PROBLEMS FROM THE AIM CHIP-FIRING WORKSHOP

These problems come from the "Generalizations of chip-firing and the critical group" workshop held from July 8th–12th, 2013 at the American Institute of Mathematics in Palo Alto, California. A reading list for the workshop included the papers [5], [6], [9], [11], [15], and [17], as well as David Perkinson's website devoted to sandpiles [16]. The problem session was moderated by Vic Reiner. These notes were recorded by Sam Hopkins. Quotations below are not direct; they are meant to summarize each speaker's key point (and are certainly subject to my not understanding this point or the mathematics behind it). The problems presented, and comments made about these problems, were as follows:

(1) Spencer Backman: "Is there a generalization of chip-firing to k-uniform hypergraphs, where chips are represented by kth roots of unity instead of ± 1? I have the general set-up. You need some additional data: a cyclic ordering of each edge. When a vertex fires, it sends the appropriate root of unity according to this order to its adjacent vertices. There are some encouraging signs: the complex Laplacian matrix is positive-semidefinite and the process is invariant under which total ordering of the vertices you take. I wrote down my thoughts about this problem in my statement of interest on the workshop's website.[1]"
   (a) Andrea Sportiello: "Perhaps we need only that the vectors sum to 0, not necessarily that they be roots of unity."
   (b) Spencer Backman: "When is a chip value 'non-negative'? I believe it is when the argument lies in some range, say between ω and ω^{n-1}. But what is the kernel of the Laplacian in this case? What is a good notion of reduced divisor, recurrent configuration, and so on?"

(2) Jeremy Martin: "We can efficiently sample random spanning trees of a graph in a uniform way by sampling the Jacobian group randomly. Can this approach be made to work when I change the graph to something more general like a simplex?"
   (a) Farbod Shokrieh: "The class of regular matroids is a good place to start looking for such a generalization. In this case we can define the Jacobian and we know the size is correct (that is, is equal to the size of a basis of the matroid). The problem is finding an efficient bijection. We have no analog of Dhar's burning algorithm at this level."

[1] The wokshop's website is aimath.org/WWN/chipfiring/.

(b) Jeremy Martin: "I can state the problem more concretely as follows. Let D be an $n \times m$ matrix over $\mathbb{Z}$ of rank n, and let $L\colon \mathbb{Z}^n \to \mathbb{Z}^n$ be given by $L := DD^t$. Define

$$J(D) := \mathbb{Z}^n/\mathrm{im}L = \mathrm{coker}(L).$$

We can uniformly sample from $J(D)$. But can we sample uniformly from the column bases of D?"

(c) Matt Baker: "If D is unimodular, then the usual Cauchy-Binet proof of Matrix-Tree theorem works and shows that $J(D)$ is in bijection with a column basis of D. We are really asking: is this bijection efficiently computable?"

(d) Jeremy Martin: "Actually, I believe you do not need unimodularity; you just need the determinants of all the maximal minors to be ± 1."

(3) Persi Diaconis: "Say you play the explorers game [8] on a circle with n vertices labeled cyclically 0 through $n-1$ and you repeatedly explore from 0. The probability that the last point occupied before the circle is filled is j is given by $A(n,j)/n!$, where $A(n,j)$ are the Eulerian numbers.[2] This fact was shown to me by Andrea Sportiello during the lunch break. What is interesting is that the situation is very different from a random walk, in which the distribution is uniform. It may be worth looking at the probability distribution of last occupied spot in the explorers game for other sets (for example in higher dimensions or on circulant graphs)."

(a) Lionel Levine: "Jim Propp made this observation about the Eulerian numbers a while ago."

(b) James Propp: "There are connections here to other models such as Pólya's urn and Bernard Friedman's urn. Are these connections useful?"

(c) Andrea Sportiello: "If you change graph to something like $\mathbb{Z}^2$, I would guess that things become ugly very fast."

(d) Lionel Levine: "It is not clear how to generalize to higher dimensions because so much is going on, but sticking to the one-dimensional case and putting weights on the edges may give some new q-analog."

(4) Sam Hopkins: "This question concerns a recent preprint of Venkatesh and Viswanath [19], which shows that for a finite graph G on n vertices, the dimension of the root space associated to the root that is the sum of all simple roots in the Kac-Moody algebra having Cartan matrix $2I_n - A(G)$ is the number of maximal G-parking functions. (Here $A(G)$ is the adjacency matrix of G.) Can this be exploited further? Are all G-parking functions present somewhere? Does the sandpile group act on some such root space?"

[2] For a definition of Eulerian numbers, see en.wikipedia.org/wiki/Eulerian_number.

   (a) Sam Hopkins: "I should also mention that there is a natural labeling of a basis of this root space by maximal parking functions."
   (b) Anton Dochtermann: "The number of maximal parking functions is also the number of last syzygies in a minimal free resolution of the toppling ideal, and also the number of chambers of a certain slice of the graphic arrangement. Can we connect the Lie theory to commutative algebra or hyperlane arrangements?"

(5) Lionel Levine: "How can we uniformly and efficiently sample from the set of maximal G-parking functions?"
   (a) Persi Diacons: "There may be a connection to my paper with Christos Athanasiadis [3]."

(6) Persi Diaconis: "When we play the explorers game and repeatedly explore from the origin on a square lattice in any dimension, the limiting shape is round. In dimensions $d \geq 3$, the fluctuations after n steps from this ball of radius $n^{1/d}$ are tiny, on the order of $\sqrt{\log(n)}$. For $d = 2$, the fluctuations are of size $\log(n)$, and for $d = 1$ the fluctuations are of size $\sqrt{n}$. These results in the higher dimensional cases have only been obtained recently; see [13], [2], and [12]. Are there other growth models where the fluctuations take on some intermediate value, say between $\sqrt{n}$ and $\log(n)$?"
   (a) Andrea Sportiello: "If we consider a wedge of the two dimensional lattice $\mathbb{Z}^2$, is it clear that the limiting shape is the sector of the circle and does not depend on the angles?"
   (b) Charles Smart: "Away from the border we should be fine."
   (c) Andrea Sportiello: "Then if you take the part of the lattice inside a parabola of exponent γ (that is, where $|y| \leq x^{1/\gamma}$), which is a further generalization of a wedge, this interpolates between one and two dimensions and is thus a candidate for intermediate growth of fluctuations."
   (d) Igor Pak: "Yes, in general if we consider the part of the lattice given by $|y| \leq f(x)$, we should be able to interpolate however we want between $\sqrt{n}$ and $\log(n)$ by choosing the appropriate function f."

(7) Matt Macauley: "Consider the equivalence relation on the set of all acyclic orientations of a graph G of converting sources to sinks. This relation encodes the chip-firing game on G among a special class of divisors. In this special case, there is a nice geometric way to view chip-firing via passing from the graphic arrangement (whose regions are in bijection with all acyclic orientations) to the toric graphic arrangement. Can we find a similar geometric way of viewing chip-firing equivalence for other (larger) classes of divisors?"
   (a) Andrea Sportiello: "I think we need to look at multi-toppling, or what has also been called hereditary chip-firing, to accomplish this."

   (b) James Propp: "Does multi-toppling have the abelian property?"
   (c) Andrea Sportiello: "It has the abelian property precisely when it is hereditary, when we fire clusters and then sub-clusters and so on."
   (d) Lionel Levine: "A good source for hereditary chip-firing is a paper of Spencer Backman [4]."

(8) Jordan Ellenberg: "By the Matrix-Tree theorem, the number of spanning trees of a graph G equals the size of the critical group of G. When are the spanning trees a torsor for the critical group? Making G into a ribbon graph, and given choice of sink, we do get a torsor structure from the rotor-router model [11]. But under what conditions can we get a natural torsor action without this additional data?"
   (a) Jordan Ellenberg: "If you do not know what a torsor is, I am asking when there is a simply-transitive action of the critical group on the set of spanning trees. Simply-transitive means that for any pair of spanning trees T and T', there is a unique group element taking T to T'. When I say I want this action to be 'natural', I mean at least that I want it to respect the automorphism group of G."
   (b) Farbod Shokrieh: "David Wagner has an argument (although not a proof) that we cannot expect to get a torsor in general."
   (c) Matt Baker: "A recent paper of mine and several coauthors [1] almost achieves this through a polyhedral decomposition of the real g-dimensional torus (where g is the genus of the graph). The cells of this decomposition are canonically in bijection with the spanning trees, the vertices are canonically in bijection with $\mathrm{Pig}^g(G)$, and $\mathrm{Pic}^g(G)$ is canonically in bijection with the critical group. The only thing that is not canonical here is the bijection between vertices and cells of the decomposition."
   (d) Lionel Levine: "Why is the genus special here?"
   (e) Matt Baker: "The set of reduced divisors depends on the choice of sink vertex q. It turns out that there is another nice set of coset representatives (well not really coset representatives because they do not form a group, but representatives of the set of degree g divisors modulo chip-firing equivalence) that can be defined in a canonical way."
   (f) Melody Chan: "In my recent paper with Church and Grochow [7], we look at when the choice of sink is needed in the rotor-router model. We show that the resulting torsor is independent of the choice of sink if and only if the graph G is planar."

(9) Lionel Levine: "Is there an efficient way to compute the rank (in the sense of Baker and Norine [5]) of a divisor?"

(10) Lionel Levine: "This question is inspired by Criel Merino's work [14] relating the Tutte polynomial to chip-firing. The Tutte polynomial is the most general invariant of a graph that has a deletion-contraction recurrence. Is there a deletion-contraction recurrence of some sort for critical groups? It would have to be at the level of group algebras rather than groups themselves."
  (a) Jeremy Martin: "Start by considering just a cycle graph."
  (b) Lionel Levine: "For hyperplane arrangement, there is a nice deletion-contraction rule that gives an exact sequence of algebras; is there a corresponding exact sequence for critical groups?"

(11) Lionel Levine: "Suppose I give you a stable, periodic sandpile configuration c on the square grid $\mathbb{Z}^2$. Is it decidable whether there is an integer n such that $c + n\delta_0$ (i.e. the addition of n grains of sand to the origin) is not stabilizable?"
  (a) Matt Baker: "What do the sets of n such that $c + n\delta_0$ is not stabilizable look like when there is such an n?"
  (b) Lionel Levine: "In that case there is a minimal such n."
  (c) Andrea Sportiello: "This minimal n must depend on the period; what else could it depend on?"
  (d) Lionel Levine: "Well that would certainly make the problem decidable."

(12) Persi Diaconis: "Is there an abelian growth model for non-overlapping discs growing in the plane?"
  (a) James Propp: "There is such an abelian model in the case of *non-nesting* discs; firing means if you have a disc that is totally contained in another one, you move it in some fixed way."
  (b) Anton Dochtermann: "We could even ask if there is an abelian model of the one-dimensional case of non-overlapping intervals on a line."

(13) Lionel Levine: "What is the right definition of an abelian network with shared memory?"
  (a) Lionel Levine: "I have unified some subset of the class of models that behave like chip-firing does, but a lot of models with an abelian property still fall outside this definition and seem to require some kind of shared memory. An example of shared memory is cluster-firing, as investigated by Spencer Backman [4], who in particular studied when the abelian property is preserved while allowing firing of multiple vertices at once."
  (b) Jordan Ellenberg: "This seems formally similar to Persi Diaconis's question on non-overlapping discs."

(14) David Perkinson: "This question is due to Richard Stanley; it is problem 4 of chapter 6 in his notes on hyperplane arrangements [18]. Is there a natural bijection ϕ between labeled trees T on $\{0, 1, 2, ..., n\}$

rooted at 0 and parking functions of length n such that

$$\deg(\phi(T)) = g - \operatorname{inv}(T),$$

where g is the genus of the complete graph K_{n+1} and $\operatorname{inv}(T)$ is the number of inversions of T? I believe such a bijection would extend to an arbitrary graph by replacing the inversion statistic with the κ statistic of Ira Gessel [10]. No bijection I have tried works (and there are many bijections between spanning trees and parking functions in the literature)."

(15) Jeremy Martin: "What is the average value of the scaling limit of the identity element of the sandpile group on $\mathbb{Z}^2$?"
  (a) Wesley Pegden: "It might be possible to explicitly construct the limit of the identity in the case of the square lattice, and then compute this average and other numbers as well such as the side length of the square of constant value 2 that appears in the middle. We do not know how to do this yet."

(16) Wesley Pegden: "Are all the ways to hexagonally tile the plane with polyominoes that are 90°-symmetric given by the tilings (up to dilations, say) considered in my recent paper with Charles Smart on the Apollonian circle-packing patterns that emerge in the scaling limit of the sandpile model? We believe this conjecture (and in fact state a stronger conjecture in the paper)."

(17) Caroline Klivans: "What are higher dimensional (in the sense of [9]) critical configurations?"
  (a) Sam Hopkins: "Are you looking specifically for analogs of stable recurrent configurations? Or do analogs of parking functions or superstable elements also interest you?"
  (b) Caroline Klivans: "We would like to generalize any of these notions from the graph case to the simplicial complex case."

(18) Caroline Klivans: "How can we extend to higher dimensions Merino's result [14] that the generating function of the critical configurations (by degrees) is equal to an evaluation of the Tutte polynomial of G (in particular, is equal to $T(1, y)$)?"
  (a) David Perkinson: "So you have a notion of external activity in this case?"
  (b) Caroline Klivans: "We can look at the cellular matroid corresponding to the simplex. However, the size of our critical group is a *weighted* sum of spanning trees, so we will need to look at the arithmetic Tutte polynomial instead of the normal Tutte polynomial."
  (c) Igor Pak: "The degree should be the same across each coset, and in this case we can define the generating function formally without choosing representatives. So we may be able to answer this question without first answering the question about what the right notion of critical configurations is."

(19) Andrea Sportiello: "Can we define a notion of chip-firing in different-dimensional faces of the same (simplicial/cell) complex?"
   (a) Gregg Musiker: "Is there a Chow ring lurking here?"
(20) Andrea Sportiello / Jordan Ellenberg: Develop a chip-firing model for cell complexes that includes chips on cells of different dimensions and features interaction between dimensions; probably, chip configurations and firing in lower dimensions influence what can happen in higher dimensions.
(21) Many participants: Generalize Baker and Norine's graphical Riemann-Roch theorem from graphs to cell complexes. Solving this problem may require a good combinatorial definition of critical configurations / superstables / G-parking functions / reduced divisors in arbitrary dimension, and/or understanding higher-dimensional generalizations of the Riemann-Roch theorem in algebraic geometry.
(22) Caroline Klivans: "Develop a chip-firing model for directed rooted forests (as defined by Olivier Bernardi and Klivans) in higher dimension."

References

[1] Yang An, Matthew Baker, Greg Kuperberg, and Farbod Shokrieh. Canonical representatives for divisor classes on tropical curves and the matrix-tree theorem. `arXiv:1304.4259`, April 2013.
[2] Amine Asselah and Alexandre Gaudillière. From logarithmic to subdiffusive polynomial fluctuations for internal DLA and related growth models. *Ann. Probab.*, 41(3A):1115–1159, 2013.
[3] Christos A. Athanasiadis and Persi Diaconis. Functions of random walks on hyperplane arrangements. *Adv. Appl. Math.*, 45:410–437, 2010.
[4] Spencer Backman. A bijection between the recurrent configurations of a hereditary chip-firing model and spanning trees. `arXiv:1207.6175`, July 2012.
[5] Matthew Baker and Serguei Norine. Riemann-Roch and Abel-Jacobi theory on a finite graph. *Adv. Math*, 215:766–788, 2007.
[6] Ben Bond and Lionel Levine. Abelian networks i: Foundations and examples. eprint, `http://www.math.cornell.edu/~levine/abelian-networks-I.pdf`, July 2013.
[7] Melody Chan, Thomas Church, and Joshua Grochow. Rotor-routing and spanning trees on planar graphs. eprint, `http://www.math.harvard.edu/~mtchan/rotor-routing_2013_07.pdf`, July 2013.
[8] Persi Diaconis and William Fulton. A growth model, a game, an algebra, Lagrange inversion, and characteristic classes. *Rend. Semin. Mat. Univ. Politec. Torino*, 49:95–119, 1991.
[9] Art M. Duval, Caroline J. Klivans, and Jeremy L. Martin. Critical groups of simplicial complexes. *Ann. Comb.*, 17:53–70, 2013.
[10] Ira M. Gessel. Enumerative applications of a decomposition for graphs and digraphs. *Discrete Math*, 139:257–271, 1993.
[11] Alexander E. Holroyd, Lionel Levine, Karola Mészáros, Yuyal Peres, James Propp, and David B. Wilson. Chip-firing and rotor-routing on directed graphs. *Prog. Probab.*, 60:331–364, 2008.
[12] David Jerison, Lionel Levine, and Scott Sheffield. Internal DLA in higher dimensions. `arXiv:1012.3453`, December 2010.

[13] David Jerison, Lionel Levine, and Scott Sheffield. Logarithmic fluctuations for internal DLA. *J. Amer. Math. Soc.*, 25:271–301, 2012.

[14] Criel Merino-Lopez. Chip firing and the Tutte polynomial. *Ann. Comb.*, 1:253–259, 1997.

[15] Srdjan Ostojic. Patterns formed by addition of grains to only one site of an abelian sandpile. *Physica A*, 318(1):187–199, 2003.

[16] David Perkinson. Sandpiles. `http://people.reed.edu/~davidp/sand/`, July 2013.

[17] David Perkinson, Jacob Perlman, and John Wilmes. Primer for the algebraic geometry of sandpiles. `arXiv:1112.6163`, December 2011.

[18] Richard P. Stanley. An introduction to hyperplane arrangements. In *Geometric combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 389–496. Amer. Math. Soc., Providence, RI, 2007.

[19] R. Venkatesh and Sankaran Visawanath. Chromatic polynomials of graphs from Kac-Moody algebras. `arXiv:1303.1148`, March 2013.