THE FURSTENBURG CONJECTURE AND RIGIDITY

The American Institute of Mathematics

Introduction

Dynamics studies the iterates of a single transformation, or more generally the joint action of several transformations. One of the most striking differences between these is a range of phenomena loosely grouped under the rubric of rigidity.

A single transformation typically has no rigidity, in the sense that it posessed a multitude of invariant measures, or a continuum of compact invariant sets, and so on. However, it often happens that under mild hypotheses the joint action of several transformations will have very few such invariant objects, often only one, but at least they have a simple classification.

For example, a single automorphism of a finite-dimensional torus has infinitely many non-atomic invariant measures, and infinitely many uncountable compact invariant sets. At the other extreme, for action of the group $GL(n,\mathbb{Z})$ on the n-torus, the only compact invariant sets are either finite or all of the torus, and the only invariant measures are Lebesgue measure and linear combinations of measures supported on finite sets.

One of the most active areas in dynamics today is finding the boundaries of rigidity behavior, and Furstenberg's Conjecture lies in the intermediate area where rigidity might possibly continue to hold, but no one is sure. Roughly speaking, it says that if two commuting endomorphisms of a torus are incommensurable (no power of one is a power of the other), then the their joint action should be rigid, in the sense that the compact invariant sets and invariant measures should be the obvious algebraic ones. Furstenberg himself [35#4369] showed that on the one-dimensional circle the only compact sets invariant under both multiplication by p and by q, where $\log p$ and $\log q$ are irrationally related, are the whole circle and finite sets (whose elements are necessarily roots of unity). The problem whether Lebesgue measure is the only atomless probability measure invariant under these transformations has come to be known as Furstenberg's Conjecture.

This conjecture is a special case of more general rigidity conjectures in the dynamics of commuting group endomorphisms.

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Chapter A: Statement of the Furstenberg Conjecture

Furstenberg's Conjecture, in its most concrete form, is the following. Let $\mathbb{T}$ denote the additive circle group, and for each integer $m > 1$ let $\phi_m : \mathbb{T} \to \mathbb{T}$ be defined by $\phi_m(t) = mt \pmod 1$. Then the only atomless probability measure on $\mathbb{T}$ that is simultaneously invariant under both ϕ_2 and ϕ_3 is Haar measure.

To say that a measure μ is invariant under a map ϕ means that $\mu(\phi^{-1}(E)) = \mu(E)$ for all measureable sets E.

A more general version of this conjecture raises the same question for the maps ϕ_p and ϕ_q, where p and q are positive integers satisfying the necessary condition that no power of p equals a power of q, i.e. that $\log p$ and $\log q$ are rationally independent.

Chapter B: History and past results

Furstenburg himself never published an explicit statement of "Furstenburg's Conjecture", although he discussed it in lectures as a prototype of a more profound intuition about normal numbers. Recall that a real number is normal base b if its b-adic expansion has the property that every finite block of digits has the expected frequency. Roughly speaking, this intuition says that it is very hard for a number to be abnormal with respect to two incommensurable bases.

The first explicit statement of Furstenburg's Conjecture occurs in the paper of Russell Lyons [89e:28031], who shows via elementary means that if μ is a measure on $\mathbb{T}$ that is jointly invariant under ϕ_p and ϕ_q, and if ϕ_p is assumed to also have the strong property that it is a Kolmogorov automorphism of $(\mathbb{T}, \mu)$ (i.e., that it has completely positive entropy), then μ must be Lebesgue measure.

The breakthrough came in 1990 with the paper of Dan Rudolph [91g:28026]. He showed using more sophisticated ergodic theory that if we merely assume that ϕ_p has positive entropy on $(\mathbb{T}, \mu)$, then μ must be Lebesgue measure. Since then, several alternative proofs of Rudolph's theorem have been given, for example by Host [96g:11092] and Parry [97h:28009]. As the review of the latter states, " It is striking that all the different approaches to the problem of the existence of non-Lebesgue, non-atomic, Borel measures invariant under ϕ_p and ϕ_q come up against the same entropy criterion."

A 200-page account of the mathematical ideas surrounding Furstenburg's Conjecture and related topics by Klaus Schmidt is currently in draft form, entitled "$\times 2, \times 3$, and $\times\beta$."

Chapter C: Analogous problems

One sign of a good problem is that it suggests lots of other analogous problems, and this is certainly the case with Furstenburg's Conjecture. The key feature of the conjecture is the joint action of two (or more) commuting algebraic mappings. We give two examples.

Katok and Spatzier ([97d:58116] and errata in [99c:58093]), while trying to understand Rudolph's ideas, developed them in the context to two commuting automorphisms A and

B of a finite-dimensional torus, which are of course assumed be independent enough to rule out obvious exceptions. They showed, for example, that any measure that is simultaneously invariant under A and B, and that has positive entropy with respect to, say, A, must be a multiple of Lebesgue measure on the unstable foliation of A on the torus. This does not show directly that the measure is Lebesgue measure in all cases, but does in some cases with further assumptions.

There are also analogous questions for totally disconnected groups, although here the statements are more intricate because of the existence of many more "obvious" invariant sets. A typical example is due originally to Ledrappier [80b:28030]. Let X be the subgroup of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{Z}^2}$ defined by the condition that $x_{i,j} + x_{i+1,j} + x_{i,j+1} = 0$ for all $i, j \in \mathbb{Z}$. Let A shift an array one place to the left, and B shift it one place down, so that A and B are commuting automorphisms of X. What are the invariant measures for the joint actions of A and B? Here there are many more compact invariant sets than just finite sets, and each supports at least one invariant measure. Nevertheless, there may well be a simple algebraic classification of the invariant measures, even in this case.

Chapter D: Approaches to a proof

Essentially all approaches to Furstenburg's Conjecture and related problems have so far been the same, namely the use of isometric directions for the action, where the action acts as a translation on a certain foliation, and invoking the observation that the only translation-invariant measures are Lebesgue on this foliation. The positive entropy assumption is then used to show that this forces the measure itself to be Lebesgue.

For example, if m and n are integers such that n/m is close to $-(\log 2)/(\log 3)$, then $\phi_2^m \circ \phi_3^n$ is nearly an isometry on the circle, or more accurately on the 6-adic solenoid since we must invert one of the two maps. This solenoid has a copy of the reals wrapping densely through it, and the conditional measures on nearby pieces of leaves induced by an invariant measure must have the property that under iterates in the isometric direction they are nearly translation-invariant. This is the key idea in Rudolph's proof.

The existence of isometric directions for certain $\mathbb{Z}^d$-actions is a special case of a very general phenomenon called "subdynamics", or the study of such actions along subgroups of $\mathbb{Z}^d$, or more generally along subspaces of $\mathbb{R}^d$, introduced by Boyle and Lind [97d:58115]. Every topological $\mathbb{Z}^d$-action on an infinite compact space has a non-empty set of lower-dimension subspaces, closed in the Grassmann topology, along which the action is nonexpansive. Dynamical properties within a connected component of the complement of this set vary nicely or are constant, while passing from one component to another typically results in abrupt changes, roughly analogous to a phase transition. It is these nonexpansive directions that have been the key to all attempts so far to prove Furstenburg's Conjecture.

Chapter E: Approaches to a counterexample

Some have expressed serious doubts whether Furstenburg's Conjecture is even true. Mainly this is based on the observation, mentioned above, that every proof so far runs up against the same positive entropy barrier. Either there is a zero entropy counterexample, or a genuinely new idea is needed for a proof.

One approach to constructing an atomless measure, invariant under ϕ_2 and ϕ_3, which is not Lebesgue measure is as follows. We start by observing that there is a Markov partition that simultaneously works for both maps, namely the partition of $\mathbb{T}$ into six equal intervals $[j/6,(j+1)/6)$ for $0 \le j \le 5$. Start the construction by assigning weights to each of these, giving six numbers a_0 through a_5. Invariance under ϕ_2 gives linear relations between the a_j, and invariance under ϕ_3 gives further linear relations. Additionally, the a_j must add up to 1. Together these cut the dimension of possible solutions from six to two.

Each interval is subdivided into six equal subintervals, so let each a_j be divided into weights a_{j0} through a_{j5}. Again, invariance under ϕ_2 gives linear relations between the a_{jk}, and invariance under ϕ_3 gives further linear relations. In addition, the sum of the a_{jk} must equal a_j. All these together give a set of equalities and inequalities, which can be solved by linear programming software. The result is that the set of solutions is a convex object in a 10-dimensional subspace 36-dimensional space with 876 vertices. Each represents a potential start for a counterexample.

The idea is to try to continue this process a few more levels, to see what it takes for an assignment of weights to be continued to the next level consistently. Some form of the zero entropy hypothesis on the measure (which is certainly necessary by Rudolph's theorem) should guide the iterative construction from one level to the next. In the end, one would end up with an assignment of weights to all the 6-adic intervals, consistent with defining a jointly invariant measure, and for which at some stage not all intervals are given equal weight. To make this measure atomless, one needs to require further that the maximum measure of the intervals at stage n must tend to 0 as $n \to \infty$.

References

References not implemented yet.