The following questions were proposed during the 2010 workshop "Emerging Applications of Complexity for CR Mappings". The name in bold preceding each problem statement belongs to the person who proposed the question. In what follows, manifolds will be assumed to be smooth unless otherwise stated and $\mathbb{B}^n$ will denote the unit ball in $\mathbb{C}^n$.

# Mappings between balls

1. **(B. Lamel)** Let $n \geq 2$.
   - Given a proper rational map $f : \mathbb{B}^n \to \mathbb{B}^N$, find the smallest number $k(n, N)$ such that $f$ is determined by its $k$-jet at the origin.
   - Let $\Omega, \Omega'$ in $\mathbb{C}^n$ and $\mathbb{C}^N$ respectively be strongly pseudoconvex domains. If $f : \Omega \to \Omega'$ is a proper holomorphic map which extends smoothly to $\partial\Omega$, does there exist a $k$ such that $f$ is determined by its $k$-jet at a point?
2. **(F. Meylan)** Assume $M \subseteq \mathbb{C}^n$ is a real-analytic generic submanifold, $f = \frac{p}{q} : \mathbb{C}^n \to \mathbb{C}^N$ is the germ of a meromorphic map at $z \in M$, and $f$ is holomorphic in a one-sided wedge $W$ attached to $M$ near $z$. Assume that $\|f(w)\|^2$ increases to 1 as $w \to M$ in $W$ . Does $f$ extend to a full neighborhood of $z$ in $\mathbb{C}^n$?
   - **Remark:** the codimension 1 case is known (Chiappari, '91).
3. **(X. Huang)** Does there exist a universal constant $t$ such that a proper holomorphic map between $\mathbb{B}^n$ and $\mathbb{B}^N$ (with $1 < n < N$) which is $\mathcal{C}^t$ up to the boundary is actually a rational map?
4. **(X. Huang)** Assume $\Gamma \subseteq \mathrm{Aut}(\mathbb{B}^N)$ is a discrete subgroup and $f : \mathbb{B}^n \to \mathbb{B}^N$ is a proper holomorphic embedding such that $\Gamma(f(\mathbb{B}^n)) \subseteq f(\mathbb{B}^n)$ and $f(\mathbb{B}^n)/\Gamma$ is compact. Is $f$ linear?
5. **(J. D'Angelo)** Let $f : \mathbb{B}^n \to \mathbb{B}^N$ be a proper rational map. Is $f$ homotopic to a polynomial map $p$ of the same degree? That is, $\forall t \in [0, 1]$, each $f_t : \mathbb{B}^n \to \mathbb{B}^N$ is proper with $f_0 = f$ and $f_1 = p$.

# Squared norms

1. **(D. Grundmeier and J. Lebl)** For $n \geq 2$, define the rank of a real polynomial $p(z, \bar{z})$ on $\mathbb{C}^n$ to be the smallest integer $N$ such that $p(z, \bar{z}) = \sum_{j=1}^{N} \pm|p_j(z)|^2$. Given $p : \mathbb{C}^n \to \mathbb{R}$ real analytic, suppose that for all complex hyperplanes $H$, $p|_H$ has rank $\leq k$. Does there exist $c(n, k)$ such that $\mathrm{rank}(p) \leq c(n, k)$?
   - **Remark:** The question is worthwhile even if $p$ is a polynomial and a sum of squares.
2. **(J. D'Angelo)** Find necessary and sufficient conditions on a real-valued polynomial $R(z, \bar{z})$ such that there exists $N$ with $R^N = \sum_{j=1}^{k} |p_j(z)|^2$.
3. **(P. Ebenfelt)** Let $S(z, \bar{z})$ be a harmonic hermitian polynomial of degree $(d, d)$ on $\mathbb{C}^n$. For any given $k$, how "many" polynomials $A(z, \bar{z})$ of type $(d - 1, d - 1)$ exist such that $S + A\|z\|^2 = \sum_{j=1}^{k} \|f_j\|^2$? Which properties of $S$ determine this?

- **Remark:** In the special case where $S = 0$, if $k < n$ then the only solution is 0 by a lemma of Huang. If $S$ is arbitrary but $k < \frac{n}{2}$ there is at most one solution.
- **Follow up:** If $S$ is fixed, can you determine a borderline $k$ for which finiteness of the number of solutions $A(z, \bar{z})$ breaks down? What is the behavior of this borderline $k$?

# Plurisubharmonic polynomials

1. **(B. Stensones)** Let $p(z, \bar{z}, w, \bar{w})$ be a plurisubharmonic, homogeneous, degree $2k$ polynomial on $\mathbb{C}^2$. Let $\Sigma$ be an algebraic curve not containing the origin. If $p|_\Sigma \equiv 0$, then is $\Sigma$ the level set of a homogeneous holomorphic polynomial?

2. **(B. Stensones)** Let $p(z, \bar{z}, w, \bar{w}) = p_{\alpha\beta\gamma\delta} z^\alpha \bar{z}^\beta w^\gamma \bar{w}^\delta$ be a plurisubharmonic real-valued polynomial on $\mathbb{C}^2$. Write $N(p) = \{(a, b) \in \mathbb{N}^2 : a = \alpha + \beta, b = \gamma + \delta, p_{\alpha\beta\gamma\delta} \neq 0\}$. Let $\Gamma_1, \Gamma_2$ be extreme edges of $N(p)$ such that $\Gamma_1 \cap \Gamma_2 \neq \emptyset$, where an extreme edge is a line in $N(p)$ with no points in $N(p)$ below. Can $q = \sum_{(\alpha+\beta,\gamma+\delta)\in\Gamma_1\cup\Gamma_2} p_{\alpha\beta\gamma\delta} z^\alpha \bar{z}^\beta w^\gamma \bar{w}^\delta$ be written as the sum of two plurisubharmonic weighted homogeneous polynomials?

   - **Remark:** the polynomial $q$ is plurisubharmonic but not necessarily weighted homogeneous.

# CR manifolds and mappings

1. **(S. Berhanu)** Let $M^{2n-1} \subseteq \mathbb{C}^n$ be a real hypersurface satisfying the maximum principle for CR functions in the strong sense (continuous CR functions attaining a (weak) local maximum must be constant). Is it necessary that any local continuous CR function is open?

   - **Remark:** In $\mathbb{C}^2$ the hypothesis cannot hold.

2. **(D. Zaitsev)** Let $n \geq 2$ and $M \subseteq \mathbb{C}^n$ be a compact CR manifold which is maximally complex (so the CR codimension is 1). Suppose in the class of maximally complex CR manifolds, $M$ is homotopic to a sphere in the complex $m$-plane, where $m - 1$ is the CR dimension. Must $M$ be the boundary of a smooth analytic submanifold?

3. **(S. Ji)** Let $M \subseteq \mathbb{C}^n$ be a compact CR manifold defined by polynomials, with $n \geq 2$ and CR codimension greater or equal to 1. The Harvey-Lawson variety may have isolated singularities. What is the relationship between the degree of the defining polynomials and the number of singularities?

4. **(D. Zaitsev)** Let $M$ be a locally homogeneous CR manifold. That is, suppose for any two points $p, q \in M$, there exists a CR diffeomorphism $h : (U, p) \to (V, q)$ for some neighborhoods $U$ of $p$ and $V$ of $q$. Does it follow that the Lie algebra of infinitesimal automorphisms spans $T_pM$ for one (and hence all) $p \in M$.

   - **Remark:** In the analytic case, the answer is yes.

5. **(D. Zaitsev)** Given $M$ a CR manifold and $p \in M$, let $O(p)$ be the set of points $q \in M$ for which there exists a CR automorphism $h : (U, p) \to (V, q)$ for some neighborhoods $U$ of $p$ and $V$ of $q$. Is $O(p)$ always a manifold? In that case, is the span of the Lie algebra of infinitesimal automorphisms of $M$ equal to $T_q O(p)$ if $q \in O(p)$?

   - **Remark:** If $M$ is minimal, holomorphically nondegenerate, real analytic, the answer (to both parts of the question) is yes.

6. **(D. Zaitsev)** Let $(M, p), (M', p')$ be germs of CR submanifolds in $\mathbb{C}^n$ which are formally equivalent. Are they biholomorphically equivalent?

   - **Remark:** The non-CR case doesn't necessarily have this property. The answer in the CR case is yes in many situations but the general question is open.

7. **(D. Zaitsev)** Let $M$ be a real submanifold of $\mathbb{C}^n$ which is generically minimal and holomorphically nondegenerate. What can you say about $\mathrm{Aut}(M, p)$ at nonminimal points?

# CR embeddings

1. **(M. Kolar)** Assume $M \subseteq \mathbb{C}^2$ is a real analytic hypersurface that admits a nonlinearizable automorphism $H$. Does there exist a hyperquadric $Q \subseteq \mathbb{C}^n$ such that $M \hookrightarrow Q$ nontrivially and $H$ is induced by an automorphism of $Q$? Can a bound on $n$ be found?

2. **(D. Zaitsev)** Assume $M \subseteq \mathbb{C}^2$ admits a nontrivial map tangent to the identity of order 2. Does $M$ have an embedding into a hyperquadric $Q$ as in the preceding problem?

3. **(S. Dragomir)** Which Pontrjagin forms of the Fefferman metric of a strictly pseudoconvex (abstract) hypersurface $M$ are obstructions to (global) embeddability of $M$?

# Approximation

1. **(J.E. Fornaess)** Let $\omega$ be a holomorphic 1-form on a polydisc in $\mathbb{C}^3$ with $\omega \wedge d\omega = 0$. Can $\omega$ be approximated on compact subsets by polynomial 1-forms $h$ satisfying $h \wedge dh = 0$?

   - **Remark:** A positive answer would lead to a nontrivial foliation of $\mathbb{CP}^3$ by hypersurfaces.