Perfect Graphs

The American Institute of Mathematics

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Version: Tue Aug 24 11:37:57 2004

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The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum number of colors needed to color the vertices of G in such a way that no two adjacent vertices receive the same color. Clearly $\chi(G)$ is bounded from below by the size of a largest clique in G, denoted by $\omega(G)$. In 1960, Berge introduced the notion of a perfect graph. A graph G is *perfect*, if for every induced subgraph H of G, $\chi(H) = \omega(H)$.

A *hole* in a graph is a chordless cycle of length greater than 3, and it is *even* or *odd* depending on the number of vertices it contains. An *antihole* is the complement of a hole. It is easily seen that odd holes and odd antiholes are not perfect. Berge conjectured that these are the only minimal imperfect graphs, i.e., a graph is perfect if and only if it does not contain an odd hole nor an odd antihole. (When we say that a graph G *contains* a graph H, we mean as an induced subgraph). This was known as the Strong Perfect Graph Conjecture (SPGC), whose proof has been announced recently.

[0]This document was organized by Maria Chudnovsky as part of the lead-in and followup to the ARCC focused workshop "The Perfect Graph Conjecture," October 29 to Nove‍nber 2, 2002. The workshop was made possible by the American Institute of Mathematics and an NSF Mathematical Sciences Institutes grant.

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Can one decide in polynomial time if a graph is perfect?

A.1 Polynomial Recognition Algorithm Found

A polynomial algorithm to test whether a graph is Berge was found in November 2002. A paper summarizing the work of two groups— Chudnovsky and Seymour and Cornuejols, Liu and Vuscovic— is due to appear in *Combinatorica*. The algorithm is independent of the proof of the strong perfect graph conjecture.

A.2 Interaction Between Skew-Partitions and 2-joins

One can think of algorithms for testing for odd holes if you use only 2-joins to decompose a graph, or if you use only skew partitions. However, the interaction between skew partition steps and 2-join steps adds another level of difficulty. Can one argue that this can be reduced only to testing for holes as you decompose using skew partitions only and testing for holes using 2-joins only?

Contributed by Jeremy Spinrad

Similar approach worked in algorithms for recognizing even-hole-free graphs: first the graph is decomposed via vertex cut-sets, and only then via 2-joins. The 2-join decomposition blocks are defined in such a way that no new vertex cut- set is introduced.

Contributed by Kristina Vuskoic

A.3 The Perfect-Graph Robust Algorithm Problem

An algorithm, which for any easily recognizable input, A, finds either an easily recognizable B or an easily recognizable C, is sometimes called a "robust algorithm" – to be distinguished from a non-robust algorithm, which, for any A without a B, finds a C. Either provides a proof of the "existentially polytime (EP) theorem": For any A, there exists a B or a C. In other words: For any A without a B, there is a C.

In [92i:68043], Jack Edmonds and Kathie Cameron advocated seeking a robust algorithm which, for any graph G, finds either a clique and colouring the same size or else finds an easily recognizable combinatorial obstruction to G being perfect. The obstruction might be specified to be a "alpha-omega partitioned subgraph", or it might be specified more particularly to be an odd hole or odd antihole.

Such an algorithm might be simpler than an algorithm for recognizing whether or not a graph is perfect, in view of precedents, and since what it would do is incomparable with perfect-graph recognition. Such an algorithm could end up giving a clique and colouring the same size in a non-perfect graph.

Here are two examples of similar problems which have been solved. Edmonds has given a simple robust algorithm which, for any graph G, either finds an odd cycle > 3 with at most one chord (a defining obstruction to G being Meyniel) or else finds a clique and colouring the same size. This is an improvement on the non-robust algorithms of Hoang and Hertz which, assuming a graph is Meyniel, find a clique and colouring the same size. Edmonds' algorithm is much simpler than the Burlet-Fonlupt decomposition algorithm for recognizing Meyniel graphs, which was motivated by an interest in optimizing in Meyniel graphs, and which is used by Hoang and Hertz.

Conforti and Cornuejols give a complicated decomposition algorithm for recognizing whether or not a matrix is balanced. At about the same time, to motivate the advocacy of a robust algorithm for either node-colouring a graph or recognizing it to be not perfect, Cameron and Edmonds presented a simple algorithm which, for any 0-1 matrix M, either finds, where x is the largest number of ones in any row, an x-colouring of the columns so that the 1's of any row are in different coloured columns, or else finds "an odd hole" in M (the defining obstruction to M being balanced). This introduced the "EP - robust algorithm" paradigm which is followed in Edmonds' Meyniel-related algorithm, and is related to the Conforti-Cornuejols-Rao treatment of balanced matrices in the same way that Edmonds' is related to the Burlet-Fonlupt treatment of Meyniel graphs. Following the same paradigm we expect there to be a robust algorithm proving the SPCG, related in the same way to the Chudnovsky-Robertson-Seymour-Thomas decomposition of Berge graphs.

In conclusion, we know the following

EP Theorem 1: For any graph, there is either a clique and a colouring of the same size, or there is an alpha-omega partitioned subgraph (or both).

EP Theorem 2 (SPGT): For any graph, there is either a clique and a colouring of the same size, or there is a odd hole or odd antihole (or both). So: Give a combinatorial polytime algorithm to find what the EP theorem asserts to exist.

Contributed by Kathie Cameron and Jack Edmonds

A.4 NP Description of Perfect Graphs

Give an NP description of perfect graphs.

Contributed by Jack Edmonds.

A.5 Recognition Algorithm Given the List of Maximal Cliques

Is there a polytime recognition algorithm for perfect graphs where the input is the list of all maximal cliques in the graph? This probelm has been resolved, since a perfect graph can itself now be recognized.

Contributed by Bruce Shepherd

A.5.a Berge Graphs with Poly-bounded Number of Max Cliques. Give a polynomial time recognition algorithm for Berge graphs with polynomially bounded number of maximal cliques.

Contributed by Jeremy Spinrad.

A.6 TDI Matrices

1. *Given an $m \times n$ $0-1$ matrix A and an m-dimensional vector b, decide whether the system $Ax \leq b$ is totally dual integral (TDI), that is, is there an integer dual solution for every objective function for which the dual optimum exists.*

This is the $0-1$ special case of the well-known problem of TDI system recognition. Let $P := Ax \leq b, x \geq 0$. Let $A(u)$ be the matrix whose rows are the normal vectors of facets of P containing u, each row is integer and the gcd of its entries is 1.

There are several known relations between TDI and unimodular systems. As Serkan Hoşten pointed out, 'nondegenerate' TDI matrices are exactly those in which $A(u)$ is an $n \times n$ matrix having determinant 1 for every vertex u.

The following problem involves unimodularity in perfectness test:

INPUT: $m \times n$ 0-1 matrix A and an m-dimensional positive vector b,
QUESTION: *Is the matrix $A(u)$ for every vertex u of P a square matrix of determinant 1 ?*

2. *Can this problem be solved in polynomial time ?*

For reducing the test for the perfectness of matrix A to this problem define b with a 'lexicographic perturbation' from the all 1 objective function.

Contributed by András Sebő

A.7 Fixed Parameter Algorithms

In the context of fixed parameter algorithms, which was recently introduced by Downey and Fellows, it would be interesting to design an algorithm with running time $O(f(k)|V|^c)$ where k is the size of the maximum clique, c is a small constant independent of k and $f(k)$ is a (exponential) function of k. Such an algorithm can work for small k even for large n.

Contributed by Mohammad Taghi Hajiaghayi

A.8 Clique Joins

A k-clique-join of $G = (V, E)$ is a set of pairs $\{(A_0, B_0), (A_1, B_1), \ldots, (A_k, B_k)\}$, where $\{A_0, B_0\}$ is a partition of V, both A_0 and B_0 contain at least one ω-clique, and $A_i \subseteq A_0$, $B_i \subseteq B_0$ $(i = 1, \ldots, k)$ (not necessarily disjoint), moreover

(i) If $x \in A_i$ and $y \in B_i$, then $xy \in E$

(ii) If K is an ω-clique of G that meets both A_0 and B_0, then there exists i so that $K \subseteq A_i \cup B_i$.

A partitionable graph does not contain a k-clique-join for $k < 2(\omega - 1)$, on the other hand odd holes, odd antiholes do all contain $2(\omega - 1)$-clique-joins.

Could the minimum of k for which a k-clique-join exists be computed (or well-characterized)? For Berge-graphs? Is there a variant of this operation that would allow to compose perfect graphs and keep perfectness ?

Contributed by András Sebő

A.9 Polynomial Size Decomposition Tree

The problem with using skew-paritions (or star cutsets) for recognition algorithms is that the decomposition tree they induce does not have polynomial size. The following questions have been suggested during the workshop:

1. Suggest different endblocks of the decompostition (other than basic perfect graphs), that can be recognized in polynomial time and yet make the decomposition tree polynomial (Bruce Reed)

2. Normally every skew-partition has 4 decomposition blocks. What if we could prove that it is enough to consider only two blocks for each skew-partition. Would that imply a polynomial size decomposition tree? Possibly introducing new endblocks or using cleaning? (Kristina Vuskovic).

3.Does decomposition of C_4-free graphs via star-cutsets induce a polynomial size decomposition tree? Possibly using cleaning? (Kristina Vuskovic).

Chapter B: Structural Characterization of Perfect Graphs

Possible structural characterization of perfect graphs. Give explicit constructions for subclasses of Berge graphs.

Contributed by Paul Seymour

Chapter C: Coloring Perfect Graphs

Can one find an efficient algorithms to color a perfect graph?

C.1 Uniquely colorable perfect graphs

Uniquely colorable perfect graphs (in which there is a unique partition into ω stable-sets) are closely related to minimal imperfect graphs: according to a result of Padberg (Perfect zero-one matrices, Math Programming, 6, (1974)) if G is minimal imperfect then for all of its vertices v, the graph $G - v$ is uniquely colorable.

There is also a combinatorial good characterization theorem for unique colorability of perfect graphs, and a polynomial algorithm for testing the property using the ellipsoid method (IPCO 1, Kannan, Pulleyblank eds, Waterloo Univ. Press, 1990). In other words *UNIQUE COLORABILITY is a tractable property for perfect graphs*, closely related to minimal imperfect graphs.

Yet, some simple conjectures related to the SPGT, resist through the years. The following one arises both by specializing more general conjectures occurring in various papers, and does not seem to trivially follow from the SPGT:

If G is perfect and uniquely colorable, does there exist two ω-cliques which meet in $\omega - 1$ points ?

Let us call the two vertices in the symmetric difference of two such cliques *forced*.

Is it true that every known uniquely colorable perfect graph collapses to an ω-clique by successive identification of forced vertices ?

It can be simply proved that a minimal imperfect graph with three forced vertices in particular positions is an odd hole or an odd antihole. A simpler proof of the following statement would shortcut the proof of the SPGT:

If G is minimal imperfect, there is a vertex v so that $N(v)$ is uniquely colorable.

Contributed by Jean Fonlupt and András Sebő

Chapter D: Optimization on Perfect Graphs

Optimization on perfect graphs without using the ellipsoid method.

D.1 New Optimization Problems on Perfect Graphs

Are there any new optimization problems (other that coloring and finding the size of the max clique) that are easier to solve for a perfect graph that for a general graph? Can we use any of the existing (future) recognition algorithms in order to do that?

Contributed by Mohammad Hajiaghayi

D.1.a A Possible New Problem. Solve in a perfect graph: do two given vertices belong to an induced hole?

Contributed by Bruce Reed

E.1 Extending a Skew -Partition

When can a skew partition of an induced subgraph be extended to a skew partition of G? Algorithmically this is answered by the algorithm of de Figueiredo, Klein, Kohayakawa and Reed [2001j:05114], but what about a theorem? Is there some theorem that says "either the skew partition is extendable, or there is a reason why not (an obstruction)"? Contributed by Paul Seymour

E.2 Graphs Without Skew-Partitions

Is there a structure theorem for such graphs? Can they all be constructed somehow? Maybe by starting with a small one, and adding little bits so that at each stage there is no skew partition?

Contributed by Paul Seymour

E.3 Graphs Without Star Cutsets

Is there a structure theorem for such graphs? Can they all be constructed somehow? Maybe by starting with a small one, and adding little bits so that at each stage there is no star cutset?

Contributed by Bruce Reed

Conjecture If neither G nor G^c has a star cutset then the disk-structure of G is connected

(A *disk* is a hole or an antihole. Two disks are adjacent in the disk structure if they share at least 2 vertices).

Contributed by Ryan Hayward

E.4 Finding Skew-Partitions in Berge Graphs

Is it easier to detect skew partitions in Berge graphs than in general ones?

Contributed by Paul Seymour

E.5 Interaction Between Different Skew-Partitions in a Graph

For $1 \leq i \leq n$, let (A_i, B_i, C_i, D_i) be a skew partition of G, where there are no edges between A_i and B_i, and C_i is complete to D_i. For each i, choose one of A_i, B_i, C_i, D_i, say X_i, and let X be the union of all these X_i. Call $G \setminus X$ a *chunk*. If there is an odd hole or antihole in G, then it belongs to the chunk (for some choice of the X_i's), so to check Bergeness of G, it is enough to check Bergeness of all the chunks. But even if n is linear in the size of G, the number of chunks can be exponential. Maybe there is a way around this. With decomposition theorems that come up from excluded minors, using separations instead of skew partitions, the same exponential blowup happens, but it can be avoided by using separations that are pairwise noncrossing - then you only get linearly many pieces. Is there an analogous nice way for skew partitions to fit together, so that we only get linearly many (or polynomially many) chunks?

Contributed by Paul Seymour

E.6 Skew -Partitions of Balanced Size

The problem with recursive skew decomposition is that at least at first glance, you get exponential behavior. This would not occur if you were always able to find a decomposition in which each of A,B,C,D have at least n/c vertices for some c. Call this a skew partition of balanced size.

a) Can you find a skew partition of balanced in polynomial time, if one exists?

b) Will always looking for a balanced skew partition if possible lead to a polynomial size decomposition tree for perfect graphs (ie always use the skew partition which maximizes the size of the smallest set)

Contributed by Jeremy Spinrad

Answer: There exists a graph admitting no skew-partition of balanced size: take a clique and for some edges $e_1, \ldots, e_k$ of it add a vertices $v_1, \ldots, v_k$ s.t. each v_i has degree 2 and is adjacent to both ends of e_i.

E.7 Recognizing Balanced Skew-Partitions

Given a skew-partition, can one check in polynomial time whether it is balanced.

Contributed by Jeremy Spinrad

E.8 Even-Pair Skew-Partition

An *even-pair skew-partition* is a partition of the vertex set of a graph G into four sets A, B, C, D s.t. A is complete to B and C is anti-complete to D, and any two non-adjacent vertices in A or B are an even pair.

Question 1 Is it true that every Berge graph is either basic or has a 2-join or has an even-pair skew-partition?

Question 2 Is even-pair skew-partition a composition?

Contributed by Bruce Reed

Chapter F: Even Pairs in Berge Graphs

An even pair is a pair of vertices such that each chordless path between them has even length. Results of Fonlupt and Uhry, Meyniel, and also Bertschi and Reed imply that no minimal imperfect graph contains an even pair. A graph G is called quasi-parity (QP) if, for every induced subgraph H of G on at least two vertices, either H or its complement has an even pair. A graph G is called strict quasi-parity (SQP) if every induced subgraph H of G either H is aclique or has an even pair.

In the past 20 years many classical families of perfect graphs were proven to be SQP, which shows the interest of this class. However there are perfect graphs with no even pairs, e.g., all the line-graphs of 3-connected bipartite graphs.

None of the following questions or conjectures has been settled in full. Solutions are known only for special subclasses of graphs, e.g., planar graphs, claw-free graphs, bull-free graphs, etc.

Contributed by Frédéric Maffray

F.1 Coloring Berge Graphs Using Even Pairs

It is known (from Fonlupt and Uhry) that contracting an even pair in a perfect graph yields a perfect graph with the same chromaticnumber. This idea can be used as the basis for a conceptually simple coloring algorithm.

Contributed by Frédéric Maffray

F.2 Recognizing Even Pairs

Can one decide in polynomial time if a given Berge graph has an even pair? (The general problem, i.e., not restricted to Berge graphs, is known to be co-NP-complete.)

Contributed by Frédéric Maffray

Can one find even pairs using balanced skew-partitions? (Is there always an even pair in the cutset of a balanced skew-partition?)

Contributed by Bruce Ree

F.3 Quasi-Parity and Strict Quasi-Parity Graphs

A graph G is called quasi-parity (QP) if, for every induced subgraph H of G on at least two vertices, either H or its complement has an even pair. A graph G is called strict quasi-parity (SQP) if every induced subgraph H of G either H is a clique or has an even pair. In the past 20 years many classical families of perfect graphs were proven to be SQP, which shows the interest of this class.

Contributed by Frédéric Maffray

F.3.a Forbidden Subgraphs for The Class of Strict Quasi-Parity Graphs. Hougardy conjectured that the minimal forbidden induced subgraphs for the class SQP are odd hole, antiholes, and some line-graphs of bipartite graphs (not determined explicitly).

Contributed by Frédéric Maffray

F.3.b Recognition of Quasi-Parity and Strict Quasi-Parity Graphs. Can one decide in polynomial time if a given Berge graph is in the class QP, or SQP?

Contributed by Frédéric Maffray

F.4 Perfectly Contractile Graphs

Bertschi called a graph G even-contractile if there exists a sequence of even-pair contractions that turn G into a clique, and he called a graph G perfectly contractile (PC) if every induced subgraph of G is even-contractile. Many classical families of graphs (Meyniel graphs, weakly chordal graphs, perfectly orderable graphs, etc) are perfectly contractile, and for some of them (Meyniel graphs, weakly chordal graphs) the coloring algorithm based on even-pair contractions is the most efficient that is known so far.

Everett and Reed conjectured that a graph is PC if and only if it contains no odd hole, no antihole, and no odd prism (two disjoint triangles with three disjoint chordless odd paths between them).

Maffray and Trotignon proved a weaker form of this conjecture, also due to Everett and Reed: if a graph contains no odd hole, no antihole, and no prism, and it is not a clique, then the graph admits an even pair whose contraction yields a graph with no odd hole, no antihole, and no prism. The proof is an algorithm to find such a pair. Here is a link to a preprint[1].

The same authors found a polynomial-time algorithm to decide if a graph belongs to that class (the class of graphs with no odd hole, no antihole, and no prism). Here is a link to a preprint[2].

Contributed by Frédéric Maffray

F.4.a Perfectly Contractile Graphs and the Decomposition Method. The following conjecture, due to Everett and Reed attempts to characterize perfectly contractile graphs.
Perfectly Contractile Graph Conjecture (PCGC) *A graph is perfectly contractile if and only if it does not contain an odd hole, an antihole nor an odd prism.*

We propose to investigate the PCGC and a possible construction of a polynomial-time recognition algorithm for perfectly contractile graphs, through the decomposition method.

The decomposition method is based on a decomposition theorem of the following form, for the class of graphs $\mathcal{C}$ we want to analyse.
Decomposition Theorem *If $G \in \mathcal{C}$, then G is either basic or it contains certain types of cutsets.*

Basic stands for a certain "simple" subclass of $\mathcal{C}$.

The idea of a decomposition based recognition algorithm for the class $\mathcal{C}$ is as follows. In a connected graph G, a node set (or an edge set or a combination of the two) is a *cutset* if its removal disconnects G into two or more connected components. From these components *blocks of decomposition* are constructed by adding some more nodes and edges. A decomposition is *$\mathcal{C}$-preserving* if it satisfies the following: G belongs to $\mathcal{C}$ if and only if all the blocks of decomposition belong to $\mathcal{C}$. A decomposition based recognition algorithm takes an input graph G and decomposes it using $\mathcal{C}$-preserving decompositions into a polynomial number of basic blocks, which are then checked, in polynomial time, whether they belong to $\mathcal{C}$.

Such a construction of blocks works nicely for clique cutsets. A node set S is a *star cutset* of a graph G if its removal disconnects G and S contains a node that is adjacent to all the other nodes of S. With the usual construction of blocks for the node cutsets, the star cutset decomposition is not preserving for the class of perfectly contractile graphs.

A generalization of star cutsets is obtained as follows. 1-Amalgams are defined and used in for the construction of a recognition algorithm for Meyniel graphs. A graph G has a *1-amalgam* if its vertex set can be partitioned into sets V_1, V_2 and K (where K is possibly empty) in such a way that:

- for $i = 1, 2$, $|V_i| \geq 2$ and V_i contains a nonempty set A_i;
- every node of A_1 is adjacent to every node of A_2 and these are the only adjacencies between the nodes of V_1 and the nodes of V_2; and
- if $K \neq \emptyset$, then it induces a clique, and every node of K is adjacent to every node of $A_1 \cup A_2$.

A graph G has a *2-join* if its node set can be partitioned into sets V_1 and V_2 so that for $i = 1, 2$, V_i contains disjoint nonempty sets A_i and B_i, and the following properties hold:

[1] http://www-leibniz.imag.fr/LesCahiers/2002/Cahier67/ResumCahier67.html
[2] http://www-leibniz.imag.fr/NEWLEIBNIZ/LesCahiers/Cahier106/ResumCahier106.html

- every node of A_1 (resp. B_1) is adjacent to every node of A_2 (resp. B_2), and these are the only adjacencies between the nodes of V_1 and the nodes of V_2;
- for $i = 1, 2$, let $\mathcal{P}_i$ be the set of all chordless paths in $G[V_i]$ with one endnode in A_i, the other endnode in B_i, and no intermediate node in $A_i \cup B_i$. For $i = 1, 2$, $\mathcal{P}_i \neq \emptyset$ and $G[V_i]$ is not isomorphic to a path in $\mathcal{P}_i$.

Let $\mathcal{C}^{pc}$ denote the class of perfectly contractile graphs.

We have the following conjectures.

Conjecture 1-Amalgam decomposition is $\mathcal{C}^{pc}$-preserving.

Conjecture 2-Join decomposition is $\mathcal{C}^{pc}$-preserving.

Conjecture No minimal non perfectly contractile graph has a star cutset.

Note that the PCGC implies all three of these conjectures.

Contributed by Claudia Linhares Sales.

F.5 Possible Structure Theorem for Berge Graphs

Conjecture For every even-pair- free Berge graph, either it or its complement is the line graph of a bipartite graph, or has a 2-join.

A direct proof (if there is one) might give a shorter proof of the SPGC.

Contributed by Robin Thomas

A general even pair is not a composition- a non-Berge graph may become Berge by contracting an even pair.

How much more do we need in order to be able to find a construction for Berge graphs using even pairs?

Question Give a sufficient condition such that if x, y is an even pair satisfying the condition then G is Berge if and only if the graph obtained from G by contracting x, y is Berge.

Contributed by Paul Seymour

F.6 Odd holes and odd walks

Given a graph $G = (V, E)$ and two vertices a, b can the following problem be solved in polynomial time ?

Find a triangle-free odd (a, b)-walk in G, where a walk can also contain repetitions of edges, and triangle-free means that the vertex-set of the walk does not contain any triangle (but can contain an odd hole).

A polynomial algorithm for this problem would specialize to a polynomial algorithm for finding odd holes (and even pairs in odd-hole-free graphs). Bienstock proved that it is NP-hard to find odd holes containing a given $a \in V$. However, a triangle-free odd (a, a)-walk exists in a 2-connected graph if and only if there exists an odd hole in G (not necessarily containing a).

Contributed by András Sebő and Nicolas Trotignon

G.1 2-divisible Graphs

A 2 division of a graph is a partition of its vertex set into two parts neither of which contains a maximum clique. Hoang and McDiarmid call a graph 2-divisible if all of its induced subgraphs permit 2-division. Every perfect graph is 2-divisible, and an odd hole has no 2-division. Thus 2-divisible graphs are odd-hole-free.

Hoang and McDiarmid made the following conjectures:

(1) G is 2-divisible iff G is odd-hole-free

Contributed by Bruce Reed

G.2 Clique Coloring of Perfect Graphs

It has been asked by Duffus et al [92e:06009] whether the clique hypergraph of perfect graphs is colorable with a constant number of colors. Several results followed showing that for some classes of perfect graphs this constant is actually 2 or 3 :

They proved it for comparability and cocomparability graphs, Bacsó et al proved the same for some more perfect graphs, and noticed that 'almost all' perfect graphs are 3-clique-colorable (applying Prömel and Steger's result (Probability and Computation, 1, 1992)); they ask whether this bound holds for all perfect graphs:

Can the vertex-set of any perfect graph be partitioned into three classes, so that no (inclusionwise) maximal clique (of size > 1) is included in any of these ? Is the same true already for odd hole free graphs ?

If the clique-hypergraph of a graph and of all of its subgraphs can be colored with k colors, that is, there exists a partition of the vertex-set into k parts so that none of them contains an (inclusionwise) maximal clique, then Hoàng and McDiarmid [2002j:05110] say the graph is *strongly k-divisible*. This is indeed a sharpening of k-divisibility where 'maximal' is replaced by 'maximum' (cardinality). In these terms the above conjecture states that perfect graphs are strongly 3-divisible. We formulate another problem in this language:

Can strong 2-divisibility be decided in polytime ?

A major difficulty with the coloration of the maximal clique hypergraph is that it is NP-hard to decide whether a partition of the vertices is a clique-coloration, and even in very particular classes of perfect graphs.

Contributed by Myriam Preissmann and András Sebő

G.3 Recognition of Odd-Hole-Free Graphs

Find a polynomial algorithm to recognize odd-hole-free graphs.

Contributed by Chinh Hoang

G.4 Even-Hole-Free Graphs

A k-division of a graph G is a partition of its vertex-set into sets $V_1, \ldots, V_k$ such that no V_i contains a largest clique of G. A graph is k-divisible if each of its induced subgraphs with at least one edge has a k-division.

Conforti, Cornuéjols, Kapoor and Vušković designed a polynomial algorithm to recognize even-hole-free graphs.

Conjecture 2 (Hoàng). Every even-hole-free graph is 3-divisible.

Conjecture 3 (Hoàng). If G is an even-hole-free graph then $\chi(G) \leq 2\omega(G) - 1$.

Hayward and Reed proposed the following.

Conjecture 4 (Hayward and Reed). An even-hole-free graph contains a vertex whose neighbourhood can be partitioned into two cliques.

Conjecture 4 implies Conjecture 3 which in turn implies Conjecture 2. Let G be an even-hole-free graph. Conjecture 4 implies that each induced subgraph H of G has a vertex of degree at most $2\omega(H) - 2$, and therefore $\chi(G) \leq 2\omega(G) - 1$. Since any graph F is $\frac{\chi(F)}{\omega(F)-1}$-divisible, G is 3-divisible.

Contributed by Chinh Hoang.

G.5 Even-hole-free circulants,

Given interer $k \geq 1$ and $m \geq 0$, let us introduce a graph $G(k,m) = (V,E)$ with circular symmetry as follows: $V = \mathbf{Z}_n = \{1, ..., n\}$, where $n = k(2m+1)$, and $(i,j) \in E$ iff

$$i - j + t\,(2m+1) = 0 \;or\; +1 \;or\; -1 \;\; (mod\; n)$$

for some integer t. (For convenience, the loops $i = j$ are included.) E.g. if $k = 5, m = 2$ then $n = 25$ and $(i,j) \in E$ iff $i - j (mod\; 25) \in \{4,5,6; 9,10,11; 14,15,16; 19,20,21; 24,0,1\}$.

It is not difficult to check that $G(k,m)$ has no even holes, (in fact, it can only have holes of length $2m+1$); furthermore,

$\omega(G(k,m)) = 2k,\;\; 2k + \lceil k/m \rceil \leq \chi(G(k,m)) \leq 2k + \lceil k/m \rceil + 1,$

and $G(k,m)$ satisfies Conjectures 2,3,4 from the section "Even-Hole-Free Graphs".

Conjecture. Every non-empty even-hole-free circulant is isomorphic to a $G(k,m)$.

Contributed by Diogo Andrade, Endre Boros, and Vladimir Gurvich

G.6 beta-perfect graphs

Definition $\beta(G) = max_{G' \subseteq G}(mindeg(G') + 1)$

(the maximum is taken over all induced subgraphs).

Note that $\beta(G) \geq \chi(G)$.

A graph G is called *β-perfect* if $\beta(G') = \chi(G')$ for all induced subgraphs G' of G.

Question Characterize β-perfect graphs.

Even holes and graphs obtained from odd holes by replacing every vertex by two adjacent vertices preserving the adjacencies in the hole (so every edge is replaced by a K_4) are known not to be β-perfect. So a β-perfect graph has no induced subgraph of those types.

Contributed by Bruce Reed

Chapter H: Partitionable Graphs

H.1 Perfect, Partitionable, and Kernel-Solvable Graphs

Given a graph G = (V,E), assign to every its edge e = (u,v) either the directed arc [u,v), or [v,u), or both. The obtained directed multi-graph D = (V,A) is called an orientation of G .

A vertex-subset K of V is called a KERNEL if K is (i) independent and (ii) absorbant, that is for each u from V K there is an arc [u,v) in A such that v in K.

Orientation D is called clique acyclic if every clique of G has a kernel in D. Orientation D is called kernel-less if it has no kernel.

Graph G is called kernel-solvable if every its clique-acyclic orientation has a kernel.

Berge and Duchet (1983) conjectured that (BD1) Perfect graphs are kernel-solvable, and (BD2) Kernel-solvable graphs are perfect.

BD1 was proved by Boros and Gurvich (1996) and by Holzman and Aharoni (1998), BD2 follows from the SPGC but no independent proof is known.

An orientation D of a PARTITIONABLE graph G is called UNIFORM if D is

(0) kernel-less and clique acyclic;

(a) for each maximum stable set S there exists a unique unabsorbed vertex v(S);

(b) v(S) belongs to the vis-a-vis clique C(S) of S;

(c) for each vertex v there exists a unique maximal stable set S(v) which does not absorb v .

Sebo (1998) proved that every kernel-less and clique-acyclic orientation of a minimal imperfect graph is uniform.

Conjecture. Each partitionable graph has a uniform orientation.

This, if true, implies BD2

Contributed by Boros and Gurvich

H.2 Partitionable graphs and odd holes

Let $G = (V, E)$ be a graph, and α, ω arbitrary natural numbers. Assume that $a, v, b \in V, av \in E, vb \notin E$ are such that $G - a$, $G - v$ have a partition of size α into ω-cliques, and $G - v$, $G - b$ have a partition of size ω into α-stable sets. It is easy to show then that G is not perfect.

Given the four partitions, find an odd hole or an odd antihole.

This would imply SPGC.

This contains the following:

Given a partitionable graph (with all the partitions), find an odd hole or an odd antihole. Does the fact that the partitions are given make the task easier ?

Contributed by András Sebő

H.3 A Property of Partitionable Graphs

We say that a graph satisfies the "no-week-pair" property if each pair of vertices of a graph is either in a maximum clique or in a maximum stable set

Conjecture: If a partitionable graph satisfies the "no-week-pair" property then the graph is an odd hole or an odd anti-hole.

The following graph is a counter-example to this conjecture:

take a 17-gon and and add all 3- 4- and 5-chords.

(This 17-vertex graph is the only known partitionable graph without a small transversal.)

It's still interesting if there are other such graphs (i.e. partitionable graph satisfying "no-week-pair" property). If there are then it would be nice to characterize them.

Contributed by Ara Markosian

H.4 Small Transversals in Partitionable Graphs

Following Bland, Huang, and Trotter [80g:05034];[86e:05075] a graph is called *partitionable* if, for some r and s, it has $rs+1$ vertices and, no matter which vertex is removed, the set of the remaining rs vertices can be partitioned into r pairwise disjoint cliques of size s and also into s pairwise disjoint stable sets of size r. Odd holes and odd antiholes are partitionable; many additional partitionable graphs have been constructed by V. Chvtal, R. L. Graham, A. F. Perold, and S. H. Whitesides [81b:05044].

A *small transversal* in a graph G is a set of $\alpha(G)+\omega(G)-1$ vertices which meets all cliques of size $\omega(G)$ and all stable sets of size $\alpha(G)$. The following problem is an easier variation on a conjecture contributed to the 1993 workshop on perfect graphs[3] by Gurvich and Temkin and on two conjectures proposed by Bacso, Boros, Gurvich, Maffray, and Preissmann [2000h:05116].

Conjecture. Every partitionable graph G with $\alpha(G) > 2$ and $\omega(G) > 2$ has a small transversal or else contains a hole of length five.

One of the milestones in the development of our understanding of perfect graphs was the theorem of Lovasz [46 #8885], asserting that every minimal imperfect graph G has precisely $\alpha(G)\omega(G)$ +1 vertices. This theorem implies that every minimal imperfect graph is partitionable and that – as pointed out by Chvatal [86h:05091] – no minimal imperfect graph contains a small transversal. It follows that a proof of the conjecture would provide another proof of the Strong Perfect Graph Theorem.

A partitionable graph without a small transversal has been constructed by by Chvatal, Graham, Perold, and Whitesides (op.cit.). Its vertices are $0, 1, ..., 16$; vertices i and j are adjacent if and only if $|i-j|$ *mod* 17 is one of $1, 3, 4, 5, 12, 13, 14, 16$.

Ara Markosian claims here[4] that this is the only known partitionable graph without a small transversal. One of the many holes of length five in this graph is $1-4-8-12-15-1$

Additional information on related results and problems can be found here[5]

Contributed by Vasek Chvatal

Chapter I: The Imperfection Ratio

A *demand vector* for a graph G with node set V is a non-negative vector of integers indexed by nodes of G. Given a graph G and a demand vector $x = (x_v \ : \ v \in V(G))$ a coloring of the pair $(G.x)$ is an assignment of a set of x_v colors to each node v of G such that two adjacet nodes receive disjoint sets of colors. Coloring the pair (G, x) corresponds exactly to usual proper coloring of the replicated graph G_x. Let G be a graph.Define the *imperfection ratio* of G by setting

$$imp(G) = max_x\{\frac{\chi_f(G_x)}{\omega(G_x)}\}$$

[3] http://dimacs.rutgers.edu/"

[4] http://www.aimath.org/WWN/perfectgraph/articles/html/46a/

[5] http://www.cs.rutgers.edu/~chvatal/perfect/problems.html#partitionable

where the maximum is over all non-zero integral demand vectors x and $\chi_F(G_x)$ and $\omega(G_x)$ are the fractional chromatic number and the clique number of the replicated graph G_x respectively. (The ratios on the right-hand side above do indeed attain a maximum value). Observe that $imp(G) \geq 1$.

The next result establishes the connection of the imperfection ratio with perfection.

Proposition For any graph G, $imp(G) = 1$ iff G is perfect.

One of the motivations for studying graph imperfection is its connection to frequency assignment. With this motivation in mind one is particularly interested in bounding the imperfection ratio for graphs of relevant graph classes. One relevant graph class are for example unit disk graph, that is graphs the node set of which can be represented by unit size disks such that two nodes are adjacent if and only if the corresponding disks intersect. It is known that $imp(G) \leq 2.155$ for any unit disk graph G, and that there exists a unit disk graph G with $imp(G)$ arbitrarily close to $3/2$.

Conjecture: For any unit disk graph G, $imp(G) \leq 3/2$.

A subclass of unit disk graphs are the induced subgraphs of the triangular lattice. These graphs are of importance for channel assignment, since a pattern of omni-directional transmitters in two dimensions laid out like nodes of the triangular lattice in the plane give good coverage.

Let us now consider an induced subgraph G of the triangular lattice T. Such a graph has a natural 3-coloring. It is possible to have $\omega(G_x) = 3$ and $\chi(G_x)$=4 for such a graph G. There is a polynomial-time coloring algorithm (by McDiarmid and Reed) which shows that for such a graph G we always have

$$\chi(G_x) \leq \frac{4\omega(G_x) + 1}{3}$$

Thus $imp(G) \leq \frac{4}{3}$ for any finite induced subgraph G of the triangular lattice T. The 9-cycle C_9 is an induced subgraph of the triangular lattice T. For any integer k the graph obtained from C_9 by replicating each of its nodes k times has clique number $2k$ and chromatic number $\lceil \frac{9k}{4} \rceil$ Is the ratio $\frac{9}{8}$ of chromatic number to clique number asymptotically the worst (greatest) possible with large demands? This questions may be rephrased in terms of $imp(G)$ as follows:

Conjecture For any induced subgraph G of the triangular lattice T, we have $imp(G) \leq \frac{9}{8}$

This would imply the following weaker and perhaps more tractable conjecture:

Conjecture If G is a trianlgle-free induced subgraph of the triangular lattice then $|V(G)| \leq \frac{9}{4}\alpha(G)$ (where $\alpha(G)$ is the size of the maximum stable set in G), and indeed $\chi_f(G) \leq \frac{9}{4}$

To get a feeling for the behavior of the imperfection ratio, we mention the following elementary decomposition result: if G is composed of two parts G_1 and G_2 that are either disjoint or overlap in a clique, then

$$imp(G) = max\{imp(G_1), imp(G_2)\}$$

The following is another property of imperfection which is desirable for any graph invariant related to perfection:

Proposition For any graph G $imp(G) = imp(G^c)$ where G^c denotes the completemt of G.

Another graph class of interest are planar graphs. It follows from the 4-colour theorem that $imp(G) \leq 2$ for any planar graph G. It is known that one can improve a little on this but we conjecture that the true value is 3/2.

Conjecture: For any planar graph G, $imp(G) \leq 3/2$.

Another area of interest are complexity issues concerning imperfection. It is known that it is NP hard to determine the imperfection ratio. One open question is whether for a fixed k one can determine in polynomial time whether a graph G satisfies $imp(G) \leq k$. For the special case of $k = 1$, this is the recognition problem for perfect graphs. Another open question is how hard is it to approximate the imperfection ratio of a graph.

Initial contribution by Bruce Reed, extended by Stefanie Gerke

Chapter J: Integer Programming

J.1 Partitionable Graphs as Cutting Planes for Packing Problems?

As is well known, the strong perfect graph conjecture has been of interest to the integer programming community as well as the combinatorics community. Now that the SPGC has been established I wanted to mention another problem which may shed light on cutting plane approaches to packing problems. Sewell (and later Bram Verweij and Aardel) gave successful cutting plane codes for solving maximum stable set problems in sparse graphs by adding odd hole inequalities as they are violated by fractional solutions. Moura studied this approach for some problems arising in design theory, but met with much less success since the graph instances were much more dense (and hence odd hole inequalities were unlikely to be violated).

Can we extend the class of odd cycle inequalities to the class of partitionable graph inequalities

$$\sum_{v \in I} x_v \leq \alpha(I)$$

for each partitionable subgraph. I.e., can we develop algorithms to solve the separation problem for this class of inequalities. One positive result is that partitionable graphs themselves can be recognized in polynomial time (another problem is to find a combinatorial algorithm to recognize partitionable graphs).

Contributed by Bruce Shepherd.

J.2 Feasibility/Membership Problem For the Theta Body

Find a polynomial time algorithm to solve the (exact) feasibility/membership problem for the theta body.

Contributed by Bruce Shepherd

Chapter K: Balanced Graphs

Definition A graph is *balanced* if every induced cycle has length $0 (mod 4)$.

Clearly balanced graphs are bipartite.

A balanced graph is *basic* if all its vertices on one side of the bipartition have degree at most 2 or G contains a hole H such that the vertices of $G \setminus H$ induce a complete bipartite graph.

Here are two conjectures concerning balanced graphs.

Conjecture 1 (Conforti, Cornuejóls,Rao) Every balanced graph is either basic or has a 2-join or a skew-partition.

Conjecture 2 (Conforti, Rao) In every balanced graph, there exists an edge that can be deleted, so that the resulting graph remains balanced.

Contributed by Gerard Cornuejòls

K.1 Balanced circulants

Given interer $k \geq 1$ and $m \geq 1$, let us introduce a graph $G(k, m) = (V, E)$ with circular symmetry as follows: $V = \mathbf{Z}_n = \{0, 1, ..., n-1\}$, where $n = 4km$, and $(i, j) \in E$ iff

$$i - j + 4mt = +1 \ \ or \ -1 \ \ (mod\, n)$$

for some integer t. E.g. if $k = 5, m = 2$ then $n = 40$ and $(i, j) \in E$ iff $i - j(mod\ 40) \in \{7, 9; 15, 17; 23, 25; 31, 33; 39, 1\}$.

It is not difficult to check that $G(k, m)$ is balanced, i.e. it has no odd cycles, nor $(4i + 2)$-holes. In fact, it may only contain holes of length 4 and $4m$. Moreover, every $(4i+2)$-cycle has at least two chords. Hence, $G(k, m) - e$ is still balanced for any edge e, in agreement with Conjecture 2 from the section "Balanced Graphs". Conjecture 1 of the same section also holds for $G(k, m)$. Indeed, if $k > 1$ then $S = \{0; 4mi+1, 4mi-1 \mid i = 1, \ldots, k\}$ is a star cutset: 0 is an isolated vertex in $\bar{G}[S]$, while $4mj$ is an isolated vertex in $G[V \setminus S]$ for every $j = 1, \ldots, k$; and if $k = 1$ then $G(k, m)$ is $4m$-cycle, that is a basic graph

CONJECTURE. Every non-empty balanced circulant is isomorphic to a $G(k, m)$.

Contributed by Diogo Andrade, Endre Boros, and Vladimir Gurvich

Chapter L: P4-structure and Its Relatives

The P_4*-structure* of a graph G is the 4-uniform hypergraph whose vertex-set V is the vertex-set of G and whose hyperedges are the subsets of V that induce P_4's (chordless paths on four vertices) in G. The graph property of being Berge can be formulated directly in terms of P_4-structure: a graph is Berge if and only if its P_4-structure contains no induced *odd ring*, meaning a 4-uniform hypergraph with vertices

$$u_0, u_1, \ldots, u_{k-1},$$

where k is odd and at least five, and with the k hyperedges

$$\{u_{i+1}, u_{i+2}, u_{i+3}, u_{i+4}\},$$

where the subscripts are taken modulo k. (The "if" part is trivial and the "only if" part is easy, even though a little tedious; see [MR 86j:05119] for a sketch of the argument.)

Can the Chudnovsky-Robertson-Seymour-Thomas decomposition theorem for Berge graphs be reformulated directly in terms of P_4-structure?

The five classes of basic graphs featured in the theorem lend themselves nicely to such reformulations: there are classes $\mathbf{C}$ of 4-uniform hypergraphs such that

- the P_4-structure of every basic graph belongs to $\mathbf{C}$,
- no 4-uniform hypergraph in $\mathbf{C}$ contains an induced odd ring,
- membership in $\mathbf{C}$ can be tested in polynomial time.

One such class is defined in terms of a certain directed graph, $D_6(H)$, associated with every 4-uniform hypergraph H: $\mathbf{C}$ consists of all 4-uniform hypergraphs H such that

- H contains no induced ring with five vertices and
- all strongly connected components of $D_6(H)$ are bipartite.

The vertices of $D_6(H)$ are all the ordered 6-tuples

$$(u_1, u_2, u_3, u_4, u_5, u_6)$$

of distinct vertices of H such that the sub-hypergraph of H induced by the set

$$\{u_1, u_2, u_3, u_4, u_5, u_6\}$$

consists of the three hyperedges

$$\{u_1, u_2, u_3, u_4\},\ \{u_2, u_3, u_4, u_5\},\ \{u_3, u_4, u_5, u_6\};$$

there is a directed edge from vertex

$$(u_1, u_2, u_3, u_4, u_5, u_6)$$

of $D_6(H)$ to vertex

$$(v_1, v_2, v_3, v_4, v_5, v_6)$$

of $D_6(H)$ if and only if

$$v_1 = u_2,\ v_2 = u_3,\ v_3 = u_4,\ v_4 = u_5,\ v_5 = u_6.$$

Trivially, membership in $\mathbf{C}$ can be tested in polynomial time; trivially, no 4-uniform hypergraph in $\mathbf{C}$ contains an induced odd ring; a proof that the P_4-structure of every basic graph belongs to $\mathbf{C}$ is easy, even though a little tedious (here[6] is a sketch of the argument).

The four kinds of structural faults featured in the decomposition theorem suggest the following three problems.

Problem 1: Find a class $\mathbf{C}_1$ of 4-uniform hypergraphs such that

- the P_4-structure of every graph with a 2-join belongs to $\mathbf{C}_1$,
- no odd ring belongs to $\mathbf{C}_1$,
- $\mathbf{C}_1$ belongs to $\mathbf{NP}$.

Problem 2: Find a class $\mathbf{C}_2$ of 4-uniform hypergraphs such that

- the P_4-structure of every graph with an M-join belongs to $\mathbf{C}_2$,
- no odd ring belongs to $\mathbf{C}_2$,
- $\mathbf{C}_2$ belongs to $\mathbf{NP}$.

Problem 3: Find a class $\mathbf{C}_3$ of 4-uniform hypergraphs such that

- the P_4-structure of every graph with a balanced skew partition belongs to $\mathbf{C}_3$,
- no odd ring belongs to $\mathbf{C}_3$,
- $\mathbf{C}_3$ belongs to $\mathbf{NP}$.

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[6] http://www.cs.rutgers.edu/∼chvatal/perfect/problems.html#basic

Chính Hoàng introduced additional graph functions that are invariant under complementation and determine whether or not a graph is Berge: these are the *co-paw-structure* ([MR 2001a:05065]), the *co-*C_4*-structure* (Discrete Math. **252** (2002), 141–159), and the *co-*P_3*-structure* (to appear in SIAM J. Discrete Math.). Can the decomposition theorem be reformulated directly in terms of one of Hoàng's invariants?

Contributed by Vašek Chvátal