GENERALIZED QUASISYMMETRIC INVARIANTS

This problem was contributed by Nantel Bergeron and Adriano Garsia (and transcribed by Nick Loehr, who takes full responsibility for any errors appearing herein).

Fix an integer n, and let $R = \mathbb{Q}[x_1, \ldots, x_n]$ be the polynomial ring in n variables. For each i and r, define generalized transposition operators $s_i^{(r)}$ by setting

$$s_i^{(r)}(x_i^a x_{i+1}^b) = \begin{cases} x_{i+1}^a x_i^b & \text{if } a < r \text{ or } b < r; \\ x_i^a x_{i+1}^b & \text{otherwise.} \end{cases}$$

Let $\mathrm{QSym}_n^{(r)}$ consist of the polynomials in R left invariant by the algebra of operators generated by all $s_i^{(r)}$ for $1 \leq i < n$. It is known that $\mathrm{QSym}_n^{(r)}$ is an algebra for all r. For example, $\mathrm{QSym}_n^{(0)} = \mathbb{Q}[x_1, \ldots, x_n]$ and $\mathrm{QSym}_n^{(1)}$ consists of the usual quasisymmetric polynomials in n variables.

Problem: Let I be the ideal of R generated by the symmetric polynomials of positive degree. Prove *Hivert's conjecture*:

$$\dim_{\mathbb{Q}}(\mathrm{QSym}_n^{(r)}/I) = n!.$$

The result is known for $r = 0$ and $r = 1$, the latter result being due to Garsia and Wallach. The best solution to this problem would exhibit an explicit basis for the quotient ring, preferably one compatible with the given filtration.