HOLOMORPHIC CURVES IN CONTACT GEOMETRY

The American Institute of Mathematics

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Chapter A: Open problems

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These questions are organized chronologically, as they arose in the workshop discussions.

First day

If M is a smooth manifold, let $J^1(M)$ denote the 1-*jet space* of M, i.e. the manifold $\mathbb{R} \times T^*M$ with the contact structure given by the contact form $du - p\,dq$, where u denotes the $\mathbb{R}$ coordinate and $p\,dq$ is the standard symplectic form on T^*M.

Question 1.1. *Tobias Ekholm asked [something approximating] the following: If $L \subset J^1(M)$ is $\{0\}$ cross the 0-section of T^*M and $L' \subset J^1(M)$ is Legendrian isotopic to L, must the projections of L and L' to T^*M intersect?*

The discussion apparently concluded that the answer is yes (if M is compact?), so at least one question seems to have been answered at the workshop! [Eliashberg points out that the answer to the question as written above is obvious from the existence of a generating function, so I probably don't have the statement of the question exactly right, and there was some nontrivial proof of something which I missed.]

If $E \to M$ is a fiber bundle and $f : E \to \mathbb{R}$ is a function, then the set of fiberwise critical values of f (or the Cerf diagram) defines a subset of $M \times \mathbb{R}$, which is the front projection of a Legendrian submanifold $L \subset J^1(M)$. We say that f is a *generating function* for L.

Question 1.2. *Which L admit generating functions? One wants to impose some restrictions on the generating function. (Typically one wants the generating function to be quadratic at infinity, although it might be interesting to explore other conditions at infinity. Quadratic at infinity imposes some restrictions on the Legendrians that can be realized, for example stabilizations (zig zags) are somehow prohibited in dimension 3.) As Yasha Eliashberg explained, there is a theorem of Giroux which gives a necessary and sufficient condition for existence of a finite-dimensional generating function f if no condition on the behavior of f at infinity is imposed. If one allows an infinite-dimensional generating function, then one can write an explicit formula for it using the action functional.*

When the Legendrian L admits a [suitable?] generating function, one can use Morse theory of the generating function to define various invariants of L.

Theme 1.3. *Lisa Traynor discussed how there is a mysterious [or not?] connection between her polynomial invariants of Legendrian submanifolds defined in terms of generating functions and other polynomial invariants defined out of Linearized Contact Homology [definition?]. In some sense the two invariants encode the same information. So maybe there is some more precise question to ask here about sorting this out.*

Question 1.4. *Ekholm asked:*
(a) Is Linearized Contact Homology always equal to Generating Function Homology [definition?] for a (compact?) Legendrian $L \subset J^1(\mathbb{R}^n)$? As evidence for this conjecture, it was proved in the case where L is the double of a compact n-manifold with boundary $M \subset \mathbb{R}^n$.

(b) For a Legendrian $L \subset \mathbb{R}^n$, does L admit a generating function [satisfying what restrictions?] if / only if the Linearized Contact Homology of L is (uniquely) defined?

Theme 1.5. *A recurrent theme discussed by several people during the first day was the possibility of understanding holomorphic curves by taking some limit in which they degenerate to simpler objects. For example, Fukaya-Oh, inspired by a paper of Witten, studied how in a certain situation, holomorphic discs degenerate to "gradient trees". This idea is currently being applied to relative contact homology in the work of Lenny Ng and Zhu Ke. Also, there is Mikhalkin's work on amoebas, which similarly understands holomorphic curves in a Lagrangian fibration by shrinking the fibers. In a similar vein, there is also recent work relating the Chas-Sullivan product on the loop space to the cup product in Floer homology. Anyway, while we are not asking a question here, this seems to be an important theme for future work.*

Question 1.6. *Ko Honda spoke on results with Etnyre that certain knot types are not transversally simple. The proof is indirect and there is no invariant here. Eliashberg asked if an invariant can be constructed [to distinguish transversal knots].*

Second day

Question 1.7. *Paul Biran asked, following up some results presented by Emmanuel Giroux, if one can give a definition of "overtwisted" in higher dimensions in terms of open books. [Giroux proposed some definition, see also what he said on the sixth day.]*

Question 1.8. *There was a fair bit of discussion on how to possibly compute contact homology in terms of (contact) open books. For example, Denis Auroux pointed out that if you have a Lagrangian L in a page, then this is a Legendrian in the open book, and the Reeb chords are the intersections of L with $\phi^k(L)$ where ϕ is the monodromy. This might be a starting point for computing the relative contact homology of L, in the complement of the binding, in terms of things related to symplectic Floer homology of Lagrangians. There is then the problem of understanding what happens when one puts back the binding, but this might correspond to an "A_∞ deformation".*

Next, Eliashberg discussed the following questions:

Question 1.9. *Conjecture: for any Stein fillable contact manifold, the cylindrical contact homology is defined. (As some have used it in the past, CCH is defined if there exists a contact form with no contractible Reeb orbits of certain Maslov indices. If such a form exists, then any two such forms give the same CCH. For this conjecture, one might not be able to eliminate bad contractible orbits, but one could hope that one can make them somehow "algebraically cancel".) If so, then counting pairs of pants etc. gives operations on the CCH, as in Floer theory of symplectomorphisms.*

Question 1.10. *This leads to the problem of computing contact homology of Stein fillable contact manifolds. If W is a Stein filling, a pseudoconvex function $\varphi : W \to \mathbb{R}$ gives a handle decomposition. One could try to understand what happens to the contact homology as one attaches handles. The subcritical case is more or less understood by the work of Mei-Lin Yau and Frederic Bourgeois. The point is that when you attach a subcritical handle, there is a contact sphere in the middle of the handle, so you basically get the homology of the manifold with coefficients in the contact homology of these spheres.*

The interesting part is to understand what happens when you attach a handle of critical index along a Legendrian sphere L. In this case a Reeb trajectory can hit L, enter the handle, and leave the handle anywhere else along L. So the new closed Reeb orbits are unions of Reeb chords of L. This is highly suggestive that there is some surgery formula in terms of the relative contact homology of L. What is the surgery formula???

Someone made some analogy with adding exceptional fibers to a Lefschetz fibration...

Question 1.11. *Generalizing the theme of computing things, one could try to extend symplectic field theory to an "extended field theory". Recall that a TQFT assigns to an n-manifold (possibly with some extra structure) a number, and to an $(n-1)$-manifold a vector space, satisfying various axioms which allow one to compute the invariant of an n-manifold by cutting it up along $(n-1)$-manifolds. But then one is the left with the problem of understanding the $(n-1)$-dimensional invariant. In an extended TQFT, one can compute the latter by cutting along $(n-2)$-dimensional manifolds, to each of which is assigned a* category*. (One can continue by assigning 2-categories to $(n-3)$-manifolds and so forth, so that the manifolds get simpler while the theory gets more complicated...) Now the question is, how can one do this for SFT?*

Question 1.12. *In setting up such an extended field theory picture, it is important to formulate holomorphic curve theory for manifolds with boundary, or open manifolds with some asymptotic conditions. The basic idea is that when you replace boundary conditions with asymptotic conditions, you get more information about and control over how you approach the boundary. For example, if you are looking at holomorphic curves with boundary in a Lagrangian, it is sometimes better to consider curves with asymptotics in the unit cotangent bundle of the Lagrangian. More generally, for the extended field theory picture, one may need to consider holomorphic curves with boundary along some Lagrangian cylinders, and then you have holomorphic curves with corners, two different types of asymptotic conditions...*

Question 1.13. *How unique are the open books corresponding to contact manifolds? Giroux explained that the open books produced by Donaldson's construction, which depend on a positive integer parameter k, are unique up to stabilization when k is sufficiently large.*

Question 1.14. *Stein is to Weinstein as plurisubharmonic is to what? [This is not a mathematical problem, just a question of what term to use; the definition we want is clear. Perhaps this question is not worthy of this problem list.]*

Takao Akahori asked the following:

Question 1.15. *(a) Make a theory of "Weinstein spaces", as opposed to Weinstein manifolds, by analogy with Stein manifolds and Stein spaces.*
(b) Is the space of psh functions on a Weinstein manifold connected? (No critical points at infinity.) What if we assume the same contact structure at infinity? There exists an example of a manifold diffeomorphic to $\mathbb{C}^n$ for which there is more than 1 critical point for any psh function, and this bears on uniqueness.

Question 1.16. *Auroux asked (and others participated in the discussion) if there is some category for Legendrians in open books, by analogy with the Fukaya category of Lagrangians in a sympletic manifold. It is difficult to separate out the two Legendrians (i.e. just look at Reeb chords from one to the other), so you would have to have coefficients in the contact homology of the individual Legendrians. One can make use of the extra function direction in the open* *book, e.g. one can filter CH by the amount of rotation of the Reeb chords around the open book. If the two Legendrians are on the same page, the CH should not agree with the Lagrangian Floer homology, because one has to take into account the images of the Legendrians under the iterated monodromy. And again, putting in the binding might somehow correspond to a deformation of an A_∞ category. Also note that in general, the relative contact homology should be a module over the absolute contact homology.*

Eliashberg discussed an approach to the geometry of gluing in the binding. For the symplectization of the complement of the binding, you have convex, flat, and concave boundary components. However you can round corners to absorb the flat component into the convex component. Then, when you glue in the binding, you "partially glue" onto part of the convex component.

Third day

At the end of his talk, Paul Biran asked the following questions:

Question 1.17. *Let P^5 be the contact S^1-bundle over T^4 with c_1 equal to the class of the symplectic form on T^4. Conjecture: P has no Stein filling.*

Question 1.18. *Let $Q^n = \{z_0^2 + \cdots + z_{n+1}^2 = 0\} \subset \mathbb{C}P^n$. Conjecture: Q^n does not contain two disjoint Lagrangian spheres. (This is trivial if n is even.)*

At the end of his talk, Leonid Polterovich asked the following question:

Question 1.19. *(a) Let M be the unit ball in $\mathbb{R}^4$ and let $L = \{p_1^2+q_1^2 = p_2^2+q_2^2 = 1/2\} \subset \mathbb{R}^4$ be the Clifford torus. Is L weakly boundary rigid? That is, if K is Hamiltonian isotopic to L in M, then must $K \subset \partial M$? (There are several proofs that one cannot have $K \subset M \setminus \partial M$.) [See Eliashberg's discussion at the end of the day.]*
(b) Similarly investigate Lagrangian tori in other ellipsoids in $\mathbb{R}^{2n}$, and explore nonremovable intersections in this setting.

At the end of his talk, Alex Ivrii mentioned the following open questions:

Question 1.20. *(a) If $L \subset T^*\Sigma_g$ is a Lagrangian that is homologous to the 0-section, and $g > 1$, must L be Lagrangian isotopic to the 0-section? (This is now known for $g = 0$ and $g = 1$ by the works of Richard Hind and Alex Ivrii respectively.)*
(b) Is there a Lagrangian $T^n \subset \mathbb{R}^{2n}$ which is not Lagrangian isotopic to the Clifford torus when $n > 2$? The answer is no for $n = 2$ by Ivrii. However current holomorphic curve techniques do not seem applicable to higher n. We do not even know:
(c) Are there any local Lagrangian knots in $\mathbb{R}^{2n}$ for $n > 2$, i.e. Lagrangian submanifolds that are asymptotic to a Lagrangian n-plane but not Lagrangian isotopic to one? (Eliashberg and Polterovich showed that the answer is no when $n = 2$.)

Question 1.21. *Margaret Symington asked a series of questions centered around the question of whether a locally toric structure is helpful for counting holomorphic curves. This is inspired by the success of using amoebas in Mikhalkin's work; locally toric pictures have a lot of the same structure Mikhalkin used. [Unfortunately I do not have good notes on this.]*

Question 1.22. *Polterovich mentioned spectral invariants in Floer homology. Namely, for $f \in \mathrm{Ham}(M)$ and $\alpha \in QH^*(M)$, define $c(f,\alpha)$ to be the smallest c such that α appears in $HF^*(\mathscr{A} \leq c)$, where $\mathscr{A}$ is the action functional on a suitable cover of the loop space. Question: study the asymptotic properties of $c(f^n,\alpha)$.*

Question 1.23. *Polterovich also suggested that it if $\Phi : M \to B$ is a [what kind of?] fibration, then the set $\{\Phi^* H \mid H : M \to \mathbb{R}\}$ is a "maximal torus" in* $\mathrm{Ham}(M)$ *and should provide a good source of examples for calculations in Floer homology. [There was then some further discussion of locally toric fibrations by various people of which I do not have good notes.]*

Question 1.24. *Polterovich then suggested that Hofer's geometry on the space of Lagrangian submanifolds (as opposed to Hamiltonian symplectomorphisms) has not really been studied properly. In particular, the following 2-dimensional problem is unsolved. Let $\mathscr{L}$ denote the space of simple closed curves in S^2 that divide S^2 into two regions of equal area. Define a metric d on $\mathscr{L}$ as follows. Suppose $F(x,t)$ is a Hamiltonian generating an isotopy $\{L_t\}$ from L_0 to L_1. Define the length of the path $\{L_t\}$ by*

$$\mathrm{length}\{L_t\} := \int_0^1 (\max F_t|_{L_t} - \min F_t|_{L_t})\, dt.$$

Finally, define

$$d(L_0, L_1) := \inf_\gamma \mathrm{length}(\gamma)$$

where the infimum is taken over any path γ from L_0 to L_1.

Question: what is the diameter of the metric space $(\mathscr{L}, d)$? It is known that $\mathrm{Ham}(S^2)$ *has infinite diameter, but there is no natural candidate for a path going to ∞ in $\mathscr{L}$. (The following possibility was proposed: take a vertical great circle and then stretch it by spinning a neighborhood of the equator a lot. However there was some skepticism that this would give a path of infinite length.)*

For a given pair of curves in $\mathscr{L}$, one can apparently get an upper bound on the distance between them in terms of combinatorics (meanders).

If this diameter is finite, then it might also be interesting to study the analogous invariant for a general pair (M, L) where M is a symplectic manifold and L is a Lagrangian submanifold of M.

For example it is interesting to ask the same question for $(\mathbb{C}P^2, \mathbb{R}P^2)$.

Also note that there is a natural map

$$\mathrm{Ham}(M, \omega) \longrightarrow \mathscr{L}(M \times M, \omega \oplus -\omega).$$

This preserves the lengths of smooth paths, but is not an isometry; the work of Ostrover shows that this map is highly distorting.

Question 1.25. *Biran pointed out that the following conjecture of Audin has not yet been proved in full generality: if L is a Lagrangian torus in $\mathbb{C}^n$ then the minimal Maslov number $N_L = 2$. In the monotone case, this follows by a simple argument using Oh's spectral sequence.*

Question 1.26. *Eliashberg discussed using SFT to prove the Audin conjecture and the weak boundary rigidity of the Clifford torus in S^3. [How much of this requires the as-yet unfinished foundations of SFT, and how much of it is merely inspired by SFT and does not require that much machinery?] The basic idea is that instead of considering holomorphic curves with boundary in L one should consider holomorphic curves with asymptotic conditions in T^*L. One then has to study holomorphic curves in $T^*(T^n)$ just once. However even this remains "stupidly open". (It was also asked if one can do something with CR structures...)*

Fourth day

Question 1.27. *At the end of his talk, Hutchings mentioned that one could try to generalize the methods therein to compute the Embedded Contact Homology of $S^1 \times S^2$, torus bundles over S^1, or unit cotangent bundles of surfaces of genus $g > 1$. It might also be interesting to try to further understand the holomorphic curves in $\mathbb{R} \times T^3$ in terms of amoebas etc.*

Question 1.28. *Peter Ozsvath discussed the problem of computing Ozsvath and Szabo's invariants combinatorially or axiomatically. Their theory lies somewhere between combinatorial invariants, which are easy to compute but not very useful, and invariants defined in terms of PDE's, which are very useful but hard to compute. "Right next door" is Khovanov's categorification of the Jones polynomial, which looks like a Floer theory but is defined purely combinatorially. In fact there is the following bridge between them: given a link K in S^3, there is a spectral sequence whose E^2 term is the Khovanov homology of K (with $\mathbb{Z}/2$ coefficients) and which converges to $\widehat{HF}(\Sigma_2(K))$, where $\Sigma_2(K)$ denotes the double cover of S^3 branched along K. Question: can one compute the differentials in the spectral sequence combinatorially? Also, $\widehat{HF}(\Sigma_2(K))$ has applications to slice genus bounds: can Khovanov homology say anything about the 4-ball genus?*

Question 1.29. *A big open problem is to prove the conjectured equivalence between Ozsvath-Szabo theory and Seiberg-Witten theory. Yi-Jen Lee discussed a possible approach to this.*

Fifth day

Question 1.30. *(Eliashberg) Do symplectic and contact invariants of the cotangent bundle of a smooth manifold M remember the topology of M? For example, if M_1 and M_2 are homeomorphic but not diffeomorphic (e.g. exotic spheres), can one detect this by showing that T^*M_1 and T^*M_2 are not symplectomorphic, or that ST^*M_1 and ST^*M_2 are not contactomorphic?*

Related question: is the natural map from knots in S^3 to Legendrian tori in $ST^*S^3 \simeq S^3 \times S^2$ injective? Ng's work is the first result in this direction.

Question 1.31. *Eliashberg discussed trying to understand the Gopakumar-Vafa picture in terms of symplectic field theory. [I missed a lot here...] One question which is important for this is to try to define Relative Contact Homology using higher genus curves, not just rational curves.*

Sixth day

Question 1.32. *Akahori discussed some problems involving understanding the geometry of Kohn-Rossi cohomology.*

Question 1.33. *(Giroux)*
(a) A contact structure on a closed manifold is equivalent to a symplectomorphism of a Stein manifold, equal to the identity on the boundary, modulo some kind of stabilization. What are interesting examples of symplectomorphisms of Stein manifolds that are equal to the identity on the boundary? One example of such a symplectomorphism is the symplectic Dehn twist around a (parametrized) Lagrangian sphere. Do these generate the group of all such symplectomorphisms?

Of course, this would imply that the group of all such symplectomorphisms of the 6-*ball is trivial, which we do not know. (Eliashberg: is there some formulation of the above question modulo the seemingly hopeless problem of understanding symplectomorphisms of higher-dimensional balls?)*

For example, consider a closed symplectic manifold (W, ω) *with* $[\omega]$ *integral. Let* H_k *denote the degree* k *Donaldson hyperplane section for* $k >> 0$*. Then* $F_k := W \setminus H_k$ *is Stein. A neighborhood of* ∂F_k *is a symplectic annulus bundle over* H_k*. Suppose you do a Dehn twist in each annulus fiber, then you get a symplectomorphism of* F_k *which is the identity on the boundary. (This is the monodromy of a certain canonical open book...) Question: is this symplectomorphism a product of symplectic Dehn twists?*

In some cases, F_k *is a subcritical submanifold, so by Biran-Cieliebak, contains no Lagrangian sphere. In particular, in this case, is the above symplectomorphism isotopic (even topologically) to the identity?*

(b) Here is a possible procedure for constructing Ustilovsky's infinitely many contact structures on S^{4n+1} *in terms of open books. The standard contact structure on* S^{2n+1} *comes from the positive symplectic Dehn twist* ϕ *on* T^*S^n*. Think that* $\phi^3, \phi^5, \ldots$ *give Ustilovsky's contact structures. Can one distinguish* ϕ *from* ϕ^3 *using Floer homology? Is there a stabilized version of Floer homology distinguishing the contact structures, and how does this compare to contact homology? More generally, this is a procedure for constructing many new contact structures out of an old one, by replacing each Dehn twist by an odd iterate of it.*

(c) Here is a way to possibly produce nonfillable contact structures in higher dimensions, analogous to overtwisted contact structures. Let W *be a Stein manifold,* ϕ *a symplectomorphism of* W *equal to the identity on the boundary, and* D^n *a Lagrangian ball with* $\partial D \subset \partial W$*. On* ∂D*, attach a handle outside, to get a new Stein manifold with a Lagrangian sphere. Compose* ϕ *with a* left-handed *Dehn twist. We get the same manifold, but are the contact structures different? Unfillable? In dimension 3, one can get all overtwisted contact structures by this construction. (Eliashberg: Try* S^5*, one left-handed twist in* $UT^*(S^2)$*.)*

Question 1.34. *Auroux discussed how, roughly, a certain subgroup of the braid group gives automorphisms of a symplectic Lefschetz pencil, asked whether we can do something like this with open books or contact pencils, and in connection with this explained how Giroux's question (a) above has a positive answer in a certain precise case.*

[The following is from the notes of John Etnyre and David Farris, since I wasn't there.]

Question 1.35. *Eliashberg made some remarks on the above topics.*

Banyaga discussed locally conformal symplectic geometry (c.s.s.). A manifold has a local conformal symplectic structure if it supports a non degenerate 2-form Ω such that $d\Omega = -\omega \wedge \Omega$ for some closed 1-form ω. An example is constructed starting with a contact manifold (N, α). Let $X = N \times S^1$, θ be the pull back of α and set $\Omega = d\theta + \omega \wedge \theta$ where ω is the pull back of the volume from on S^1.

Define Lichnerowicz cohomology as follows. Fix a closed 1-form ω now define d_ω by $d_\omega \theta = d\theta + \omega \wedge \theta$. This is a differential on forms and if Ω is a c.s.s. then $d_\omega \Omega = 0$. So a conformal symplectic form is closed with respect to some differential.

Another example of a c.s.s is constructed by starting with a symplectic manifold (X, ω) then let $\Omega = f\omega$ for any positive function f on X. Such a c.s.s. is called a *global* c.s.s.

Question 1.36. *Given a c.s.s. is it global?*

There is the following result: a c.s.s. Ω is global if and only if ω is exact. Are there other conditions?

Banyaga has constructed c.s.s. for which Ω is non zero in the Lichnerowicz cohomology, but such examples seem few and far between.

Question 1.37. *Find other constructions of c.s.s. that represent a non zero class in Lichnerowicz cohomology.*

Given a c.s.s. one can consider compatible almost complex structures, just as one does for symplectic structures.

Question 1.38. *What can be said about holomorphic curves for theis compatible almost complex structure? Are they useful tools for studying c.s.s.? Eliashberg thinks it is unlikely they will be able to say much.*

Question 1.39. *Polterovich made the following conjecture and gave some discussion of examples and related results. Conjecture: let (M, ξ) be a closed contact manifold and $\varphi \in \mathrm{Cont}(M, \xi)$ a contactomorphism. Suppose that the induced action on the contact homology $CH(M, \xi)$ is hyperbolic. (I think part of the problem is to formulate what "hyperbolic" should mean in this situation.) Then φ is strongly dissipative, i.e. there does not exist a smooth φ-invariant volume form.*

This question is part of a larger theme. There is a kind of map from the smooth topological world to the contact world. For example given a manifold M one can consider its unit cotangent bundle X. X is a contact manifold. Given a submanifold K of M one can consider its unit conormal bundle L_K in X. This is a Legendrian submanifold of X. Given a diffeomorphism f of M one can lift it to a contactomorphism of X. These are just a few examples of the "functor" between the smooth world and the contact world. [Compare Eliashberg's first question on the fifth day.] The main question of Polterovich is: How much information is lost by this functor? The above question is just one possible way of showing that some subtle information is preserved. [How exactly does this relate to the previous paragraph?]

Question 1.40. *Mitsumatsu discussed something about linking numbers of orbits of vector fields in connection with Reeb vector fields of contact structures and foliations.*

Question 1.41. *Yi-Jen Lee gave some further discussion of how to possibly relate Ozsvath-Szabo and Seiberg-Witten theory.*

Question 1.42. *Peter Ozsvath said that it would be nice to see a spectrum theory (a la C. Manolescu) for Lagrangian intersections.*