Problems from the AIM workshop

"Fourier analytic methods in convex geometry"

Most of the questions are formulated for people working in the area of convex geometry. An interested reader can find the necessary definitions in the following books:

R. Gardner, *Geometric Tomography*, Cambridge University Press, 1995.

A. Koldobsky, *Fourier Analysis in Convex Geometry,* Math. Surveys and Monographs, AMS (2005).

V. Milman, G. Schchtman, *Asymptotic theory of finite-dimensional normed spaces.* Lecture Notes in Mathematics, 1200. Springer-Verlag, 1986.

R. Schneider, *Convex bodies: The Brunn-Minkowski Theory,* Cambridge University Press, 1993.

The transparencies of several of the presentations will be posted at the Workshop website, `http://www.aimath.org/pastworkshops/fourierconvex.html`

Problems:

1. (R. Schneider) Let $Z_n \subset \mathbb{R}^n$ be a zonoid whose polar Z_n^o is also a zonoid. Let d denote the Banach-Mazur distance. Does $d(Z_n, B_2^n)$ converge to 1 as the dimension n tends to infinity?
2. (R. Schneider) For an origin symmetric convex body $K \in \mathbb{R}^n$, let $\lambda(K)$ be the smallest number $\lambda \geq 1$ such that there exists a zonoid Z with $K \subset Z \subset \lambda K$. For example, for the cross-polytope C^n we have

$$\lambda(C^n) = \frac{n}{2^{n-1}} \binom{(n-1)}{[(n-1)/2]} \approx \sqrt{\frac{2n}{\pi}}$$

as $n \to \infty$. Is $\lambda(K)$ maximal for the cross-polytope?

3. Let $\mathcal{K}^n$ denote the class of convex bodies in $\mathbb{R}^n$. A Minkowski class $\mathcal{M}$ is a subset of $\mathcal{K}^n$ that is closed in the Hausdorff metric, under Minkowski linear combinations and translations. A body K is a generalized $\mathcal{M}$-body if there are two bodies $M_1, M_2 \in \mathcal{M}$ such that $K + M_1 = M_2$.

   Let G be a subgroup of $GL(n)$. We say that $\mathcal{M}$ is G-invariant if whenever $K \in M$ and $g \in G$, we have $gK \in \mathcal{M}$. Given $B \in \mathcal{K}^n$, $G \subset GL(n)$, we define $M_{B,G}$ as the smallest Minkowski class containing B that is G-invariant.

   Theorem (Schneider, F. Schuster) Let $B \in \mathcal{K}^n$ be non-symmetric. Every neighborhood of B contains an affine image B' of B such that the generalized $M_{B',SO(n)}$ bodies are dense in $\mathcal{K}^n$.

   Problem: (R. Gardner) Dualize the theorem. If we replace generalized zonoids by "generalized intersection bodies", Minkowski sums by radial sums, etc, does a similar theorem hold?

4. In $\mathbb{R}^n$, how many segments do we need to approximate a zonoid by zonotopes? Let K be a zonoid and Z a zonotope that is the sum of M segments. If $d(Z,K) \leq 1+\epsilon$, then M is of the order $C(\epsilon)n\log n$ (Talagrand). If K is the Euclidean ball, this can be improved to $C(\epsilon)n$. Is the extra $\log n$ in Talagrand's result necessary?

5. (G. Schechtman) A theorem of Spencer states that if $\{X_i\}_{i=1}^n$ are in B_∞^n and $\epsilon_i = \pm 1$ then
$$\min_{\epsilon_i} \| \sum_{i=1}^n \epsilon_i X_i \|_\infty \leq C\sqrt{n}.$$
   Does the same hold for any n-dimensional zonoid? *i.e.*, does there exist a universal constant C such that for all n-dimensional centered zonoid Z, if $\{X_i\}_{i=1}^n \in Z$ then there are signs $\{\epsilon_i\}_{i=1}^n$ with $\sum_{i=1}^n \epsilon_i X_i \in C\sqrt{n}Z$?

   (If one can remove the log factor in problem 4 then the answer here is positive. As is, the best known substitute for $C\sqrt{n}$ is $C\sqrt{n\log\log n}$.)

6. (M. Rudelson) Let K be a symmetric convex body. There exists a linear operator T such that

$$\int_{S^{n-1}} h_{TK}(\theta)\, d\sigma(\theta) \cdot \int_{S^{n-1}} h_{(TK)^o}(\theta)\, d\sigma(\theta) \leq C \log n.$$

What is the upper bound for the above quantity for non-symmetric convex bodies K? (in this case we need to take T affine).

7. (S. Robins) Given N, can we find explicitly N points on S^{n-1} that are "uniformly" arranged? Are they the same as the vertices of the polytope with N vertices and maximum volume that is contained in S^{n-1}?

8. Let f be a function on S^{n-1}. Can we decide from the Fourier transform of f if f is the support function of a convex body? Or if f is the radial function of a convex body?

9. (R. Schneider) Classify all continuous $T : \mathcal{K}^n \to \mathcal{K}^n$ with the following properties:
   - $h_{T(K+L)} = h_{TK} + h_{TL}$
   - $T(\theta K) = \theta TK$ for every rotation θ.

10. (R. Schneider) Show that for most origin-symmetric convex K, the intersection body of K is not the polar of a zonoid. ("Most" means in the Baire category sense).

11. (S. Robins)
    (a) Let K be a polytope and consider the expansion of h_K in spherical harmonics. If all of the coefficients have small absolute value, are all the facets of K simplices or close to simplices?
    (b) If, for each integer frequency, the Fourier transform of the indicator function 1_K is small, does the same conclusion hold? (This conjecture is made by analogy with pseudorandom number sequences which exhibit this behavior in dimension 1, and the intuition here is that thegiven hypothesis on the coefficients should exhibit a "random" polytope in some sense.)

12. (Y. Gordon) Let K be a symmetric convex body. We define the zonoid ratio as

$$zr(K) = \min_{Z \in \mathcal{Z}, Z \subset K} \frac{|K|^{1/n}}{|Z|^{1/n}}.$$

The volume ratio is defined as

$$vr(K) = \min_{E\, ellipsoid,\, E \subset K} \frac{|K|^{1/n}}{|E|^{1/n}}.$$

Clearly, $vr(K) \geq zr(K)$. Y. Gordon and M. Junge proved that there is a constant $C_0 > 0$ such that every centrally symmetric convex body K in R^n has a section L of dimension $[n/2]$ such that $zr(L) \geq C_0\, vr(K)$.

Since by K. Ball the local unconditional constant $\chi_u(L)$ is $\geq zr(L)$ for every convex L, and these parameters of the local theory are always hard to compute, it follows that $C_0\, vr(K)$ yields an easy computational lower bound for some $[n/2]$ dimensional section L of K.

Question: Suppose $0 < \lambda < 1$ is given, what is the the greatest value $f(\lambda) > 0$ such that for every $n \geq 3$, every n-dimensional convex body K will have a section L of dimension $[\lambda n]$ for which $zr(L) \geq f(\lambda) vr(K)$?

13. (S. Robins) **Minkowski's Theorem in $\mathbb{Z}^d$.** Given a finite number of vectors $\{\vec{n_i}\} \subset \mathbb{Z}^d$ and integers $\alpha_i \in \mathbb{Z}$, is there a polytope such that $\{\vec{n_i}\}$ are normal to the facets and the number of integer points on the ith-facet is α_i? Do we need some extra constraints?

14. Which projection or/and section data are needed to determine non-symmetric convex bodies?

    Several known cases: A non-symmetric convex body is uniquely determined if we know:

    - The volume of sections passing through a given interior point and the centers of gravity of those sections.
    - The mean width and the Steiner points.
    - The brightness function and the illumination function.

Questions:

(a) (Schneider) Do the areas of projections plus the centers of gravity determine a body?

(b) What is the dual version of the third known case (brightness + illumination)?

15. (A. Zvavitch) Consider the Gaussian measure of sections of the cube B_∞^n. It is known that if $n \geq 3$,

$$\gamma_{n-1}\left(B_\infty^n \cap \theta^\perp\right) \leq \gamma_{n-1}\left(\sqrt{\frac{n}{n-1}}B_\infty^{n-1}\right).$$

Is this the best upper bound? If we introduce a dilation factor $r > 0$, for which $\theta = \theta(r)$ is $\gamma_{n-1}\left(rB_\infty^n \cap \theta^\perp\right)$ maximal?

16. **Minimum of slabs of the cube** B_∞^n Given $t \leq 2\sqrt{2} - 2$, the minimal slab is in the direction of $(1, 0, \ldots, 0)$. Conjecture: there are numbers t_1 and t_2 such that for $2\sqrt{2} - 2 < t < t_1$, the minimum is in the direction of $(1, 1, 0, \ldots, 0)$; for $t_1 < t < t_2$, the minimum is in the direccion of $(1, 1, 1, 0, \ldots, 0)$ and for $t_2 < t$, the minimum is in the direction of $(1, 1, \ldots, 1)$.

17. Given $n + k$ points $x_1, \ldots, x_{n+k}$ on S^{n-1}, with $k \leq n$, we want to cover the sphere with caps of radius $r = r(n, k)$ centered at those points. What is the best possible position for the points that will minimize r? How does this minimal r behave as function of n and k?

18. Given $n + 1$ points $x_1, \ldots, x_{n+1}$ on S^{n-1}, find the configuration such that the convex hull of $x_1, \ldots, x_{n+1}$ has the largest mean width. Is it maximal for the regular simplex?

19. Let $K \subset \mathbb{R}^n$ be a non symmetric set with centroid 0.

$$vol(K \cap (-K)) \geq 2^{-n} vol(K).$$

Is the simplex the extremal case? If not, what is it?

20. Find the smallest radius $R = R(n)$ such that if a convex body K contains a ball of radius R, then the number of integer points in K is equivalent to the volume of K, up to a multiplicative factor (a polynomial of n).

21. (W. Weil) Let ΠK be the projection body of K. It is known that if K is a polytope and $\Pi\Pi K$ is homothetic to K, then K is a direct sum of planar polygons (in even dimensions) or a direct sum of planar polygons and a segment (in odd dimensions). If K is an arbitrary convex body and $\Pi\Pi K$ is homothetic to K, what can we conclude about K?

22. (A. Daurat and R. Gardner) We say that a measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ is a ridge function if it is constant in one direction, *i.e.*, $f(x) = g(\langle x, v\rangle)$ for some $g : \mathbb{R} \to \mathbb{R}$, and $v \in S^1$. Given n ridge functions $f_1, \dots, f_n$ with directions $v_1, \dots, v_n$ respectively, the set $E = \{x \in \mathbb{R}^2 : (f_1 + \cdots, f_n)(x) \geq 0\}$ is uniquely determined among all measurable sets by the X-rays in the directions of $v_1, \dots, v_n$.

    It is known that there are sets of four directions such that every planar convex body can be determined by X-rays in those directions (for example, the directions of $v_1 = (1, 0)$, $v_2 = (0, 1)$, $v_3 = (2, 1)$ $v_4 = (-1, 2)$ work). However, no set of three directions is enough. Is it possible to represent any convex body $K \subset \mathbb{R}^2$ in the form $K = \{x \in \mathbb{R}^2 : (f_1 + f_2 + f_3 + f_4)(x) \geq 0\}$ where $f_1, \dots, f_4$ are ridge functions in the above four directions?

23. For $t > 0$, $a \in S^{n-1}$, consider the slab

$$Sl(a, t) = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1, |\langle x, a\rangle| \leq t\}$$

    and the section

$$S(a, t) = \{x \in \mathbb{R}^n : \|x\|_\infty \leq 1, \langle x, a\rangle = t\}.$$

    Denote $V(a, t) = vol_n(Sl(a, t))$ and $A(a, t) = vol_{n-1}(S(a, t))$.

    Let $f_m = (1, \dots, 1, 0, \dots, 0)/\sqrt{m} \in S^{n-1}$, where $1 \leq m \leq n$ and there are exactly m ones in f_m. It is known (Hensley, Ball;Oleskiewicz-Pelcynski) that $1 \leq A(a, 0) \leq \sqrt{2}$, with equality in the left attained for $a = f_1$, and on the right for $a = f_2$. (In the complex case, the result is $1 \leq A(a, 0) \leq 2$).

    (a) (V. Milman) Is the minimum of $V(a, t)$ for fixed $t > 0$ and $a \in S^{n-1}$ (variable) always attained for a standard vector $a = f_i$?

(b) If $t \le 2\sqrt{2}-2$, is

$$\min_{a\in S^{n-1}} V(a,t) = V(f_1,t)?$$

This is true if $t \le 3/4$ (F. Barthe, A. Koldobsky). In the complex case, it is known to be true for $t \le 4/5$ (H. Koenig). Is it true for $t \le t_0 \approx 0.867$? Here t_0 is the solution of

$$\pi t^2 = t(t^2-1)\sqrt{2-t^2} + (2t^2-1)\arcsin(t\sqrt{2-t^2}).$$

(c) (H. Koenig) Determine the best constants in the Khintchine-type inequality

$$c_p^{-1}|a| \le \|\sum_{j=1}^n a_j X_j\|_{L_p}$$

where X_j are i. unif. dist. on S^{m-1}, $-m+1 < p < 0$.

(d) (H. Koenig) Find an integral formula for the slab-volume in B_1^n and for the non-central sections of B_1^n. Find the extremal directions for the slabs in B_1^n.