REES CRITERIA

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Abstract. These are notes on the author's talk given at the workshop on Integral Closure, Multiplier Ideals and Cores, AIM, December 2006.

Let $(R, \mathfrak{m})$ be a Noetherian local ring, equidimensional, universally catenary, of dimension d.

Theorem 0.1 (The Rees Criterion)**.** *Let $J \subset I$ be $\mathfrak{m}-$primary. Then J is a reduction of I if and only if $e(J) = e(I)$.*

Remark 0.2. Ratliff showed this property is equivalent to R being equidimensional and universally catenary.

1. The ideal case

Let I be an ideal, and let $G = \mathrm{gr}_I(R)$. Consider $H^0_{\mathfrak{m}}(G)$ which is the set of all elements of G killed by some power of the maximal ideal $\mathfrak{m}$ of R. This is a finite G-module. Therefore, it is annihilated by a uniform power t of $\mathfrak{m}$, $\mathfrak{m}^t H^0_{\mathfrak{m}}(G) = 0$. Hence it's a finite graded module over $G \otimes_R R/\mathfrak{m}^t$, which is standard graded over an Artinian local ring. Therefore we can talk about Hilbert funciton, Hilbert polynomial, multiplicity, etc.

Definition 1.1. The j-multiplicity of the ideal I is defined as $j(I) = e(H^0_{\mathfrak{m}}(G))$ if $\dim H^0_{\mathfrak{m}}(G) = d$, and zero otherwise. Note that $j(I) = 0$ if and only if the analytic spread $\ell(I) < d$.

Remark 1.2 (Achilles-Manaresi)**.** $j(I) = \Sigma_{\mathfrak{P} \in \mathrm{Min(G)},\ \mathfrak{P} \supset \mathfrak{m}} \lambda(G_{\mathfrak{P}}) \cdot e(G/\mathfrak{P}) = e(G')$, where G' is obtained via:

$$\begin{array}{ccc} \mathrm{Proj}(\mathcal{R}(I)) & \twoheadrightarrow & \mathrm{Spec}(R) \\ \cup & & \cup \\ \mathrm{Proj}(G) & \twoheadrightarrow & V(I) \\ \cup & & \cup \\ \mathrm{Proj}(G/\mathfrak{m}G) & \twoheadrightarrow & V(\mathfrak{m}). \end{array}$$

Now $\mathrm{Proj}(G')$ is the largest subscheme of $\mathrm{Proj}(G)$, none of whose irreducible components are contained in $\mathrm{Proj}(G/\mathfrak{m}G)$.

Theorem 1.3 (Flenner-Manaresi)**.** *For $J \subset I \subset R$ the following conditions are equivalent:*

1. *J is a reduction of I.*
2. *$j(J_{\mathfrak{p}}) = j(I_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*
3. *$j(J_{\mathfrak{p}}) \leq j(I_{\mathfrak{p}})$ for all $\mathfrak{p} \in V(J)$ with $\ell(J_{\mathfrak{p}}) = \dim R_{\mathfrak{p}}$.*

Proposition 1.4. *If $a_1, \ldots, a_d$ are general elements in I, with $R/\mathfrak{m}$ infinite, and $\mathfrak{a} := (a_1, \ldots, a_{d-1})$, then*

$$j(I) = \lambda(R/(\mathfrak{a} : I^{\infty}, a_d)) = \lambda(R/(\mathfrak{a} : a_d^{\infty}, a_d)).$$

2. The module case

Let E be a submodule of a free module $F := R^e$. Then we have the induced map of symmetric algebras:

$$\begin{array}{ccccc} \mathrm{Sym}(E) & \longrightarrow & \mathrm{Sym}(F) & = R[t_1, \ldots, t_e] = A \\ & \searrow & & \nearrow & \\ & & \mathcal{R}(E) & & \end{array}$$

Definition 2.1 (Buchsbaum-Rim multiplicity)**.** If $\lambda(F/E)$ is finite, then for n large enough we have

$$\lambda(F^n/E^n) = \frac{br(E)}{(d+e-1)!}\; n^{d+e-1} + \text{ lower terms}$$

Theorem 2.2 (Kleiman-Thorup)**.** *Let $U \subset E \subset F$, with F free and $\lambda(F/U)$ finite. If $d > 0$ then $U \subset E$ is a reduction if and only if $br(U) = br(E)$.*

3. General module case

This part is based on joint work with Javid Validashti. Consider the containment $E^n \subset F^n$. If we take the length of $H^0_{\mathfrak{m}}(F^n/E^n)$ it may not behave like a polynomial for n large. Instead, we consider the following filteration:

$$E^n \subset E^{n-1}F \subset \ldots \subset EF^{n-1} \subset F^n.$$

Then consider the function

$$\Sigma_E(n) := \sum_{i=0}^{n-1} \lambda(H^0_{\mathfrak{m}}(E^iF^{n-i}/E^{i+1}F^{n-i-1})).$$

For n large enough, it turns out to be a polynomial of the form

$$\frac{j(E)}{(d+e-1)!} n^{d+e-1} + \text{lower terms}.$$

Remark 3.1. The j-multiplicity of E depends on the embedding $E \subset F$.

Example 3.2. If $\lambda_R(F/E) < \infty$ then $\Sigma_E(n) = \lambda(F^n/E^n)$, therefore $j(E) = br(E)$.

Example 3.3. If $E = I$ is an ideal in the ring $F = R$, then

$$\Sigma_E(n) = \sum_{i=0}^{n-1} \lambda(H^0_{\mathfrak{m}}(I^i/I^{i+1}))\,,$$

which is the first sum transform of the function defining $j(I)$, thus $j(E) = j(I)$.

Definition 3.4. For $1 \le i \le d$ we let $j_i(E) = \sum_{\mathrm{ht}\,\mathfrak{p}=i} j(E_{\mathfrak{p}})$

Theorem 3.5. *If $U \subset E \subset F$, with $\dim F/U < d$. Then TFAE*

1. *$U \subset E$ is a reduction.*
2. *$j(U_{\mathfrak{p}}) = j(E_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Spec}\,(R)$.*
3. *$j(U_{\mathfrak{p}}) \le j(E_{\mathfrak{p}})$ for all $\mathfrak{p}$ with $\ell(U_{\mathfrak{p}}) = \dim R_{\mathfrak{p}} + e - 1$.*
4. *$j_i(U) = j_i(E)$ for all $1 \le i \le d$.*

Remark 3.6. The analytic spread $\ell(E)$ of a module E is defined as $\dim \mathcal{R}(E) \otimes_R k$.

Remark 3.7. The main idea to show $\Sigma_E(n)$ is a polynomial is to use an "internal grading". Let $E \subset F$ be R-modules. Let $I = E \cdot A$ be the ideal generated by the image of E in $A := \mathcal{R}(F)$, a standard graded R-algebra. Now consider $\mathcal{R}_A(I) = A[It] \subset A[t]$ and assign degree 0 to t. Let $G = \mathrm{gr}_I(A)$ and define the " internal grading" on G by $[G]_n = \oplus_i [I^i/I^{i+1}]_n$. Since I is genereated by E in degree 1, we can write this sum as

$$[G]_n = \oplus_i [I^i/I^{i+1}]_n = \oplus_i (E^i F^{n-i}/E^{i+1}F^{n-i-1}).$$

With this grading, G is a standard graded algebra over $G_0 = R$. Let $D := \dim G = \dim A$ and consider the homogeneous ideal FG which is a finite graded module. Hence, for t large enough $H^0_{\mathfrak{m}}(FG)$ is a finite graded module over $G/\mathfrak{m}^t G$, a standard graded ring over an Artinian local ring. Therefore, we can talk about its Hilbert polynomial which is indeed $\Sigma_E(n)$. Therefore, $j(E) = e(H^0_{\mathfrak{m}}(FG))$ if $\dim H^0_{\mathfrak{m}}(FG) = D$ and zero otherwise.

Remark 3.8. Note that in previous remark we do not need F to be free and we can define the multiplicity $j(E|F)$ of a pair of modules.

To prove the implication $2 \Rightarrow 1$ of the main theorem, we pass to the integral closure of U in E to assume $\lambda(E/U)$ is finite and we prove the following result:

Theorem 3.9. *Let $U \subset E \subset F$ be R-modules with the length $\lambda(E/U)$ finite. If $d > 0$ then*

(1) $j(U) \geq j(U|E) + j(E)$
(2) *If $j(U) = j(E)$ then $j(U|E) = 0$, which implies $U \subset E$ is a reduction.*

Proof. For part (1), consider the following diagram of containments

$$\begin{array}{ccc} U^iF^{n-i} & \longrightarrow & E^iF^{n-i} \\ \uparrow & & \uparrow \\ U^{i+1}F^{n-i-1} & \longrightarrow & E^{i+1}F^{n-i-1}. \end{array}$$

By left exactness of $\Gamma_{\mathfrak{m}}(-)$ we have

$$\lambda_R\Big(\frac{E^iF^{n-i}}{U^iF^{n-i}}\Big) + \lambda_R\Big(H^0_{\mathfrak{m}}\Big(\frac{U^iF^{n-i}}{U^{i+1}F^{n-i-1}}\Big)\Big) \geq \lambda_R\Big(H^0_{\mathfrak{m}}\Big(\frac{E^iF^{n-i}}{U^{i+1}F^{n-i-1}}\Big)\Big).$$

By exactness of $\Gamma_{\mathfrak{m}}(-)$ on short exact sequences where the left most module has finite length we get

$$\lambda_R\Big(H^0_{\mathfrak{m}}\Big(\frac{E^iF^{n-i}}{U^{i+1}F^{n-i-1}}\Big)\Big) = \lambda_R\Big(H^0_{\mathfrak{m}}\Big(\frac{E^iF^{n-i}}{E^{i+1}F^{n-i-1}}\Big)\Big) + \lambda_R\Big(\frac{E^{i+1}F^{n-i-1}}{U^{i+1}F^{n-i-1}}\Big).$$

Now, summing up these inequalities for $i = 0, 1, 2, ..., n-1$ we get

$$\Sigma_U(n) \geq \lambda_R(E^n/U^n) + \Sigma_E(n).$$

Therefore $j(U) \geq j(U|E) + j(E)$. This proves (1).

For part (2), if $j(U|E) = 0$ then $\lambda(E^n/U^n)$ is a polynomial of degree at most $d+e-2$. Let $B = \mathcal{R}(E) \supset J = UB$, and let $G = \mathrm{gr}_J(B)$, with the internal grading. Then $\lambda(E^n/U^n) = H_{EG}(n)$. Therefore $\dim_G EG \leq d + e - 1 < \dim G$. Thus EG is a nilpotent $G-$ideal (G is equidimensional), hence $E/U \subset B/UB$ is nilpotent. Therefore $U \subset E$ is a reduction. $\square$