Classical Hall algebra" (Macdonald)

$\mathcal{A}$ = finite length $\mathcal{O}$-modules $\leadsto$

$\mathcal{O}$ = DVR
$\mathfrak{m}$ = mxml ideal
$\mathcal{O}/\mathfrak{m} = \mathbb{F}_q$

$$\mathcal{I}_{\mathcal{A}} = \{ \underbrace{\mathcal{O}/\mathfrak{m}^{\lambda_1} \oplus \cdots \oplus \mathcal{O}/\mathfrak{m}^{\lambda_r}}_{= \mathcal{O}_\lambda} \}$$

indexed by partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r)$

$$u_{\mathcal{O}_{(1)}} \cdot u_{\mathcal{O}_{(1)}} = u_{\mathcal{O}_{(2)}} + (q+1)\, u_{\mathcal{O}_{(1^2)}}$$

$$0 \to \mathcal{O}/m \longrightarrow \ ? \ \to \mathcal{O}/m \to 0$$

i) $? = \mathcal{O}/m^2$

$\# \{ K \subseteq \mathcal{O}/m^2 \text{ s.t. } K \simeq \mathcal{O}/m \} = 1$

ii) $? = \mathcal{O}/m \oplus \mathcal{O}/m$

$\# \{ K \subseteq \mathcal{O}/m \oplus \mathcal{O}/m \ \ K \simeq \mathcal{O}/m \} = q+1$

<u>EX.</u>

$$u_{\mathcal{O}_{(2)}} \cdot u_{\mathcal{O}_{(1)}} = u_{\mathcal{O}_{(3)}} + q\, u_{\mathcal{O}_{(12)}}$$

<u>THM</u> (Hall, Steinitz)

$$H_A \xrightarrow{\sim} \mathbb{C}[x_1, x_2, \dots]^{S_\infty}$$

$$u_{\mathcal{O}_{(1^r)}} \longmapsto q^{\frac{r(r-1)}{2}} e_r$$

$u_{\mathcal{O}_\lambda} \longmapsto$ Hall-Littlewood polynomials $P_\lambda(q)$

(commutativity related to duality in the category).

2. <u>Quivers</u>

$\overset{\cdot}{1} \longrightarrow \overset{\cdot}{2}$

$S_1 = k \to 0$

$S_2 = 0 \to k$

A = repn of quiver of k = finite field

Example

$$u_{S_1} \cdot u_{S_2} = u_{S_1 \oplus S_2} + u_{k \to k}$$

$$u_{S_2} \cdot u_{S_1} = u_{S_1 \oplus S_2}$$

(not commutative)

Ex $\quad u_{S_1}^2 u_{S_2} - (q+1)\, u_{S_1} u_{S_2} u_{S_1} + q\, u_{S_2} u_{S_1}^2 = 0$

(Serre relation: $\quad E_1^2 E_2 - 2 E_1 E_2 E_1 + E_2 E_1^2 = 0$)

THM (Ringel)

$$H_{\mathrm{Rep}_{\mathbb{F}_q}^{n} \vec{Q}} = \mathcal{U}_q^+(sl_3)$$

True for any quiver

$$H_{\mathrm{Rep}_{\mathbb{F}_q} \vec{Q}} \hookleftarrow \mathcal{U}_q^+(g_{\vec{Q}}) \quad \text{where}$$

$g_{\vec{Q}}$ is the KM algebra corresp. to $\vec{Q}$

(Isom. in finite case)

II) Hall algebras of curves

X smooth proj. variety / $\mathbb{F}_q$

$H_X := H_{\mathrm{coh}(X)}$ too big (depends on $\mathbb{F}_q$)

need to isolate a nice subalgebra.

Key observation (Kapranov) Function field version of automorphic forms.

set of vector bundles on X (curve)

$$\mathrm{Vec}_n(X) \xleftrightarrow{1-1} \mathrm{Gl}_n(k(x)) \backslash \mathrm{Gl}_n(\hat{k}_{x_0}) / \mathrm{Gl}_n(\hat{\mathcal{O}}_{x_0})$$

Fix $x_0 \in X$ (remark of A. Weil)

Analogue:

$$SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) / SO_2(\mathbb{R})$$

$$\Gamma \backslash G / K$$

Analogue of Eisenstein series

$$E_1(z) = \sum_{d \in \mathbb{Z}} 1_{\{\text{line bdles of deg } d\}} z^d$$

$E_1(z_1)\, E_1(z_2) \cdots E_1(z_r)$ generated by Eisenstein series.

Analogue of "Hecke operators"

$$1_{(0,1)} := 1_{\{\text{torsion sheaves of deg } 1\}}$$

some explicit constant

$$[1_{(0,1)}, u_V] = \alpha \sum_{\substack{V' \supset V \\ V'/V \text{ torsion of deg } 1}} u_{V'}$$

modification of V at a point

More generally:

$$1_{(0,d)} = \sum_{\substack{\tau \text{ torsion} \\ \deg \tau = d}} u_\tau$$

<u>Defn</u> $U_X \subseteq H_X$ subalgebra generated by:

$$1_{(1,d)} = \sum_{\mathcal{L} \in \mathrm{Pic}^d} u_{\mathcal{L}} \qquad d \in \mathbb{Z}$$

$$1_{(0,\ell)} \quad \ell \geq 1$$

<u>THM</u> (Kapranov)

Put
$$\psi(z) = \sum_{\ell \geq 1} 1_{(0,\ell)} z^\ell$$

$$E_1(z) = \sum_{d \in \mathbb{Z}} 1_{(1,d)} z^d$$

Then

1) $\zeta_X\left(\frac{z_1}{z_2}\right) E_1(z_2) E_1(z_1) = \zeta_X\left(\frac{z_2}{z_1}\right) E_1(z_1) E_1(z_2)$

2) $\psi(z)\psi(w) = \psi(w)\psi(z)$

$$\psi(w)\, E_1(z) = P_X(z,w)\, E_1(z)\, \psi(w)$$

("Eisenstein series are Hecke operators eigenvectors")

III) <u>Genus 0, k punctures</u>

$$X = \mathbb{P}^1, \quad \begin{cases} \lambda_1, \lambda_2, \ldots, \lambda_k \in \mathbb{P}^1(\mathbb{F}_q) \\ p_1, p_2, \ldots, p_k \geq 2 \end{cases}$$

<u>Defn</u> A vector bundle on $\mathbb{P}^1$ with $D = \sum p_i \lambda_i$ a <u>parab. structure</u> consists of

$$\left(V, \quad V|_{\lambda_i} = V_{ip_i} \supseteq \cdots \supseteq V_{i2} \supseteq V_{i1} \right)$$

<u>Point</u>: $T^* M^{r,\underline{d}}_{\mathbb{P}^1, D}$ closely related to (over $\mathbb{C}$ now)

$\mathcal{M}_{\mathbb{P}^1, D}$ (Higgs bundles)

Let $\mathfrak{g}_{KM}$ be the KM algebra associated to

Let $L\mathfrak{g}_{KM} = \mathfrak{g}_{KM}[t, t^{-1}] \oplus \mathbb{K}$

loopalgebra   problem?

(Garland)

<u>THM</u> (Schiffmann, Kapranov for $k=0$)

There exists a map

$$\psi : U_q^+(L\mathfrak{g}_{KM}) \twoheadrightarrow U_{\mathbb{P}^1, D}$$

& ψ is an isom if $\mathfrak{g}_{KM}$ is finite or affine Lie algebra.

Expect $L\mathfrak{g}_{KM}$ to act on $\bigoplus_{\mu} H^{mid}(M_\mu)$ (by analogy).

IV) $g=1$, <u>$k=0$</u>

E an elliptic curve / $\mathbb{F}_q$

$Coh(E)$ well understood by work of Atiyah

<u>Defn</u> $\mathcal{F} \in Coh(E)$ is semistable if $\mathcal{G} \subseteq \mathcal{F}$, $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$ where

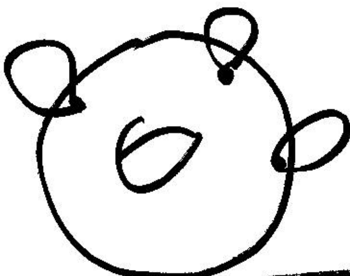
$$\mu(\mathcal{H}) \leq \deg \mathcal{H} / rk\, \mathcal{H} \in \mathbb{Q} \cup \{\infty\}$$

$\mathcal{C}_\nu :=$ full subcategory of semistable bundles of slope ν

Abelian category, stable under extensions.

E.g. $\mathcal{C}_\infty =$ Torsion sheaves on E

$$\mathrm{Tor}\,E = \prod_{x\in E}' \mathrm{Tor}_x \simeq \prod_{x\in E}' \mathcal{O}_x\text{-modules of finite length} =: \mathrm{Mod}_{fl}\,\mathcal{O}_x$$



$\mathcal{L}$ line bundle of deg d

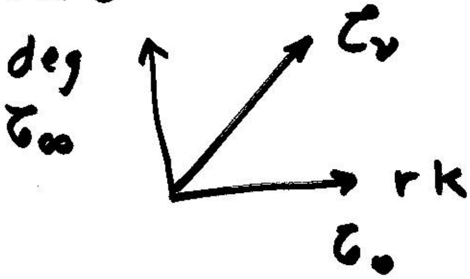
$\mathcal{L} \in \mathcal{C}_d$

<u>THM</u> (Atiyah)

i) Any indecomposable sheaf is semistable

ii) For any $\nu \in \mathbb{Q}$, $\mathcal{C}_\nu \simeq \mathcal{C}_\infty$

& Picture to keep in mind



$SL_2(\mathbb{Z})$ symmetries

$\mathcal{O}(1) \leftrightarrow \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Set $H_E^{(\nu)} := H_{\mathcal{C}_\nu}$

E.g. $H_E^{(\infty)} = H\left(\prod_x' \mathrm{Mod}_{fl}\,\mathcal{O}_x\right) = \bigotimes' H_{\text{classical}}$

Cor of Atiyah's thm

- $H_E^{(\nu)} \simeq H_E^{(\infty)}$ all $\nu \in \mathbb{Q}$.
- $H_E \simeq \overrightarrow{\bigotimes_\nu}{}' \, H_E^{(\nu)}$ (as vector spaces)

For $U_E = \langle \mathbb{1}_{(1,d)}, \mathbb{1}_{(0,\ell)} \rangle$

$$U_E^{(\nu)} = U_E \cap H_E^{(\nu)}$$

E.g.

$$\begin{tikzcd}
U_E^{(\infty)} = U_E \cap H_{\mathrm{Tor}\,E} = \langle \mathbb{1}_{(0,\ell)} \rangle \arrow[d, hook] \arrow[rr, phantom, "\simeq H_{cl}"] & & \\
H_{\mathrm{Tor}} \simeq \bigotimes_x{}' H_{cl}
\end{tikzcd}$$

classical Hall algebra

diag embedding

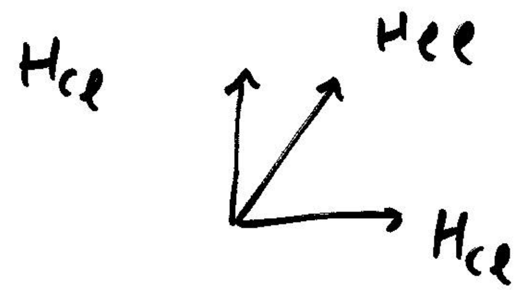
Prop (Burban–Schiffmann)

i) $U_E = \bigotimes_\nu{}' \, U_E^{(\infty)}$

(ii) $U_E^{(\nu)} \simeq U_E^{(\infty)}$ $(\simeq H_{cl})$ all ν

as vector spaces

Picture



Example

$$\underline{1}_{(0,1)} \cdot \underline{1}_{(1,0)} = \sum_{\tau, \mathcal{L}_0} u_\tau u_{\mathcal{L}_0}$$

$$0 \to \mathcal{L}_0 \to ? \to \tau \to 0$$

$$? = \mathcal{L}_0 \oplus \tau = \mathcal{L}_1$$

(unique since $\mathrm{Ext}^1(\tau, \mathcal{L}_0) = k$)

$$\# \{ \mathcal{L}_0 \mid \mathcal{L}_0 \subset \mathcal{L}_1, \ \mathcal{L}_1 / \mathcal{L}_0 \text{ torsion of deg } 1 \}$$

$$= \# \mathrm{Pic}^0(E) = \# E(\mathbb{F}_q)$$

THM (Burban - Schiffmann)
There exist a $A = \mathbb{C}[v^{\pm 1}, \tau^{\pm 1}]$ - algebra U_A (free as A-module, given by generators and relations) s.t. for all E

$$U_E = U_A \mid \begin{matrix} v = q^{-1/2} \\ t = q^{-1/2}\sigma \end{matrix}$$

σ eigenvalue of Frob in $H^1(E)$

MAIN THM (Schiffmann - Vasserot)

For all n, there exist

$$\psi_n : \mathcal{U}_A \twoheadrightarrow e\, \ddot{H}^+_{Gl_n} e \quad \text{(spherical DAHA)}$$

& $\bigcap_n \operatorname{Ker} \psi_n = \{0\}$

Remark Under ψ Hecke operator is mapped to Macdonald operator.

Eigenvectors prod of Eisenstein series $\leftrightarrow$ Eigenvectors Macdonald polynomials

Construction of Macdonald polynomials

Let $c = v t$

$$\tilde{E}_r(z) := \sum_{d \in \mathbb{Z}} \mathbb{1}_{(r,d)} (v^{1-r} z)^d$$

For any λ_i $i = 1, \dots, r$ form

$$\tilde{\mathbb{E}}_{\underline{\lambda}}(z) := E_{\lambda_1}(z)\, E_{\lambda_2}(c^{-1} z) \cdots E_{\lambda_r}(c^{1-r} z)$$

then if $\lambda_i < \lambda_{i+1}$ for some i $\tilde{\mathbb{E}}_{\underline{d}}(z) = 0$

if λ is a partition (i.e. otherw.)

$$\tilde{\mathbb{E}}(z)\Big|^{ss}_{z^0} = \omega\, P_\lambda(v, c) \quad (\omega = \text{involution } \omega s_\lambda = s_{\lambda'})$$

$\in U^{(0)}_- \simeq H_{r_0}$