PROBLEM SESSIONS: "SUBCONVEXITY BOUNDS FOR L-FUNCTIONS"

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ABSTRACT. These are the notes I extracted from the problem sessions. There may be some mistakes. So, use them at your own risk! Do not hesitate to contact me if you want to add or correct some points.

CONTENTS

1. MONDAY PROBLEM SESSION

1.1. Unknown cases of (sub)convexity.

- This point was mentioned by Reznikov. Let π be an automorphic cuspidal representation of GL_m for $m \geqslant 1$ and π' be an automorphic cuspidal representation of GL_n for $n \geqslant 1$. Do we know the convexity bound for $L(\pi \times \pi', s)$? It absolutely converges on $\Re s > 1$ and thus we know the convexity bound in the s-aspect. Do we know the convexity bound in any aspect for any $m \geqslant 1$ and any $n \geqslant 1$? No! For instance, when $m = n = 2$, it is known via L^2-theory of automorphic forms. Is there a geometric method which could give the result for any $m \geqslant 1$ and any $n \geqslant 1$?
- These subconvexity problems were mainly suggested by Michel. Let f be a Hecke Maass cusp form of level 1 and Laplacian eigenvalue $\lambda_f := 1/4 + it_f^2$. We want to break the convexity bound for $L(f, 1/2 + it_f)$ in the spectral aspect namely to find $\delta > 0$ (even microcospic) such that

$$L(f, 1/2 + it_f) \ll_\varepsilon t_f^{1/4-\delta+\varepsilon} \tag{1.1}$$

  for any $\varepsilon > 0$. About this problem, two questions arose during the discussion.
  - Is there a work on this from Luo?

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– Does somebody know an arithmetic application of such unkown convexity bound?

Note that this is a case for which the analytic conductor drops since

$$Q(f,1/2+it_f)=\left(1+\left|1/2+it_f-it_f\right|\right)\left(1+\left|1/2+it_f+it_f\right|\right)\approx t_f.$$

Thus, previous experience suggests that it should be difficult to prove. You can also think of the method of moments to be convinced: you will have to estimate an higher moment if you want to break convexity. Another example in which the analytic conductor drops is given by

$$L(f\times g,1/2+it_f)\ll_{g,\varepsilon} t_f^{1/2-\delta+\varepsilon} \tag{1.2}$$

for any $\varepsilon>0$. Here, g is a fixed Hecke Maass cusp form. Note that if g is an holomorphic Hecke cusp form of weight 4 and level q, such L-functions already appear in Phillips-Sarnak's deformation theory even if people are more interested in non-vanishing results in this context. Roughly speaking, this theory deals with the deformation of $\Gamma_0(q)$ in the direction given by g. The authors proved that a positive proportion (in a suitable sense) of $L(f\times g,1/2+it_f)$ does not vanish which entails that a positive proportion (in the same suitable sense) of f's is annihilated by such deformation.

- Let f and g be two fixed Hecke Maass forms. We want to break the convexity bound for $L(f\times g,1/2+it)$ in the s-aspect namely to find $\delta>0$ such that

$$L(f\times g,1/2+it)\ll_{f,g,\varepsilon} t^{1-\delta+\varepsilon} \tag{1.3}$$

for any $\varepsilon>0$. Jutila suggested an extra-average over the spectral parameter t_f of f to produce some saving. For instance, he proved with Motohashi that

$$L(f\times g,1/2+it)\ll_{g,\varepsilon}\begin{cases} t^{1+\varepsilon}\big/\sqrt{t_f} & \text{if } t_f^{3/2}\ll t\ll_\varepsilon t_f^{2-\varepsilon},\\ \left(t+t_f\right)^{2/3+\varepsilon} & \text{if } t\ll t_f^{3/2}.\end{cases}$$

How can we extend this range?

- Let us talk about the symmetric-square L-function $L(\mathrm{Sym}^2 f,s)$ for any Hecke Maass cusp form of level q_f, spectral parameter t_f and nebentypus χ_f. This L-function is of degree 3. Thus, the subconvexity problem in the s-aspect is given by

$$L(\mathrm{Sym}^2 f,1/2+it)\ll_{f,\varepsilon} t^{3/4-\delta+\varepsilon} \tag{1.4}$$

for any $\varepsilon>0$ and for some $\delta>0$. About the three spectral parameters at infinity of $L(\mathrm{Sym}^2 f,s)$, one is of constant size and the two others are of size t_f. Thus, the subconvexity problem in the spectral aspect is given by

$$L(\mathrm{Sym}^2 f,1/2+it)\ll_{q_f,t,\varepsilon} t_f^{1/2-\delta+\varepsilon} \tag{1.5}$$

for any $\varepsilon>0$ and for some $\delta>0$. The subconvexity problem in the level aspect is given by

$$L(\mathrm{Sym}^2 f,1/2+it)\ll_{q_f,t,\varepsilon} q^{\varepsilon}\times\begin{cases} q_f^{1/2-\delta} & \text{if } \chi_f \text{ trivial or non-quadratic},\\ q_f^{1/4-\delta} & \text{otherwise}\end{cases} \tag{1.6}$$

for some $\delta > 0$ and for any $\varepsilon > 0$ since

$$L(f \times f, s) = L(\chi_f, s) L(\mathrm{Sym}^2 f, s).$$

Michel said that if you know a subconvexity bound for $L(\mathrm{Sym}^2 f, s)$ in the s-aspect then you know (by Cauchy-Schwarz) a subconvexity bound for $L(f \times g, s)$ when $f \neq g$ and g is fixed in the s-aspect. He also said that the subconvexity problem for symmetric square L-functions could have some link with metaplectic tools on $\widetilde{GL_2}$ via an integral representation of this L-function in which an Eisenstein series on $\widetilde{GL_2}$ occurs.

- Let us talk about triple L-functions. Let f, g, h some Hecke Maass forms, f being of level q_f, spectral parameter t_f, weight k_f and the same notations for the two others. We could be interested in the following subconvexity problems

$$L(f \times g \times h, 1/2 + it) \ll_{f,g,t,q_h} k^{2-\delta+\varepsilon} \tag{1.7}$$

when h is holomorphic.

$$L(f \times g \times h, 1/2 + it) \ll_{f,g,h,\varepsilon} t^{2-\delta+\varepsilon}. \tag{1.8}$$

$$L(f \times f \times h, 1/2) \ll_{q_f,h,\varepsilon} t_f^{1-\delta+\varepsilon}. \tag{1.9}$$

This last case is again an example of situation in which the conductor drops. Reznikov said that it is easier and doable to prove

$$L(f \times g \times h, 1/2) \ll_{\varepsilon} t_f^{2-\delta+\varepsilon} \tag{1.10}$$

when f, g and h have some comparable but not equal spectral parameters at infinity such that the conductor does not drop.

1.2. **Universality of convexity breaking exponents.**

- One aims at explaining the apparently unrelated occurences of Weyl's subconvexity exponent given by $1/4(1-1/3)$ and Burgess' subconvexity exponent given by $1/4(1-1/4)$. Remember that Weyl's subconvexity exponent appears
  - in the subconvexity problem for GL_2 L-functions $L(f, s)$ in the s and spectral aspect,
  - in the subconvexity problem for twisted L-functions $L(f \times \chi, s)$ in the level aspect (f and χ are of same level),
  - in the subconvexity problem for Rankin-Selberg L-functions $L(f \times g, s)$ in the t_f aspect

  whereas Burgess' subconvexity exponent appears
  - in the subconvexity problem for Dirichlet L-functions $L(\chi, s)$ in the level aspect,
  - in the subconvexity problem for twisted L-functions $L(f \times \chi, s)$ in the conductor aspect of the character.
- A natural problem is to find particular L-functions for which we know how to prove better exponents than Weyl and Burgess' ones. Soundararajan suggested to look at $L(\chi, s)$ when χ is of conductor say 3^n and n goes to infinity by taking advantage of Vinogradov's method. We can talk about subconvexity in the depth aspect in such context. A similar example is $L(f \times \chi, s)$ by taking advantage of Graham-Ringrose's method when the modulus of χ is a higly divisible square-free integer .

- It is also natural to wonder if there exists some applications which need a better subconvexity exponent than Weyl or Burgess' one. Michel suggested an application to André Oort conjecture discovered by Edixhoven. This conjecture asserts that a curve contained in $X_0(1)\times X_0(1)$ which does not project to $X_0(1)$ itself and contains infinitely many CM points is the modular curve itself (embedded in the product as a graph of Hecke correspondence). Assuming GRH for quadratic imaginary fields, Edixhoven "proved" this conjecture. The main hole for an unconditional proof being that he needs to know that the number of primes less than $\log^2(|d|)\log_2^2(|d|)$ which are split in the quadratic imaginary field of discriminant d tends to infinity with d. Fouvry, applying a result of Linnik and Vinogradov, noticed that the number of primes less than $|d|^{1/4+\varepsilon}$ which are split in the quadratic imaginary field of discriminant d tends to infinity with d. Improving Burgess' bound for character sums will improve the previous range. Studying these small split primes is a challenging problem because it may occur when one wants to build an efficient amplifier. For instance, Duke-Friedlander-Iwaniec faced this problem when they tried to prove subconvexity bound for class group L-functions without appealing to the spectral theory of automorphic forms.

2. TUESDAY PROBLEM SESSION

2.1. General definition and interesting examples of periods.

Lindenstrauss gave the following general definition of period. Let G be a group and H be a subgroup of G. The general space is given by

$$X := G(\mathbb{Q})\backslash G(\mathbb{A}_{\mathbb{Q}}) .$$

To any automorphic form f on X (a smooth function on X which belongs to the space of an automorphic representation), we define some periods by

$$\int_{H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathbb{Q}})} f(h)g(h)\mathrm{d}h$$

for any automorphic form g on $H(\mathbb{Q})\backslash H(\mathbb{A}_{\mathbb{Q}})$. Then, people gave fundamental examples of periods.

Example 1: Fourier coefficients of cusp forms on GL_2
Here, $G = GL_2$ and H is the unipotent subgroup of G namely

$$H := \left\{\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}, x\in\mathbb{R}\right\}.$$

For any g in $GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}})$, the period

$$\int_{\mathbb{R}} f\left(g\begin{pmatrix}1 & t\\ 0 & 1\end{pmatrix}\right)e(-nt)\mathrm{d}t$$

is directly linked to the n-th Fourier coefficient of f for any $n\in\mathbb{Z}$.

Example 2: Special values of GL_2 L-functions
Here, F is a number field, $G = GL_2$ and H is the torus subgroup of G namely

$$H := \left\{\begin{pmatrix}y & 0\\ 0 & 1\end{pmatrix}, y>0\right\}.$$

The period

$$\int_{F^\times\backslash\mathbb{A}_{\mathbb{F}}^\times} f\left(\begin{pmatrix} y & 0\\ 0 & 1\end{pmatrix}\right)\mathrm{d}^\times y$$

is directly linked to the special value $L(f,1/2)$ up to Γ-factors.

Example 3: Triple product formula
Here, $G = GL_2 \times GL_2$ and $H = GL_2$ is a subgroup of G via the diagonal embedding.Thus, an automorphic form F on G is a pair of automorphic forms f_1 and f_2 on GL_2. If $h \in H$ then $F(h) = f_1(h)f_2(h)$. The period

$$\int_{GL_2(\mathbb{Q})\backslash GL_2(\mathbb{A}_{\mathbb{Q}})} f_1(h)f_2(h)f_3(h)\mathrm{d}h$$

is linked (up to some factors) to the special value $L(f_1\times f_2\times f_3,1/2)$.

2.2. **A convexity bound for periods?** The discussion was about understanding what could be a general bound for period which specializes to (sub)convex bounds for L-functions. Lindenstrauss said that a convex bound for period should be a bound that comes from general harmonic analysis results (it is the case for L-functions). Then, Miller defined an automorphic period which is our previous geometric period up to some factors (which are sometimes special values of L-functions). Thus, bounding these factors turns out to bounding both (automorphic and geometric) periods. Another problem was trying to understand if there is a canonical way to define a period such that it does not depend on the choice of test vectors in the space of representations that occur.

2.3. **List of problems to be discussed into small groups.**

- Subconvexity problems when the analytic conductor drops (in particular for the symmetric square L-function).
- Improving Weyl's exponent in various examples. In particular, investigate Soundararajan's idea about Dirichlet characters of highly composite moduli or try to prove some explicit spectral decomposition of shifted convolution sums (Harcos' suggestion).
- Develop explicit Good-Motoashi type identities.
- Develop associativity type identities in the conductor aspect.
- Formulate clearly period problems in relation to L-functions and representation theory.
- Quantitative equidistribution results to prove subconvexity bounds.

3. Wednesday problem session

Here is an incomplete list of the problems which could be understood in a close future.

- Silberman suggested to try to prove a strong hybrid subconvexity bound for standard L-functions on GL_n namely try to find $\delta > 0$ such that

  $$L(\pi,1/2+it) \ll_\varepsilon Q(\pi,1/2+it)^{1/4-\delta+\varepsilon}$$

  for any $\varepsilon > 0$. According to Garett, the Diaconu-Garrett-Goldfeld extension of Good's method to GL_nxGL_{n-1} may have something to offer in this direction.Venkatesh also suggested to try to find some applications of such subconvexity bound before proving them. For instance, it can be interesting to clarify the links between subconvexity problems and equidistribution problems in higher rank for classical groups. Duke also had in mind to extract explicit information from higher rank Artin L-functions.

- About periods, a very important point is to find an heuristic way of predicting what is an analogue of convexity bounds and Lindelöf hypothesis for periods. Also, prove some bounds for periods which specialize to subconvexity bounds for L-functions. Silberman mentioned the problem of predicting what could be the expected bound for the infinite norm of general automorphic forms when the spectral parameters go to infinity.
- Jutila suggested to prove some Ω-results about the error term which occurs in some moments of families of L-functions. For instance,
$$\int_{t\leqslant T}\left|L(f,1/2+it)\right|^2\mathrm{d}t=\mathrm{Main}(T)+\mathrm{Error}(T)$$
where $\mathrm{Error}(T)=\Omega(\sqrt{T})$ is expected to hold for any GL_2-automorphic form. What could be some applications of such Ω-results?
- Venkatesh suggested to find a way to guess when it is possible to prove some asymptotic formula for some moment given by
$$\sum_{f\in\mathscr{F}}L(f,1/2)$$
where as usual the conductor of each L-function of $\mathscr{F}$ is of size almost constant say $Q(\mathscr{F})$ in the logarithmic scale and $Q(\mathscr{F})\to+\infty$. If
$$4\log|\mathscr{F}|>\log Q(\mathscr{F})$$
then we generally can prove an asymptotic formula. At the moment, the record is
$$6\log|\mathscr{F}|=\log Q(\mathscr{F})$$
in the work of Conrey and Iwaniec on the cubic moment of automorphic L-functions. Can we do better?
- Soundararajan asked if it is possible to prove some subconvexity bound in the critical strip but outside the critical line and eventually near the edge of the critical strip. One known instance is the ζ function in the s-aspect near $\Re s=1$ via Vinogradov.
- Michel asked about an analogy of (sub)convexity for p-adic L-functions. The answer could be some integrality property (Mazur, Prasad,...).

4. THURSDAY PROBLEM SESSION

- Venkatesh convinced us that many equidistribution results are useful to prove asymptotic formula for moments of L-functions with a power saving in the error term if the suitable equidistribution results are quantitative ones. He illustrated this by two GL_2-examples namely
$$\int_{-T}^{+T}\left|L(f,1/2+it)\right|^2\mathrm{d}t$$
and
$$\sum_{\chi\mod(q)}^{\times}\int_{\mathbb{R}}\left|L(f\times\chi,1/2+it)\right|^2\mathrm{d}t.$$

- In addition, he mentioned what could be the obstacles to do the same in higher rank cases. On one hand, it is very hard to prove an integral representation (period) of L-functions which occured in higher rank since the archimedean computation may be very far away from obvious. According to Garett, the Diaconu-Garrett-Goldfeld extension of Good's spectral-theory-based method to $GL_n x GL_{n-1}$ illustrates the complications at archimedean places. On the other hand, it is necessary to analyse such integral representation via ergodic theory or spectral methods. Two difficult instances are given by $O_n \times O_{n-1}$ and $GL_n \times GL_{n-1}$.
- An application of subconvexity bounds for GL_n may be some information about the $2n$-moment of the Riemann ζ function. For instance, some Motohashi type formula make the link between GL_3 and $|\zeta|^6$. We also have to mention the work of Conrey and Iwaniec.

5. Friday problem session

- There will be a website dedicated to subconvexity for L-functions. People agreed that it should contain references to important results, a list of people with their current attempts and previous results also.
- Venkatesh put the stress on the subconvexity problem in higher-rank cases. Few methods are known which have bearing on it. After Venkatesh-Lindenstrauss and Bernstein-Reznikoff treatments of triple products, the exceptions are works in progress mentionned by Garett: Venkatesh's applications of ergodic theoretic ideas coming from Ratner and Clozel, and Diaconu-Garrett-Goldfeld's GL_n version of an old method of Good.
Venkatesh suggested that a first higher rank example to undertake should be $GL_3 \times GL_2$ with f_3 on GL_3 is fixed and f_2 is varying. In order to get some insight into that, it would be profitable for everybody that classical analytic number theorists try to understand the case when f_3 is an Eisenstein series namely

$$\sum_{\substack{f \text{ Hecke-Maass of level } 1 \\ \text{and eigenvalue } 1/4+t_f^2 \\ t_f \backsim T}} \int_{t \backsim T} \left| L(f, 1/2+it) \right|^6 \mathrm{d}t.$$

Note that the size of the family is about T^3 whereas the size of the analytic conductor is about T^{12} which reveals the level of difficulty. Also, people should understand where the GL_3-theory occurs in the analytic analysis.