– DISTANCES (METRICS)

$$d(x,y) = \text{“distance from } x \text{ to } y\text{”}$$

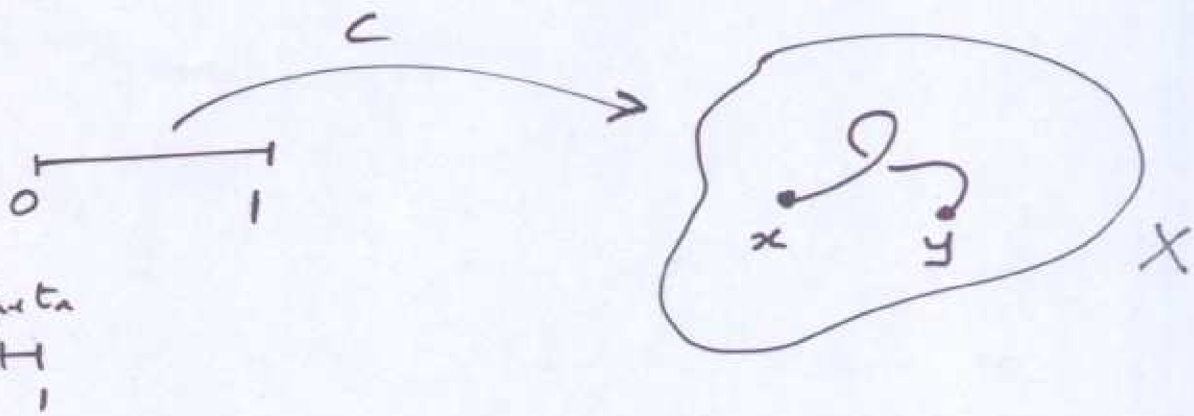
- $d(x,y) = d(y,x)$
- $d(x,y) \geq 0$, $\left(d(x,y) = 0 \iff x = y \right)$

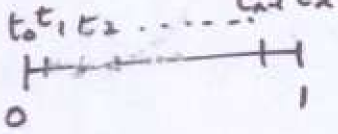
($\Delta \leq$) • $d(x,y) + d(y,z) \geq d(x,z)$

– In GOOD CASES

$d(x,y)$ = length of shortest path from x to y

LENGTH :

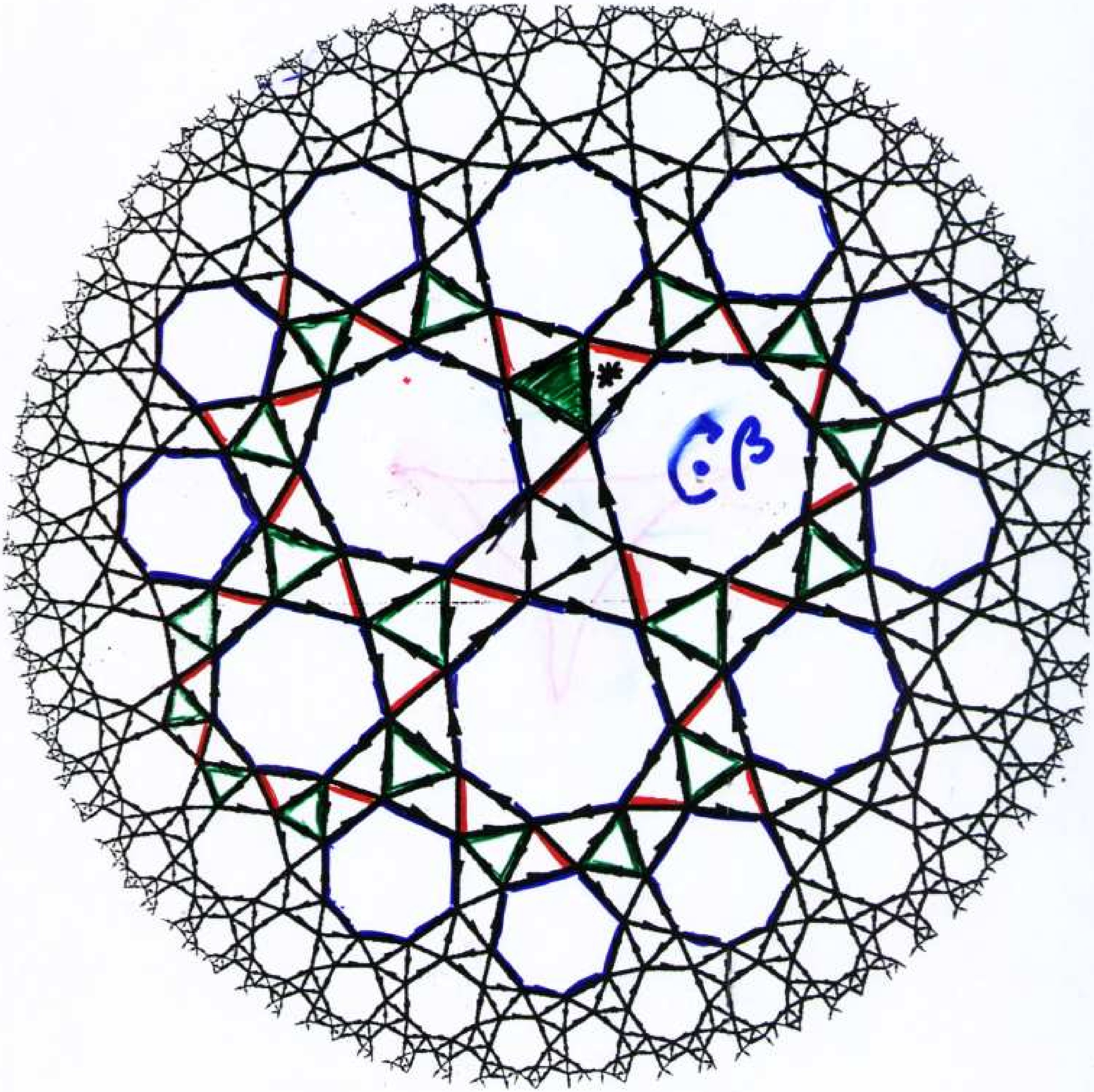




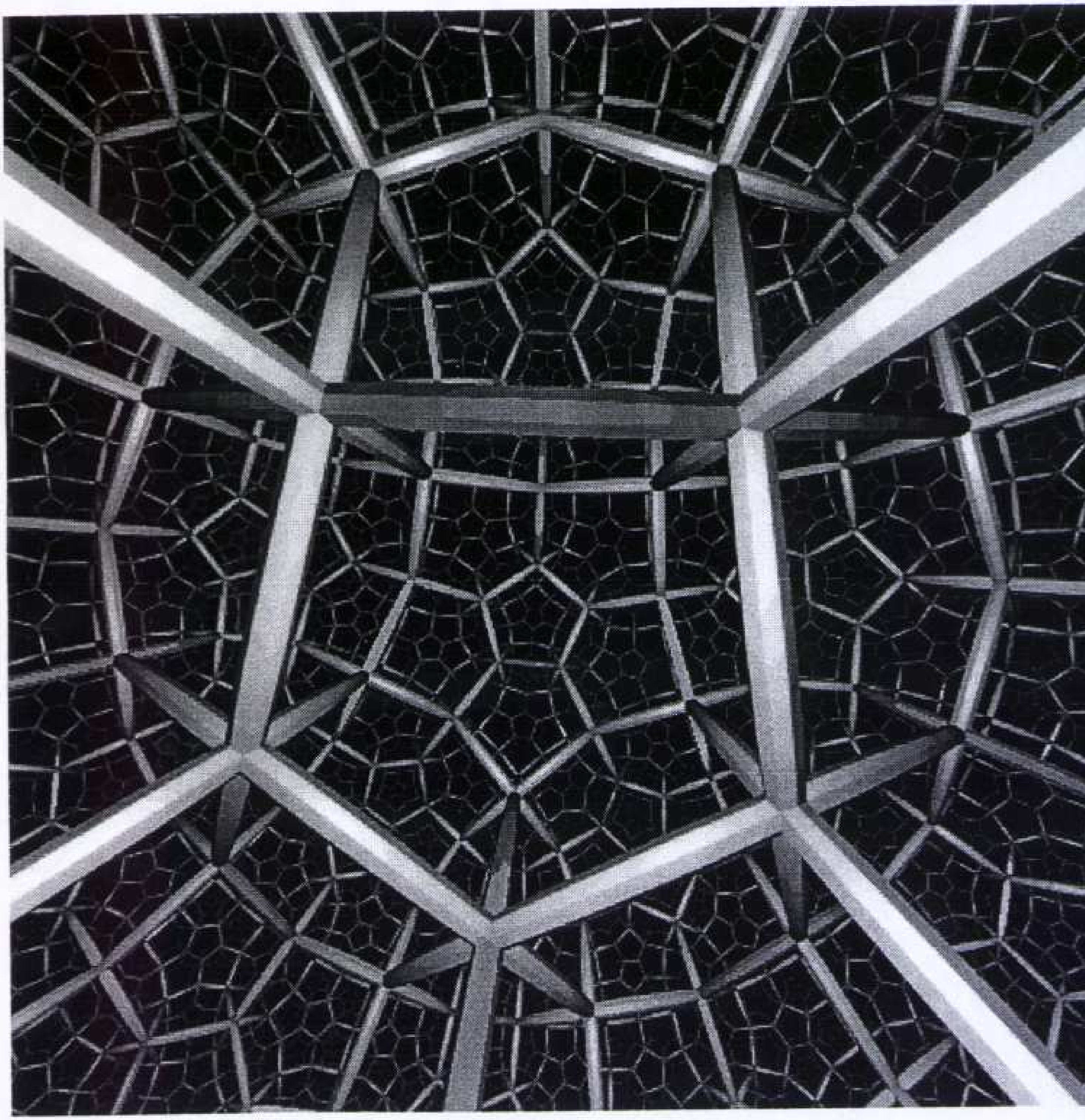
$$\text{length}(c) = \sup_{0=t_0\leq \dots \leq t_n = 1} \sum_{i=0}^{n-1} d(c(t_i), c(t_{i+1}))$$

dR 2

AHM



$$\left\langle \alpha, \beta, \gamma \;\middle|\; \beta^7 = 1,\ \alpha^2 = 1,\ \gamma^3 = 1,\ \overset{*}{\beta\alpha\gamma} = 1 \right\rangle$$



GEODESICS

A **GEODESIC** is a path from x to y such that

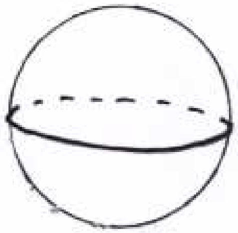
LENGTH of PATH $=$ $d(x,y)$.

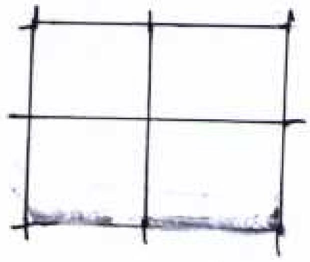
- **ISSUES**:
  - EXISTENCE ?
  - UNIQUENESS ?
  - (LOCAL) RECOGNITION ?

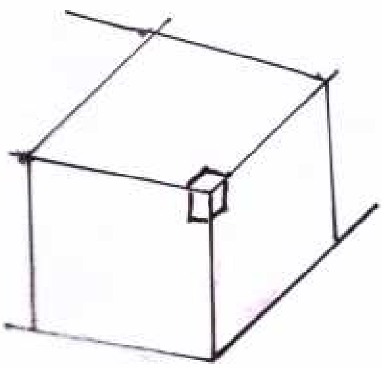
- FOR US, EXISTENCE IS NOT A REAL PROBLEM. (if we're dealing with polyhedral spaces, for example).

- COMPARISON to EUCLIDEAN GEOMETRY.

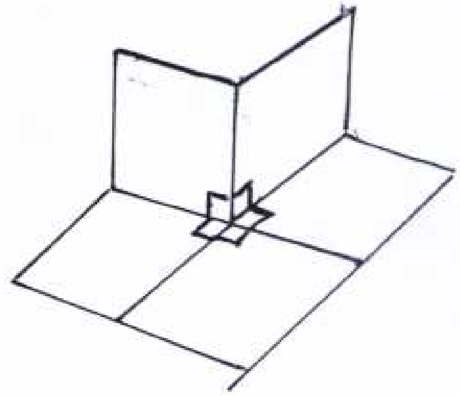
>0 $\qquad$ 0 $\qquad$ <0





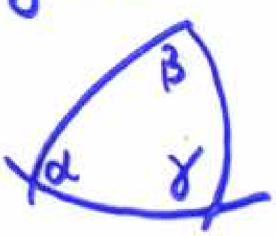


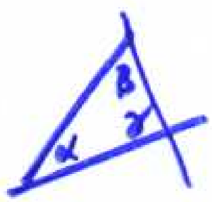
"corner of a box"

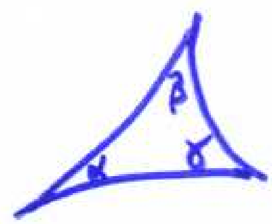


"corner of a hallway"

Triangles:

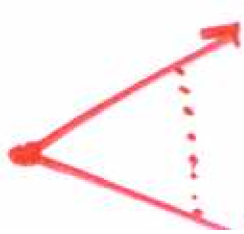


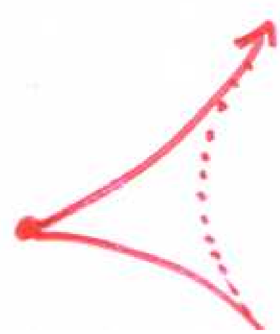




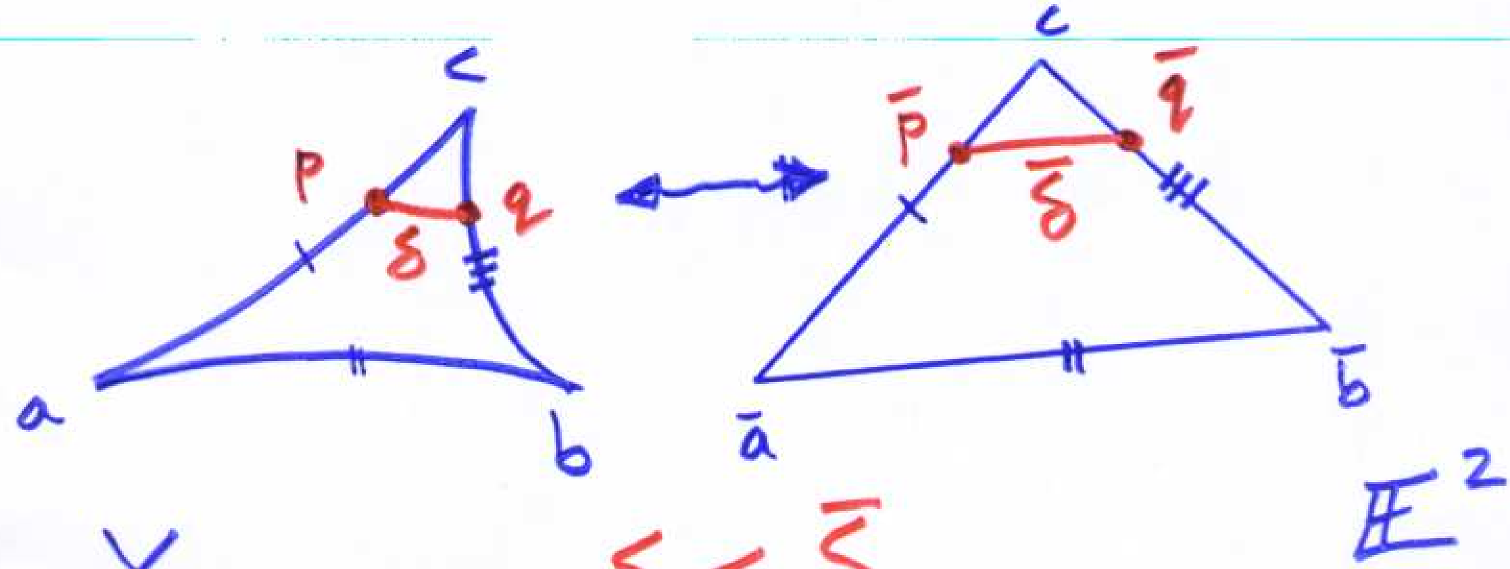
$\alpha+\beta+\gamma>\pi$ $\qquad$ $=\pi$ $\qquad$ $<\pi$

GEODESICS





of the space compares favourably to Euclidean geometry.



$\delta \leq \bar{\delta}$

<u>Rmk</u>: Convexity of $t \mapsto d(c_1(t), c_2(t))$

<u>KEY</u> <u>POINTS</u>:

(1) <u>LOCAL-to-GLOBAL</u>: If small Δs satisfy this condition (in a 1-connected space), so do large ones.

(2) <u>LOCAL</u> GEODESICS ARE <u>GLOBAL</u> GEODESICS

"No sharp corners"

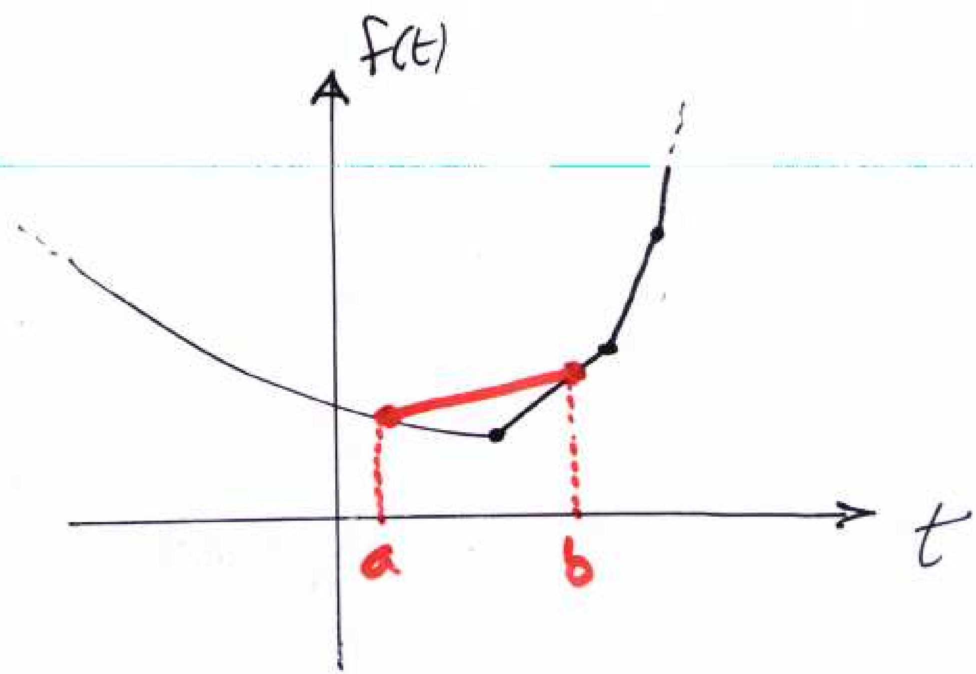
"distance minimize

- $f: \mathbb{R} \to \mathbb{R}$

LOCALLY CONVEX $\Rightarrow$ GLOBALLY CONVEX

- CONVEX + PROPER $\Rightarrow$ MINIMA (GLOBAL!) EXIST

- CONVEXITY $\Rightarrow$ MINIMA ARE UNIQUE



CONVEXITY IS YOUR FRIEND!

1.) Circumcentre :

$$\text{centre}_1(X) = \text{centre of smallest ball containing } X$$

i.e. Choose x_0 to minimize $\max_{x \in X} d(x_0, x)$

2.) Centre of Mass : With weighted points,

choose x_0 to minimize $\sum_{x \in X} d(x_0, x)^2 \mu(x)$

$\mu(x)$ = weight of x.

3) Centroid : Iteratively, replace X by set of midpoints of geodesics — this synthesizes

$$x_0 = \frac{1}{|X|}\{x_1 + \cdots + x_r\} \text{ in } \mathbb{E}^n$$

Rmk : STABILITY etc. ; CONVEXITY.

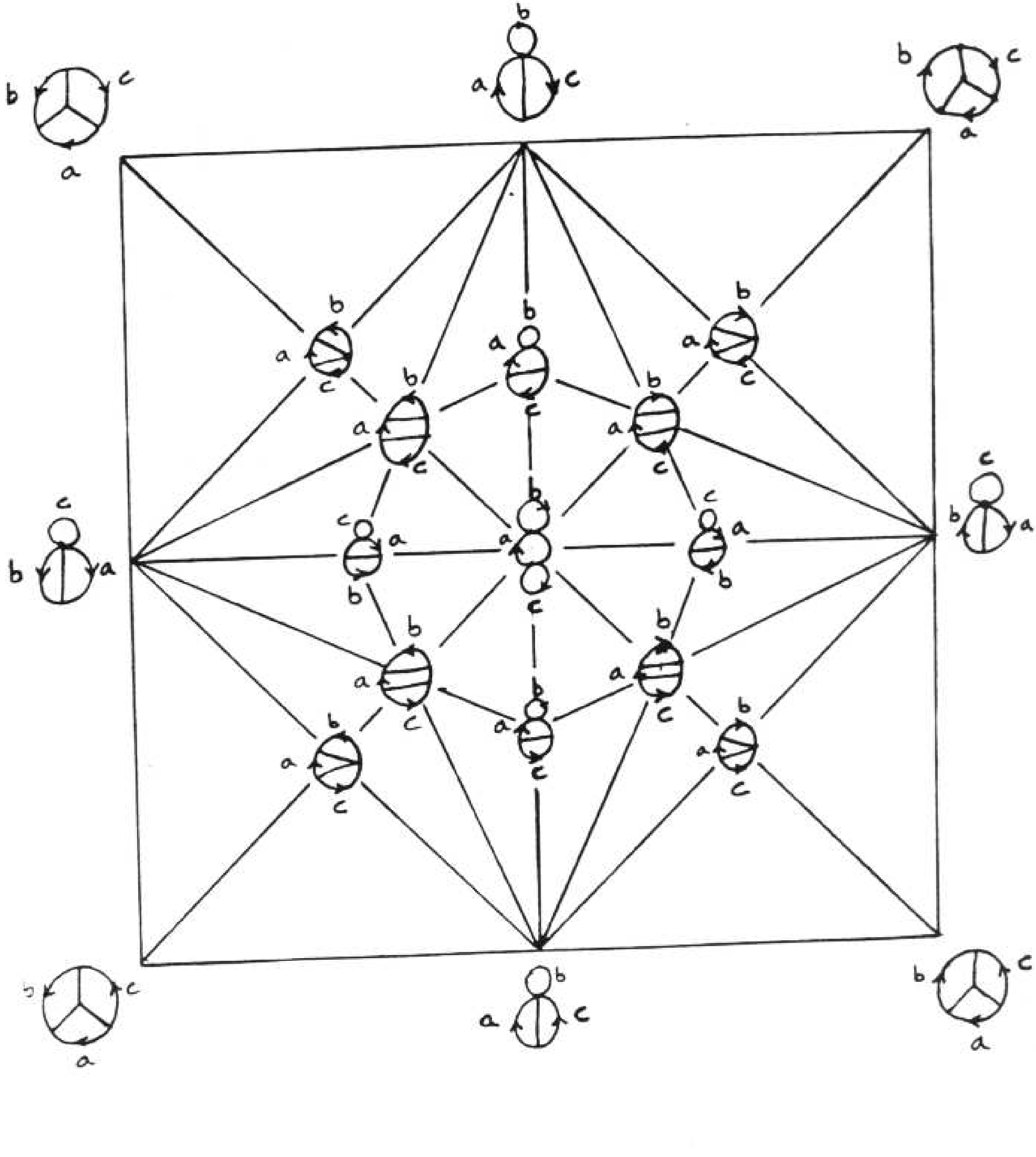


Figure 5.4: The face A(+)

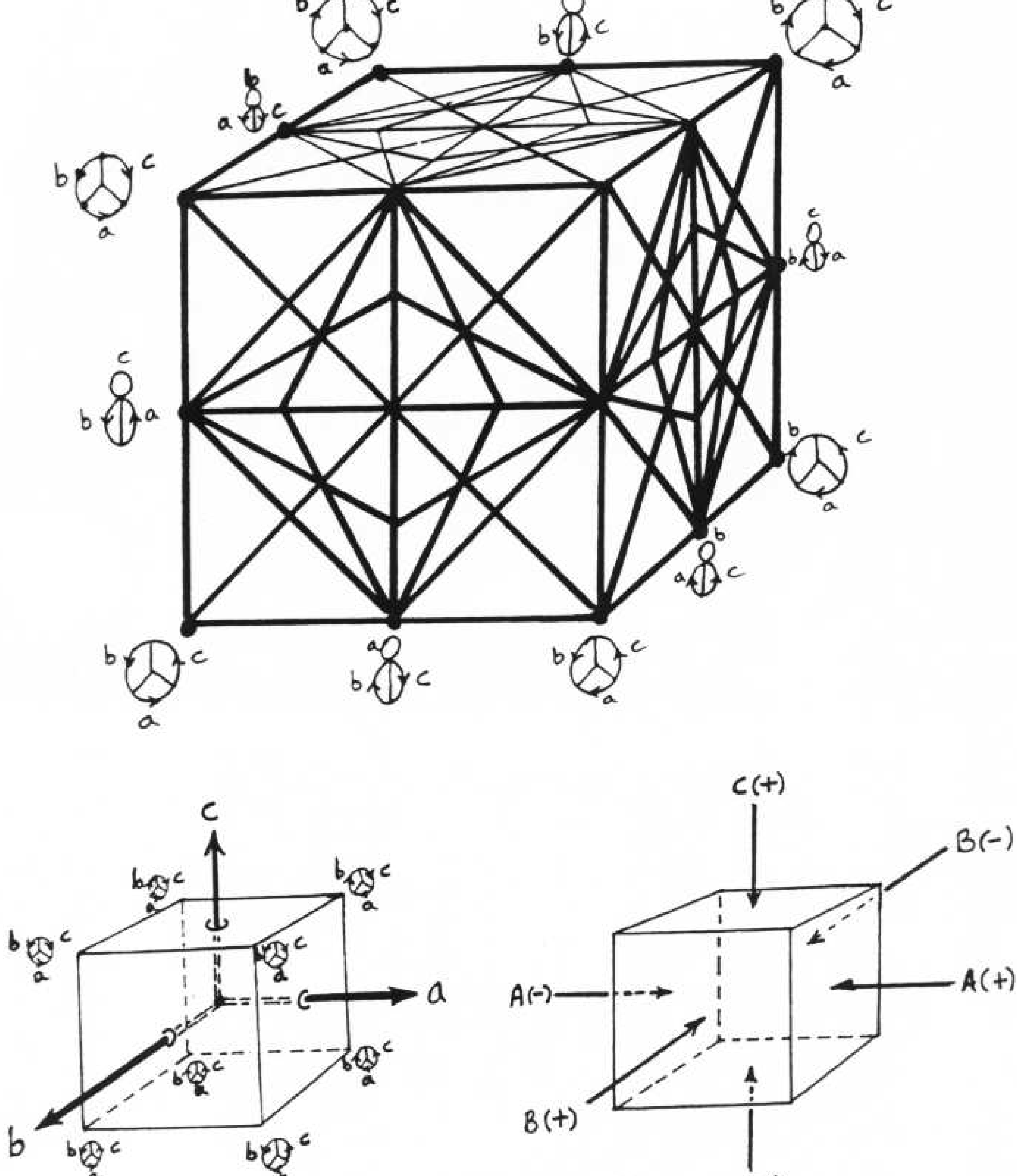
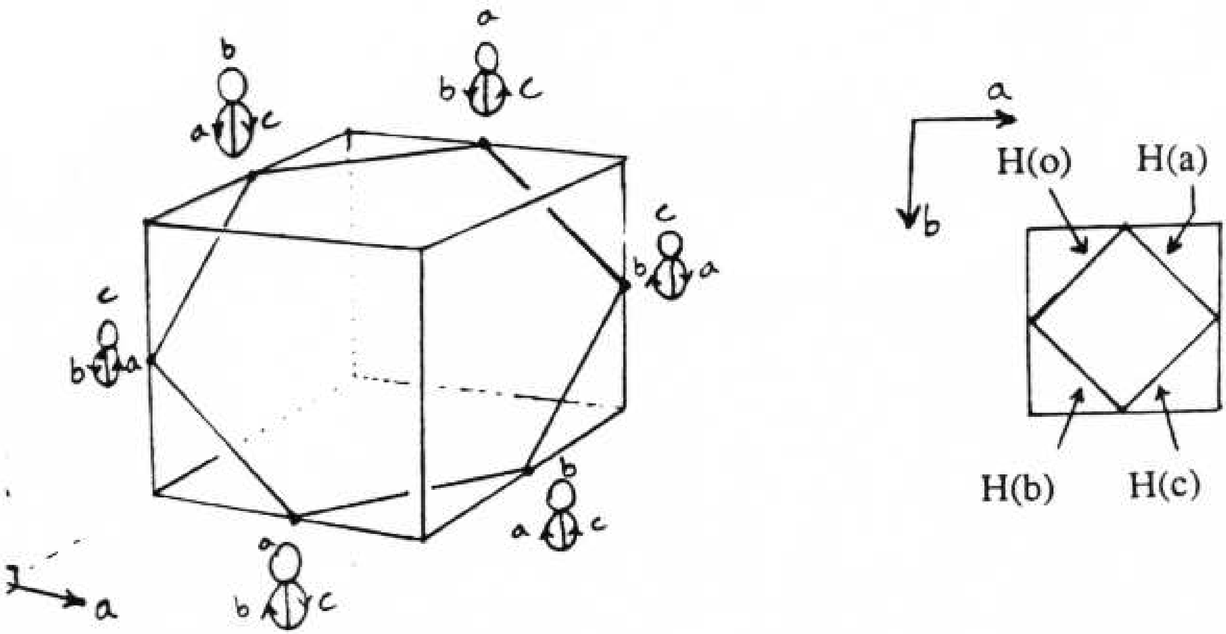


Figure 5.5: The cube



H(0) in the cube

Pattern of intersections with C(+), the top face

Figure 5.7: Fitting the hexagons into the cube

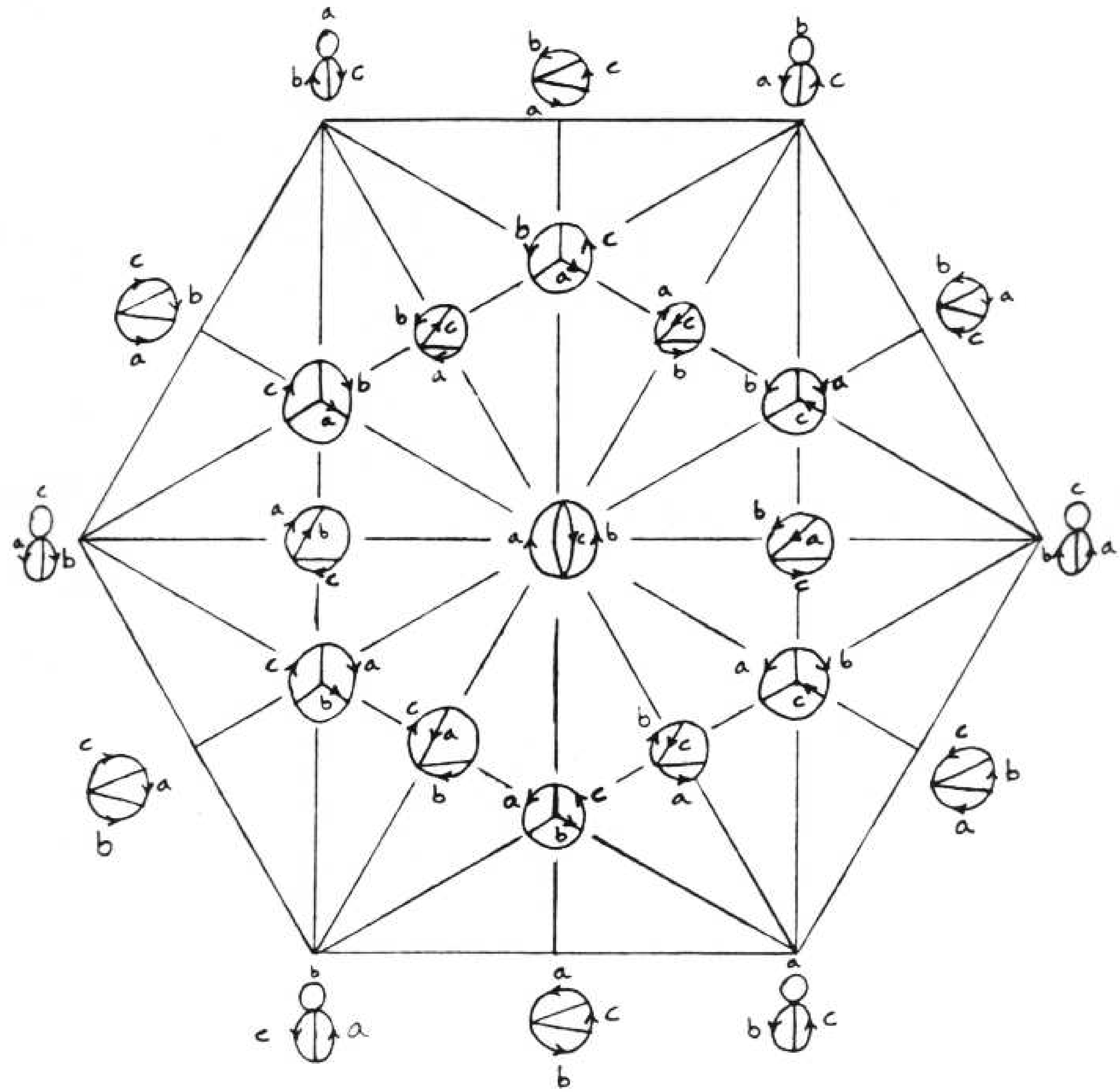


Figure 5.6: The hexagon H(c)

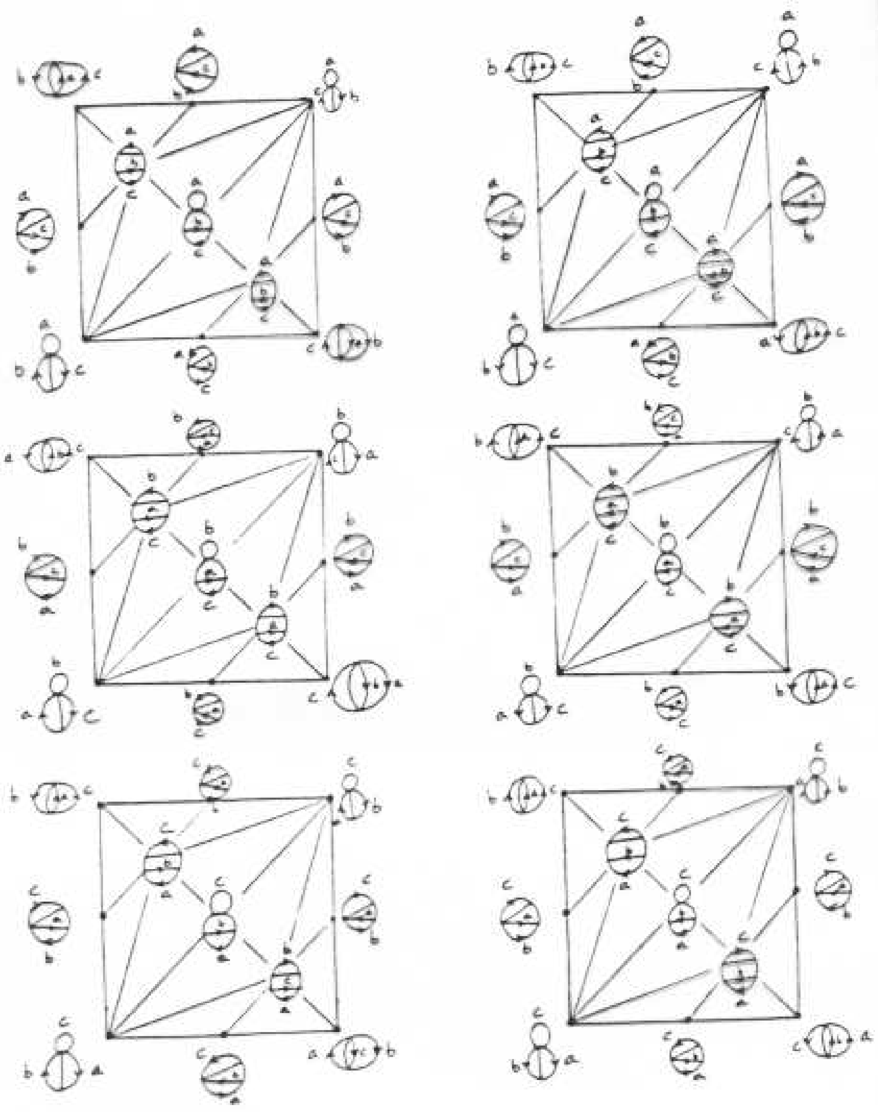


Figure 5.9: The remaining seventy two 2-cells
S(b,c) S(o,a)
S(a,c) S(o,b)
S(b,a) S(o,c)

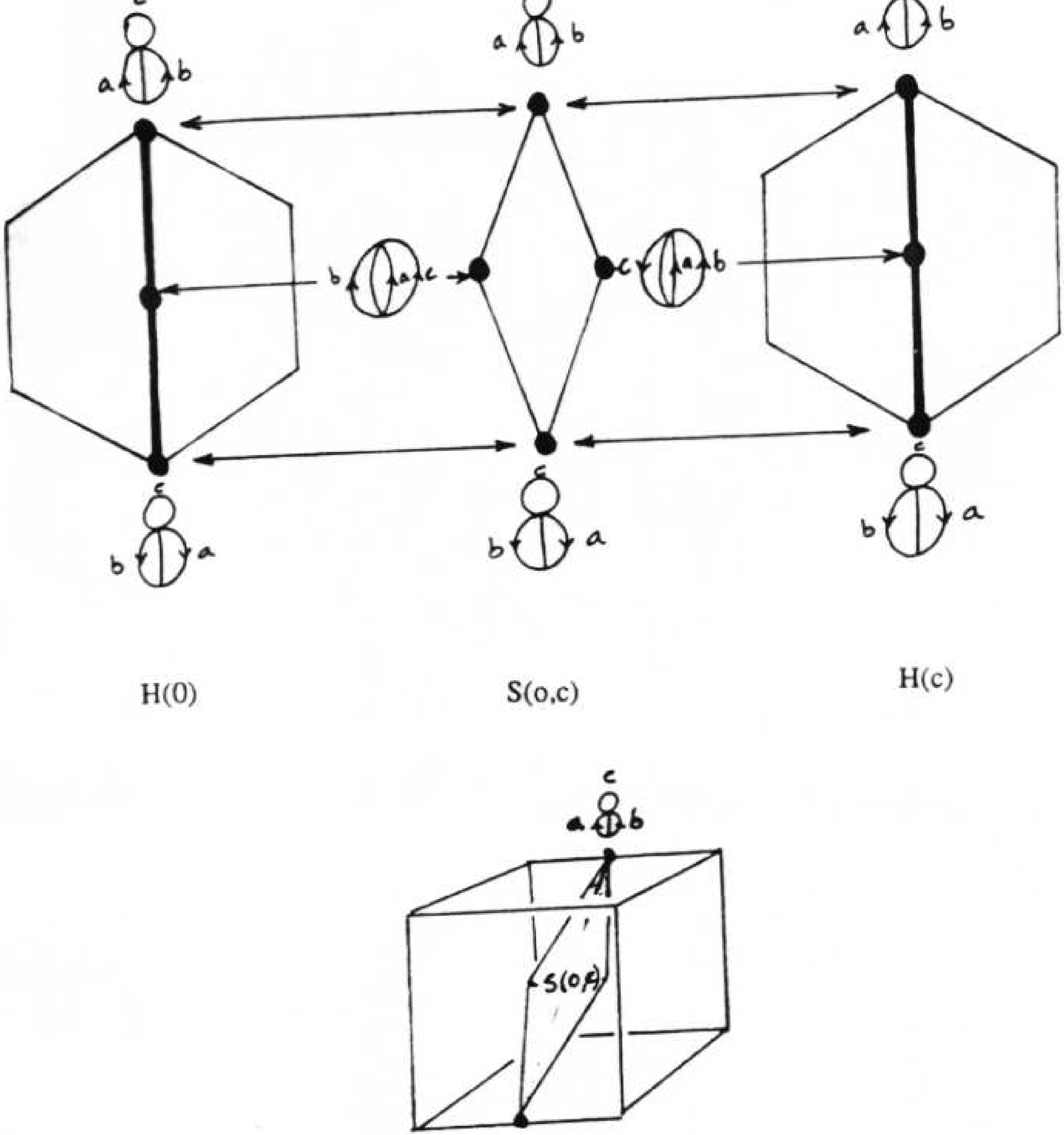


Figure 5.10: Attaching the disc S(o,c)