Open Problems in Multidimensional Stability of Waves and Patterns

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S. Benzoni-Gavage, H.K. Jenssen & M. Williams: *(Existence and Stability of Spherical Fronts)*
Setup:
We want to investigate the long time stability of curved fronts: shock, reactive or viscous. To shed some light on this problem we chose the simplest geometry, namely spherical fronts.

Consider either the viscous or inviscid equations of gas dynamics in space dimension $d = 2$ or $d = 3$. For simplicity, one can even consider barotropic flow which assumes that pressure is a function of density alone (and not temperature etc). This has the advantage that the mass and momentum equations decouple from the energy equation, which can thus be disregarded. For further simplicity, we may consider the case where the velocity is radial (i.e. no swirl) so that $\varrho(\mathbf{x}, t) = \varrho(r, t)$ and $\mathbf{u}(\mathbf{x}, t) = u(r, t)\frac{\mathbf{x}}{r}$, $r = |\mathbf{x}|$. After these simplifications, the equations one considers are (c.f. Hoff & Jenssen, *Symmetric nonbarotropic flows with large data and forces*, Arch. Rational Mech. Anal., **173** (2004), 297–343

$$\begin{aligned} \rho_t + (\rho u)_r + \frac{m\rho u}{r} &= s(r,t) \\ (\rho u)_t + (\rho u^2 + P(\rho))_r + \frac{m\rho u^2}{r} &= \nu \left(u_r + \frac{mu}{r} \right)_r + F(r,t), \end{aligned}$$

where $m = n - 1$, $n =$ space dimension, and s, F are the source terms for mass and momentum, respectively.

Open Problem:
Do there exist stationary spherical solutions to the above problem (the idea being that stationary spherical solutions should be the simplest multi-D objects to study). It is not clear how the source terms s and f must be chosen in order to have a viscous or inviscid shock. Could f or s be identically zero? What is the role of curvature in the stability of the front and existence of nearby perturbed spherical fronts? Note that in the short-time analysis of Majda, curvature does not play a role. A possible scenario is that as the radius of the spherical shock gets bigger, stability is more likely, and there is likely to be a transition from stability to instability as the radius decreases to a critical radius r^*. Can one prove or disprove this?

Open Problem:
Determine the relationship between the Lopatinski determinant for stability and recent work on stability of spherical waves using spherical harmonics (c.f. C.C. Wu & P. H. Roberts, *Bubble shape instability and sonoluminescence*, Phys. Lett. A **250**, 131 (1998)).

Open Problem:
A classical and apparently very hard problem is the analysis of *non stationary* spherical waves, e.g. a focusing inviscid shock (c.f. R. Courant & K.O. Friedrichs, *Supersonic Flow and Shock Waves*, Interscience Publishers, New York, (1948) or L.D. Landau & E.M. Lifshitz *Fluid Mechanics*, Pergamon Press, Oxford (1959)) Are such waves stable?

P. Szmolyan: *(Kinetic and Boltzmann Equations)*
Setup:
Consider the kinetic equation

$$f_t + v \cdot f_x Q(f, f)$$

where $f = f(x, v, t), x \in \mathbb{R}^d$. Profiles solve

$$(v - c)f' = Q(f, f)$$

Open Problem:
Under what conditions do there exist profiles for this equation? Are they stable to multidimensional perturbations? What does the Evans function look like. (As a first step, one might start with considering discrete velocity models.)

Y. Li: *(Stability of travelling waves of the full water wave problem near the critical case)*
Setup:
Consider the water wave equations

$$\begin{aligned} \eta_t &= \mathcal{G}\Phi \\ \Phi_t + \frac{\Phi_x^2 + 2\eta_x \Phi_x \mathcal{G}\Phi - (\mathcal{G}\Phi)^2}{1 + \eta_x^2} + g\eta &= 0 \end{aligned}$$

where $\widehat{\mathcal{G}}(k) = k \tanh(k)$. The equation admits travelling wave solutions $(\eta_c, \Phi_c) = (\eta(x - ct), \Phi_c(x - ct))$ for a variety of wave speeds.

Open Problem:
How we determine the stability of such waves near the critical speed c^*.

S. Benzoni-Gavage: *(Shocks with Capillarity)*
Setup:
Consider the equations

$$\begin{aligned} \partial_t \varrho + \nabla \cdot (\varrho \mathbf{u}) &= 0 \\ \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla P &= \nabla \left(K(\varrho)\nabla\varrho + \frac{1}{2}\frac{\partial K(\varrho)}{\partial \varrho}|\nabla\varrho|^2 \right) \end{aligned}$$

where $\mathbf{u} \in \mathbb{R}^3$ is the velocity, $\varrho > 0$ is the density of the fluid, and $P = P(\varrho)$ is the pressure. The equations admits a planar profile $(\varrho, \mathbf{u})$. It can be shown that the spectrum of the resulting linear operator obtained by linearization of the equations about the planar profile must lie on the imaginary axis.

Open Problem:
Are there any eigenvalues on the imaginary axis? Is the planar profile spectrally stable or unstable? Does linear information tell the whole story? That is, can one prove full nonlinear stability from the linear information.

R. Pego: *(Stability of Toda Lattice Solitons)*
Open Problem:
The Toda lattice

$$\ddot{q}_k = \exp(q_{k+1} - q_k) - \exp(q_k - q_{k-1})$$

has the well known solutions

$$q_k = \log\left(\frac{\cosh(\beta(k - ct + 1))}{\cosh(\beta(k - ct))}\right), \qquad c = \frac{\sinh\beta}{\beta}$$

What are their spectral stability properties?

M. Williams : *(Two Interacting Shocks in 1D)*
Setup:
Consider the inviscid conservation law

$$\partial_t \mathbf{u} + \partial_x f(\mathbf{u}) = 0 \qquad \text{(A)}$$

and the viscous regularization

$$\partial_t \mathbf{u}^\varepsilon + \partial_x f(\mathbf{u}^\varepsilon) = \varepsilon \Delta \mathbf{u}^\varepsilon \qquad \text{(B)}$$

where for simplicity we may take $\mathbf{u} \in \mathbb{R}^2$ and $x \in \mathbb{R}$. Take a two-shock solution to the viscous conservation law of the form below:

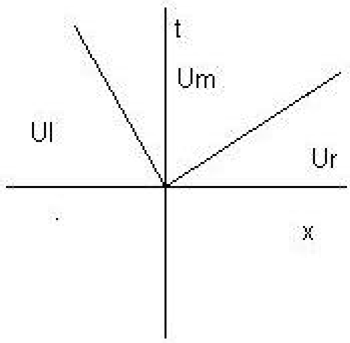

Figure 1: The Two Shock Setup

Open Problem:
Can we construct explicit solutions to the viscous problem (B), say of the form $\mathbf{u}^\varepsilon(t, x, t/\varepsilon, x/\varepsilon)$, which converge in a reasonable sense to solutions $\mathbf{u}(t, x)$ of the inviscid problem (A) as $\varepsilon \to 0$? Note: this question has been answered abstractly (c.f. the C.I.M.E. notes by A. Bressan), but our goal is to explicitly resolve the dynamics at the corner where U_l and U_r meet in the xt-plane.

J. Humphreys: *(Stability for strong viscous shocks of the p-system)*
Consider the p-system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t + p(v)_x &= (b(v)u)_x \end{aligned}$$

where $p' < 0$, $p'' > 0$. Can we show that strong shocks for the p-system are stable or unstable? What role, if any, does symmetrizability play in the onset of instability?

K. Promislow : *(MultiD front dynamics in optical resonance)*
A model for pattern formation in an optical cavity near resonance is given by

$$i\varphi_t - \frac{1}{2}\Delta_l \varphi + |\varphi|^2\varphi + (i-a)\varphi - \gamma\varphi^* = 0$$

where $\Delta_l = \partial_x^2 + l^{-1}\partial_y^2$ and a, γ are real constants. The equation admits a planar front solution. Consider a perturbation of the planar front. What is the Evans function for the *perturbed* front? What are the dynamics of the front?

B. Sandstede: *(Dynamical interpretation of the roots of the $D(\lambda)$ embedded in the absolute spectra)*
Setup:
In many circumstances, the Evans function can be extended into branch points of the linear dispersion relation. It is natural to ask what role, if any, roots of the Evans function at branch points play for the temporal dynamics of the linear or nonlinear evolution. It has been shown by MURATA [Tohoku Math J 37 (1985) 151–195] that temporal decay rates of scalar linear heat equations depend very much on the presence of these roots.

Open Problem: To what extent is this true for more general parabolic PDEs and for other nonlinear equation?

S. Malham: *(Biscale chaos)*
Setup:
Consider the coupled reaction diffusion equations

$$\begin{aligned} u_t &= \delta\Delta u - uv^2 \\ v_t &= \Delta v + uv^2 \end{aligned}$$

which is a model of autocatalysis. Here the paramteter δ is a certain ratio of the speed of the autocatalyst molecules and the fuel molecules. When $\delta \sim 8$, small perturbations of a planar front evolve into a very complex front. It has been suggested (c.f. *Biscale chaos in propagating fronts*, Phys. Rev. E **52**, (1995), pp. 4724 –4735) that the wrinkles that form from the perturbed planar interface exhibit spatial-temporal chaotic behaviour characterized by *two* length scales. This is called *biscale chaos.*

Open Problem:
What are the stability properties of the planar front? Can we understand the secondary instability by getting an Evans function for the front that arises from the secondary instability?

M. Haragus: *(Stability for KP-I profiles with periodic structure)*
Setup:
Consider the KP-I equation

$$(u_t - u_{xxx} + cu_x + uu_x)_x + u_{yy} = 0$$

Clearly the function $u(x, y, t) = \phi(z)$ where $\phi(z)$ is the usual KdV soliton solution is a solution to the KP-I equations with no variation in the y direction. This line soliton is known to be unstable to transverse perturbations. A family of y-periodic waves bifurcates from it and connects to the well-known lump solution, which is presumably stable.

Open Problem:
What is the stability of the KP-I soliton solution with periodic structure?

J. Albert: *(Benjamin-Ono type equations)*
Setup:
Consider the Benjamin-Ono equation

$$u_t + uu_x - \mathcal{K}u_x = 0$$

where $\widehat{\mathcal{K}u}(k) = |k|\hat{u}(k)$. This equation supports travelling waves of the form $\phi(z) = \frac{4}{1+z^2}$. The resulting eigenvalue problem is

$$(cv + \mathcal{K}v - \phi v)_z = \lambda v \qquad \text{(A)}$$

where v is the perturbation $v := u - \phi$.

Open Problem:
What is the spectrum of (A) and how does it relate to the spectrum of

$$cv + \mathcal{K}v - \phi v = \lambda v$$

which is well known. A possible approach would be to treat the eigenvalue equation as a two dimensional problem.

K. Zumbrun: *(Strong Shocks in one or multiD of viscous conservation laws)*
The Setup:
Consider the system of viscous conservation laws in one or several space dimensions d,

$$u_t + \sum_{j=1}^{d} f^j(u)_{x_j} = \sum_{j,k=1}^{d} (B^{jk}(u)u_{x_k})_{x_j}$$

where $u \in \mathbb{R}^n$, f is a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$ and $B^{j,k}$ is a smooth mapping from $\mathbb{R}^n$ to $\mathbb{R}^{n\times n}$. The stability of weak shocks in one spatial dimension has been investigated recently by H. Freistühler & P. Szmolyan (*Spectral stability of small shock waves, I*, Arch. Rat. Mech. Analysis, **164**, (2002), 287-309) and R. Plaza & K. Zumbrun (*An Evans function approach to spectral stability of small-amplitude shock profiles*, J. Discrete and Continuous Dynamical Systems **10**, 2004, no. 4, 885-924). The weak shock assumption induces a fast-slow structure (with the small parameter being the strength of the shock) that can be exploited in the calculations. It is not immediately apparent that there is fast-slow structure hidden somewhere for strong shock profiles that can exploited.

Open Problem:
Determine the stability properties of strong shocks for viscous conservation laws in one or several space dimensions.

M. Wechselberger: *(Loss of hyperbolicity, algebraic decay and the Evans function)*
Setup:
For certain values of p, the generalized KdV equation

$$u_t - u_{xxx} + u^p{}_x = 0$$

admits standing waves $\phi(x)$ which decay exponentially to zero as $|x| \to \infty$. However, the eigenfunctions of the operator obtained by linearizing about ϕ need not decay exponentially and depending on the power p (say p=5) may only decay algebraically as $|x| \to \infty$.

Open Problem:
Can one construct an Evans function for this problem? If so, how does it behave? Note that for this problem the absolute spectrum touches the essential spectrum so there is no exponential dichotomy that we can exploit.

P. Howard: *(Combination structures in viscous conservation laws)*
Open Problem:
The thin film equation

$$u_t + (u^2 - u^3)_x = -\varepsilon(u^3 u_{xxx})_x$$

supports solutions that are comprised of a Lax shock moving to the left with speed s_1 and an undercompressive shock moving to the right with speed s_2 as in Figure 2. Can one prove the stability or instability of such a structure?

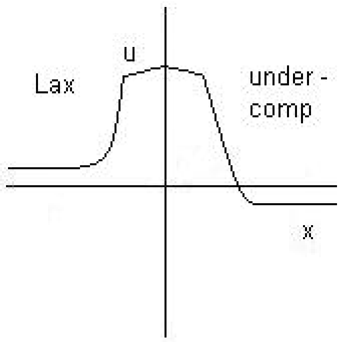

Figure 2: The Combination Structure

C. Jones: *(Stability of energized states of NLS)*
Consider

$$iu_t = u_{rr} = \frac{n-1}{r} u_r + f(|u|)u + \omega u$$

Is there really no spectrum off of $i\mathbb{R}$? What is the meaning of this for stability? Does the linear information give us full stability? Opposite Krein signature eigenvalues for NLS.

T. Kapitula: *(Transient dynamics for vortex patterns)*
Setup: Start with the Gross-Pitaevskii equation

$$iq_t + \Delta q \pm |q|^2 q = V(x)q$$

Numerical investigations show that if one starts with a "nice" vortex pattern with a lot of symmetry, it evolves to a "ugly" vortex pattern with little symmetry or structure, and then evolves further to to a "nice" pattern again.

Problem: By what mechanism does this happen and how can one capture the general dynamics?

H. Warchall: *(Stability of Encapsulated-Vortex Solutions)*
Setup:
Consider the equations

$$\begin{aligned} Ju_t &= \Delta u + g(u) && \text{(NLS)} \\ u_{tt} &= \Delta u + g(u) && \text{(NLKG)} \end{aligned}$$

where $u \in \mathbb{R}^{N+1} \to \mathbb{R}^M$, $g : \mathbb{R}^M \to \mathbb{R}^M$ continuous satisfying $g(y) = h(|y|^2)\hat{y}$, with $h : [0, \infty) \to \mathbb{R}$ and $\hat{y} \equiv y/|y|$. J is an invertible $M \times M$ skew symmetric matrix. Consider standing-wave solutions to (NLS) and (NLKG) of the form

$$u(x,t) = e^{\mu K t}\hat{\psi}(\hat{x})w(r)$$

with $\mu \in \mathbb{R}$ a constant, and K a real skew-symmetric $M \times M$ matrix with

$$\begin{aligned} K &= J^{-1} && \text{for (NLS)} \\ K^2 &= -I && \text{for (NLKG)} \end{aligned}$$

where $w : [0, \infty) \to \mathbb{R}$ and $\hat{\psi} : S^{N-1} \to S^{M-1}$. Here $\hat{\psi}$ is a unit vector valued eigenfunction of the Laplacian on the sphere $S^{N-1} \subset \mathbb{R}^N$, with

$$\Delta_S \hat{\psi} = -l(l + N - 2)\hat{\psi}$$

We remark that the possible values of l are limited by the dimension M of range space (see J. Iaia & H.A. Warchall, *Encapsulated-vortex solutions to equivariant wave equations: existence*, SIAM J. Math. Anal., **30** (1999), 118-139). If u satisfies (NLS) or (NLKG) then the spatial profile w satisfies

$$w'' + \frac{N-1}{r}w' - \frac{l(l+N-2)}{r^2}w + f(w) = 0$$

where $f(y) = g(y) + \omega y$ with

$$\omega = \begin{cases} \mu & \text{for (NLS)} \\ \mu^2 & \text{for (NLKG)} \end{cases}$$

Note: Traveling waves are generated by Galilean or Lorentz boosts. The idea is to generalize, e.g. standing wave solutions $u(x,t) = e^{i\omega t}e^{im\theta}w(r)$ of $-iu_t - \Delta u = g(u)$ whose stability was analysed by R.L. Pego & H.A.Warchall (*Spectrally stable encapsulated vortices for nonlinear Schrödinger equations*, J.Nonlinear Sci., **12** (2002), 347–394). Under appropriate conditions on f there exist smooth exponentially localized solutions w to the profile ODE. One essentially needs

$$f'(0) < 0 \qquad F(t) = \int_0^t f(s)\,ds > 0 \quad \text{for some } t > 0$$

Open Problem:
Under what conditions are the encapsulated-vortex solutions $u(x,t) = e^{\mu K t}\hat{\psi}(\hat{x})w(r)$ of (NLS) or (NLKG) with $N \geq 3$ stable or unstable?