Problems from the ARCC Workshop “Gaps Between Primes”

December, 2005

Notes by Angel Kumchev.

1. Evidence in support of the Elliott–Halberstam conjecture.

- J.B. Conrey asked what evidence exists in support of the Elliott–Halberstam conjecture (EH). He also suggested that the search for further such evidence may shed some light on the conjecture or even lead to the resolution of an “easy” case, which in turn would suffice to establish the existence of infinitely many bounded gaps between primes by the GPY method. Similar comments were also made by G. Harcos in his contributed statement.
- R.C. Vaughan observed that the Barban–Davenport–Halberstam theorem is consistent with EH.
- A. Granville noted that the Cramér probabilistic model also supports EH.
- In an earlier lecture on the Bombieri–Vinogradov theorem and the large sieve, R.C. Vaughan had remarked that progress toward EH would most likely exploit cancellation among the values of $\bar{\chi}(a)$ in the formula

$$\psi(x; q, a) = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \bar{\chi}(a)\psi(x, \chi). \tag{1}$$

  J.B. Conrey asked whether known results on the asymptotic of the sum

$$\sum_{\chi \bmod q}{}^{*} \chi(a)|L(1/2, \chi)|^2$$

  suggest that the desired cancellation in (1) may be nonexistent.
- K. Soundararajan noted that the results mentioned by Conrey can be explained via the Euler product of $L(s, \chi)$ and that there is no such phenomenon in the case of $\psi(x, \chi)$. Therefore, it seems that Vaughan’s remark remains valid until further notice.

2. On the large sieve, I.

- H. Helfgott asked whether the standard large sieve inequality for the exponential sum

$$S(\alpha) = \sum_{n=M+1}^{M+N} a_n e(\alpha n)$$

  is best possible for coefficient sequences with $a_n \in \{0, 1\}$. He noted that the usual proof that the large sieve is essentially best possible uses a sequence that does not belong to this class.
- R.C. Vaughan said that the answer to this question is most likely in the affirmative.

3. On the large sieve, II. Y. Motohashi proposed to try to improve the large sieve by restricting the moduli. He mentioned that D. Wolke has results in that direction for prime moduli.

4. $|t\zeta(1+it)| \geq 1$.

- J. Pintz asked whether it is true that

$$|t\zeta(1+it)| \geq 1 \qquad \text{for all } t \in \mathbb{R}.$$

  He noted that this inequality holds when $|t| \geq t_0$ and when $|t| \leq \delta$. A complete proof would simplify the GPY method at certain points.

- R.C. Vaughan recalled being asked a similar question in the past and that he was able to answer it in the affirmative. He said that he thinks that he has the solution written somewhere in his files.

5. The multiplicative twin-prime problem of Elliott.

- In 2003 P.D.T.A. Elliott proved that for every fixed integer $a > 1$, the equation

$$a^k = \frac{p+1}{q+1} \tag{2}$$

  has infinitely many solutions in primes p, q and integers k with $2 \leq k \leq \log q$. K.B. Ford suggested to try to adapt the GPY method to reduce the range for k in Elliott's result.

- J. Pintz remarked that an upper bound of the form $k \ll (\log q)^{1/2+\epsilon}$ seemed a reasonable goal at that stage of our understanding of the GPY method.

- Several participants noted that the adaptation of the GPY method to this problem would require an analogue of Gallagher's result on the average size of the singular series for prime k-tuples. A. Granville noted that the problem of understanding the singular series of (2) may be of independent interest.

6. Long chains of primes.

- K.B. Ford proposed the following problem: Let x be large and consider a sequence of primes $p_1, \ldots, p_k \leq x$ such that for all $j = 2, \ldots, k$,

$$p_j = m_j p_{j-1} + 1 \qquad \text{for some } m_j \in \mathbb{N}.$$

  Obviously, $k = O(\log x)$. Prove or disprove that $k = o(\log x)$.

- H.L. Montgomery noted that this question reminded him of a problem of Erdös: Show that there are finitely many n such that $n - 2^k$ is prime for all integers k with $2^k < n$.

- R.C. Vaughan recalled that Erdös thought that $n = 105$ is the largest such n.

7. Small gaps between primes in thin sequences.

- A. Kantorovich asked whether it is possible to prove the existence of small gaps between primes from the sequence

$$\{p \leq x \mid p = [n \log n] \text{ for some } n \in \mathbb{N}\}.$$

  This is a "thin" set of primes:

$$\#\{p \leq x \mid p = [n \log n] \text{ for some } n \in \mathbb{N}\} \sim x(\log x)^{-2}.$$

- Several participants noted that if one is interested in this question, then one may also investigate the analogous question for Piatetski-Shapiro primes.

8. Mersenne composites.

- A. Granville asked whether it is possible to use the GPY method to prove (under the assumption of EH, if necessary) that there are infinitely many composites of the form $2^p - 1$ (Mersenne composites).
- C. Elsholtz noted that there may exist related work by M.R. Murty under the assumption of Artin's primitive root conjecture.

9. Second order differences between primes.

- T.D. Wooley proposed the following problem: How often are the second differences
$$p_{n+2} - 2p_{n+1} + p_n$$
small (large)? He noted that Erdös has proved that
$$|p_{n+2} - 2p_{n+1} + p_n| \geq (1 + \delta) \log p_n \tag{3}$$
infinitely often.
- A. Balog added that he had thought about the problem in the past and had convinced himself that the left side of (3) is infinitely often as large as the largest known gap between consecutive primes.

10. Differences between primes and pair correlation.

- Let $F(\alpha)$ be Montgomery's pair correlation function. In 1982 D.R. Heath-Brown proved that, under RH and Montgomery's Pair Correlation Conjecture,
$$\liminf_{n\to\infty} \frac{p_{n+1} - p_n}{\log p_n} = 0.$$
T.H. Chan asked whether it is possible to prove a converse result (which, of course, would be unconditional post-GPY).
- J. Pintz expressed serious doubt.
- In the same paper, Heath-Brown proved also (under the same assumptions) that
$$p_{n+1} - p_n \ll \sqrt{p_n \log p_n}.$$
J.B. Conrey asked whether it is possible to use random matrix theory (RMT) to improve further on this result.
- K. Soundararajan felt quite strongly that RMT should not help in this problem.

11. Differences between primes and $\zeta(s)$. Traditionally, analytic information about the zeta-function has been used to derive upper bounds for the differences between consecutive primes. Y. Motohashi asked whether this process can be reversed. For example, what (if anything) can be said about $\zeta(s)$ in the critical strip under the assumption that $p_{n+1} - p_n \ll_\epsilon p_n^\epsilon$?

12. Limits of the GPY method.

- H. Helfgott noted that it seems that the GPY method is by nature a lower bound method and will most likely never yield an asymptotic formula for the number of primes with a given spacing. He asked those who were well-versed in the GPY method whether this is indeed so and also whether the GPY method is capable of producing a right-order lower bound.
- J. Pintz replied that an asymptotic formula of any kind appears to be out of the reach of the method. He also noted that the method could potentially yield a "right-order lower bound", but that that will involve considerable technical difficulties. Some ideas in that direction were sketched in C. Yildirim's talk. It should be noted that in this context, a "right-order lower bound" means that one can show that a positive proportion of the k-tuples we consider contain at least two primes.

13. Distances between Gaussian primes. Several members of the audience asked whether the GPY method can be adapted to study small distances between Gaussian primes. T.D. Wooley remarked that such an adaptation should be straightforward.

14. Small values of linear forms at prime arguments. A. Balog proposed the following problem: Show that the difference $|ap - q|$, where p, q are primes and $a > 1$ is a fixed integer, can be small. More generally, consider the case when a is real, $a \neq 1$.

15. Alternative weights in the GPY method.

- The GPY method uses the weights

$$\Lambda_R(n; \mathcal{H}) = \sum_{d | P(n; \mathcal{H})} \lambda_d, \qquad \lambda_d = \mu(d) \frac{(\log_+ (R/d))^{k+l}}{(k+l)!},$$

  where $P(n; \mathcal{H}) = (n + h_1) \cdots (n + h_k)$ and $\log_+ x = \max(0, \log x)$. These λ_d's are Selberg-type sieve weights. J.B. Conrey asked whether it is possible to strenghten the method quantitatively by using a different choice of the λ_d's. For the duration of the workshop, there was an ongoing discussion of this question. Much of it focused on weights of the form

$$\lambda_d = \mu(d) \bigg\{ \prod_{p | d} f \left(\frac{\log p}{\log R} \right) \bigg\} Q\big(\log_+ (R/d) \big),$$

  where f and Q are smooth functions at our disposal.

- R.C. Vaughan noted that it may be worth investigating how the method is affected by changing Selberg's weights to other traditional sieve weights.

- Another suggestion was to try to combine upper and lower sieves a la Chen. K. Soundararajan commented that there will be an issue calculating the negative contribution.

16. Possible improvements on the Bombieri–Vinogradov theorem.

- J.B. Friedlander posed several questions in his talk:
  - Show that for some fixed $\theta > 1/2$ and $D = x^\theta$, one has

$$\sum_{\substack{d \leq D \\ (d,a)=1}} \mu(d) \left(\psi(x; d, a) - \frac{x}{\phi(d)} \right) \ll x (\log x)^{-A}.$$

  - Prove anything beyond the Bombieri–Vinogradov theorem for the sum

$$\sum_{d \leq D} \max_{\substack{(a,d)=1 \\ |a| < (\log x)^{2005}}} \left| \psi(x; d, a) - \frac{x}{\phi(d)} \right|.$$

  - For any "reasonable" weights λ_d, show, for some fixed $\theta > 1/2$ and $D = x^\theta$, that

$$\sum_{\substack{d \leq D \\ (d,a)=1}} \lambda_d \left(\psi(x; d, \overline{a}) - \frac{x}{\phi(d)} \right) \ll x (\log x)^{-A}.$$

    Here, $\overline{a}$ is defined modulo d by $a\overline{a} \equiv 1 \pmod{d}$.

  All of these were motivated by the limitations of the Bombieri–Friedlander–Iwaniec method for primes in arithmetic progressions to large moduli.

- Several participants proposed to investigate theorems of the Bombieri–Vinogradov type for averages of arithmetic functions other than $\Lambda(n)$ (cf. known results for the divisor functions $d(n)$ and $d_3(n)$).
- A. Granville asked whether it is possible to formulate a reasonable conjecture about zeros of L-functions that would yield EH(θ) with a fixed $\theta > 1/2$.

17. Elliott–Halberstam from bounded gaps.

- During the workshop, much attention focused on how to prove EH(θ) with $\theta > 1/2$, so we can deduce the existence of bounded gaps between primes. V. Blomer asked whether it is possible to go the opposite way and derive an improvement on the Bombieri–Vinogradov theorem from a "strong" quantitative result on bounded gaps between primes.
- R.C. Vaughan commented that such an improvement would encode so much information about L-functions that the hypothesis would have to be extremely strong.
- K. Soundararajan added that the "strong quantitative result" could be as strong as the Hardy–Littlewood k-tuple conjecture with a sharp error term, say $O(N^{1/2+\epsilon})$.

18. From k-tuples to $(k+1)$-tuples.

- J.B. Conrey asked whether it is possible to deduce the existence of prime $(k+1)$-tuples from a version of the Elliott–Halberstam conjecture for prime k-tuples.
- T.D. Wooley and several others pointed out that it is not quite clear what the proper statement of EH for prime k-tuples is. It was mentioned that there are results by Balog, Kawada, and Mikawa of the form

$$\sum_{q\le Q} \max_{(a,q)=1} \sum_{\substack{1\le b\le q\\(b,q)=1}} \left| \sum_{n\le x/q} \varpi(qn+a)\varpi(qn+b) - \mathrm{MT}(x,q;a,b) \right| \ll x^2(\log x)^{-A}.$$

  Here, $\varpi(\cdot)$ is the characteristic function of the primes and $\mathrm{MT}(x,q;a,b)$ is a main term.

- R.C. Vaughan suggested that the following estimate is another possible candidate:

$$\sum_{q\le Q} \max_{(a,q)=1} \left| \sum_{\substack{1\le b\le B\\(b,q)=1}} \left(\sum_{n\le x/q} \varpi(qn+a)\varpi(qn+b) - \mathrm{MT}(x,q;a,b) \right) \right| \ll Bx(\log x)^{-A}.$$

- J. Pintz proposed to try to use EH for twin primes to obtain two pairs of twin primes that are close to each other.

19. Triples in prime-like sequences.

- C. Elsholtz asked whether it is possible to find infinitely many triples s_n, s_{n+1}, s_{n+2}, with $s_{n+2} - s_n$ bounded, in sequences that are commonly considered to be similar to but more manageable than the primes (e.g., integers that are sums of two squares).
- K.B. Ford noted that for sums of two squares the problem is trivial, because of triples of the form n^2, n^2+1, n^2+4, for example.
- J.B. Friedlander suggested that such trivialities can be avoided by requiring that the two squares be of comparable sizes. S.W. Graham and T.D. Wooley proposed another option: produce "many" gaps.
- R.C. Vaughan proposed considering the more general question of triples of values of norm-forms.