ONE-DIMENSIONAL SUMS FOR THE IMPATIENT

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Let $R = (R_1, R_2, \ldots, R_k)$ be a sequence of rectangular partitions and λ a partition. We shall give a combinatorial definition of the one-dimensional sums $X_{\lambda;R}(q)$ [1]. This definition essentially appears in [3] [4]. See [2] for the Kostka-Foulkes special case. We assume some knowledge of the Robinson-Schensted correspondence but will actually not talk about crystals at all.

Define the generating function $H_R(x;q)$ by

$$H_R(x;q) = \sum_T x^T q^{E(T)} \tag{1}$$

where $T = (T_k, \ldots, T_2, T_1)$ runs over k-tuples where T_i is a semistandard tableau of shape R_i, x^T is the monomial whose exponent is the total content of the tableau list T, and $E(T)$ is the energy of T, whose definition is given below. It can be shown that H_R is a symmetric function. Define

$$H_R(x;q) = \sum_\lambda s_\lambda(x) X_{\lambda;R}(q). \tag{2}$$

The energy function E requires two constructions, the "rectangle-switching" bijection (combinatorial R-matrix) and the local energy function. The rectangle-switching bijection $\sigma = \sigma_{(R_2,R_1)}$ sends $(T_2, T_1) \mapsto (T_1', T_2')$ where T_i and T_i' are semistandard tableaux of shape R_i. To compute T_i' form a biword from (T_2, T_1) whose lower word is the row-reading word of T_2, followed by the row-reading word of T_1. The upper word contains a letter a_i (resp. b_i) above every letter in the lower word coming from the i-th row of T_1 (resp. T_2). For example, let

$$T_2 = \begin{array}{|c|c|c|c|}\hline 1&2&5&5\\\hline 3&6&6&7\\\hline\end{array} \qquad T_1 = \begin{array}{|c|c|c|}\hline 1&1&2\\\hline 2&3&3\\\hline 4&4&4\\\hline\end{array}.$$

Then the biword of (T_2, T_1) is

$$\begin{array}{ccccccccccccccccc} b_2 & b_2 & b_2 & b_2 & b_1 & b_1 & b_1 & b_1 & a_3 & a_3 & a_3 & a_2 & a_2 & a_2 & a_1 & a_1 & a_1 \\ 3 & 6 & 6 & 7 & 1 & 2 & 5 & 5 & 4 & 4 & 4 & 2 & 3 & 3 & 1 & 2 & 2 \end{array}$$

The tableau pair (P, Q) is obtained by column inserting the lower word, starting from the right end, and recording using the upper word.

$$P = \begin{array}{|c|c|c|c|c|}\hline 1&1&1&2&3\\\hline 2&3&3&4\\\cline{1-4} 3&4&4\\\cline{1-3} 5&5&7\\\cline{1-3} 6&6\\\cline{1-2}\end{array} \qquad Q = \begin{array}{|c|c|c|c|c|}\hline a_1&a_1&a_1&b_1&b_1\\\hline a_2&a_2&a_2&b_2\\\cline{1-4} a_3&a_3&a_3\\\cline{1-3} b_1&b_1&b_2\\\cline{1-3} b_2&b_2\\\cline{1-2}\end{array}$$

The tableau Q (which has shape ν, say) is a kind of Littlewood-Richardson tableau that counts the multiplicity of s_ν in $s_{R_2} s_{R_1}$, which is 1 since products of two rectangles are multiplicity-free. The letters a_i form a canonical rectangular subtableau. The letters b_i in the columns to the right of the canonical subtableau form a Yamanouchi tableau. There is a unique way to put the rest of the letters into the remainder of the shape ν to form a semistandard tableau (namely, Q) in the alphabet $a_1 < a_2 < \cdots < b_1 < b_2 < \cdots$. Let Q' be the tableau of the same shape and content as Q that is similar but is semistandard in the alphabet $b_1 < b_2 < \cdots < a_1 < a_2 < \cdots$.

$$Q' = \begin{array}{|c|c|c|c|c|}\hline b_1 & b_1 & b_1 & b_1 & a_1 \\ \hline b_2 & b_2 & b_2 & b_2 \\ \cline{1-4} a_1 & a_1 & a_2 \\ \cline{1-3} a_2 & a_2 & a_3 \\ \cline{1-3} a_3 & a_3 \\ \cline{1-2} \end{array}$$

The tableau Q' is unique by multiplicity-freeness. A new biword is obtained by reverse column insertion for the pair (P, Q').

$$\begin{matrix} a_3 & a_3 & a_3 & a_2 & a_2 & a_2 & a_1 & a_1 & a_1 & b_2 & b_2 & b_2 & b_2 & b_1 & b_1 & b_1 & b_1 \\ 6 & 6 & 7 & 3 & 5 & 5 & 1 & 4 & 4 & 2 & 3 & 3 & 4 & 1 & 1 & 2 & 3 \end{matrix}$$

The parts of the biword below the a_i and the b_i can be put into rectangular partition diagrams to form tableaux T_1' and T_2' respectively.

$$T_1' = \begin{array}{|c|c|c|}\hline 1 & 1 & 4 \\ \hline 3 & 5 & 5 \\ \hline 6 & 6 & 7 \\ \hline \end{array} \qquad T_2' = \begin{array}{|c|c|c|c|}\hline 1 & 1 & 2 & 3 \\ \hline 2 & 3 & 3 & 4 \\ \hline \end{array} \tag{3}$$

The rectangle-switching bijection is defined by $(T_2, T_1) \to (T_1', T_2')$.

The local energy function $E(T_2, T_1)$ is the statistic on 2-tuples of rectangular tableaux defined as follows. Let ν be the shape of P or Q coming from the pair (T_2, T_1) as above. Let $E(T_2, T_1)$ be the number of cells in ν that are strictly to the right of the s-th column where s is the maximum width of R_1 and R_2.

In the running example $E(T_2, T_1)$ is 1 since the shape $\nu = (5, 4, 3, 3, 2)$ has 1 cell to the right of the 4-th column.

Finally, for $T = (T_k, \dots, T_2, T_1)$ we define

$$E(T) = \sum_{1 \le i < j \le k} E(T_j^{(i+1)}, T_i) \tag{4}$$

where $T_j^{(i+1)}$ is the rectangular tableau of shape R_j obtained by switching the tableau T_j to the right until it reaches the $(i+1)$-th position. It must be switched, one by one, past the tableaux T_{j-1}, T_{j-2}, $\dots$, T_{i+1}.

This concludes the definition of the energy function $E(T)$ and therefore of the one-dimensional sum $X_{\lambda;R}(q)$.

There is also a cocharge or coenergy version $\overline{X}_{\lambda;R}(q)$ of the one-dimensional sum. The only difference is that instead of using the local energy function E, one uses the local coenergy function $\overline{E}(T_2, T_1)$. Given the shape ν as above, $\overline{E}(T_2, T_1)$ is the number of cells in ν in rows whose index is strictly greater than r, where r is the maximum height of the rectangles R_1 and R_2. In the running example, $\overline{E}(T_2, T_1) = 5$.

It is easy to see that $E(T_2,T_1)+\overline{E}(T_2,T_1)=|R_1\cap R_2|$, the area of the rectangle formed by the intersection of the partition diagrams of R_1 and R_2. It follows that

$$E(T)+\overline{E}(T)=||R||:=\sum_{1\le i<j\le k}|R_i\cap R_j|.$$

and that the coenergy analogue $\overline{H}_R(x;q)$ of $H_R(x;q)$ and the resulting coefficient $\overline{X}_{\lambda;R}(q)$ satisfy

$$\begin{aligned}\overline{H}_R(x;q)&=q^{||R||}H_R(x;q^{-1})\\ \overline{X}_{\lambda;R}(q)&=q^{||R||}X_{\lambda;R}(q^{-1}).\end{aligned}\tag{5}$$

References

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