OPEN PROBLEMS: AIM WORKSHOP ON RANDOM MATRICES (DEC 2010)

(1) (Tao) (Solved) Consider the sum $\frac{1}{\sqrt{n}}A_n + B_n$ of an iid random matrix A_n (with all entries of mean zero and variance one), normalized by $\sqrt{n}$, plus a bounded rank matrix B_n of bounded norm. How are the outlier eigenvalues of this sum (outside the disk predicted by the circular law) distributed?

(2) (Tao) To what extent can the local universality results for Wigner matrices be extended to block-Wigner matrices, in which some of the blocks are zero? Note that the semicircular law may change as a consequence of these zero blocks, but the local behavior should be largely unchanged. Such a theory would unify the Wigner theory with the covariance matrix theory, and may also be useful in obtaining universality for (Hermitian) polynomial combinations of several independent Wigner matrices.

(3) (Tao) The eigenvalue distribution of GUE is the equilibrium measure for Dyson Brownian motion. On the other hand, the sine process is the scaling limit of the eigenvalue distribution of GUE. Is there a well-defined analogue of Dyson Brownian motion that has sine process as the "equilibrium measure"? Can this be used to give a characterization of the sine process that might be used to establish that process directly for ensembles such as GUE?

(4) (Tao) Are there analogues of the circular law or bulk universality for quaternion-valued iid or Wigner matrices, or equivalently for matrices whose 2×2 blocks are independent, but which can contain dependencies within the blocks? This may be one of the simplest models in which the independence hypothesis is relaxed slightly.

(5) (Vu) Prove universality of local statistics for random ± 1 symmetric matrix with zero row sums. (Random means with respect to the uniform distribution over all such matrices.) This is another simple model when the independence hypothesis is relaxed. It is known that this model satisfy the semi-circle law.

(6) (Vu) Prove that the adjacency matrix of a random regular graph on n points and degree at least 3 has full rank with probability tend to 1 as n tends to infinity. Here the degree can be a function of n (e.g. $n/2$).

(7) (Vu) The following problem is a slight variant of the well-known hidden clique problem in TCS. Consider the following two models of random symmetric matrices:

(A) Random Bernoulli matrices: Upper diagonal entries are iid Bernoulli random variables, taking values ± 1 with probability $1/2$.

(B) Random Bernoulli matrices with a hidden corner (clique): Fixed the entries at the top $m \times m$ minor to be all 1 and generate the rest of the entries as in (A).

Given one sample each from A and B, we would like to find a polynomial algorithm to tell A and B appart with probability at least 2/3 (2/3 can be repalced by any constant larger than 1/2). Of course, this is easy if m is very large, say $m \gg \sqrt{n}$. The question is that what is smallest value of m when this is possible. It is not known if one can do with $m = o(\sqrt{n})$.

(8) (Vu) Let M_n be a random hermitian matrix (satisfying the condition $C0$). Let λ_1 be its largest eigenvalue and λ_1' the largest eigenvalue of its $(n-1) \times (n-1)$ left-bottom principal minor. We wonder if $n(\lambda_1 - \lambda_1')$ has a limiting distribution. (Notice that the normalization factor is n, not $n^{2/3}$; it is known that with probability $1 - o(1)$, $\lambda_1 - \lambda_1' \leq n^{-1+o(1)}$ for a large class of matrix M.)

(9) (Vu) Let M_n be a random hermitian matrix. Let $u_1, \dots, u_n$ be its eigenvectors. Then with probability $1 - o(1)$, $\max_{1 \leq i \leq n} \|u_i\|_\infty = O(\sqrt{\frac{\log n}{n}})$. (Notice that u_i is not uniquely defined, but $\|u_i\|_\infty$ is).

(10) (Tao-Vu) For $\delta n \leq i \leq (1-\delta)n$ with $\delta > 0$ fixed, one has

$$\mathbf{E}\lambda_i = n^{1/2}\gamma_i + n^{-1/2}C_{i,n} + \frac{1}{4\sqrt{n}}(\gamma_i^3 - 2\gamma_i)\mathbf{E}\eta^4 + O_\delta(n^{-1/2-c})$$

for some absolute constant $c > 0$, where $C_{i,n}$ is some bounded quantity depending only on i, n (and is in particular independent of η). The same statement should also be true for the median $\mathbf{M}\lambda_i$.

(11) (Many people) Quantify convergence rate to any Universal law.

(12) (Ben Arous) Universality of smallest and largest gap distribution (in the bulk).

(13) (Ben Arous) Show Poisson law for largest spacing, in GUE of order $\sqrt{\log N}/N$. What about GOE?

(14) (Ben Arous) Consider the adjacency matrix $M(n,p)$ of $G(n,p)$ with $p = c/n, c > 1$. It is easy to see that any eigenvalue of a finite tree is an atom in the limiting spectral distribution of $M(n,p)$, Are there any other atom ? Show that the limiting distribution of the eigenvalues of the 2-core of $G(n,p)$ is continuous.

(15) Find an algebraic operation that allows one to get Tracy-Widom distributions from i.i.d. random variables.

(16) For X i.i.d. entries (not normal, as in problem 1): universality in bulk/edge? Universality of extreme e.v.? Local circular law (go from scale n^{-c} to $n^{-1/2+\epsilon}$? Is it possible to use Dyson Brownian motion to prove universality?

(17) (motivated by Gustavson's results). Compute decorelation distance for Hermitian/Orthogonal case, that is, how large should ℓ be for λ_k and $\lambda_{k+\ell}$ to have a joint Gaussian behavior?