Variational Methods in Celestial Mechanics

The American Institute of Mathematics

This is a hard–copy version of a web page available through http://www.aimath.org
Input on this material is welcomed and can be sent to workshops@aimath.org
Version: Sun Jun 22 14:05:57 2003

Table of Contents

Chapter A: Open Problems

Notes by Kuo-Chang Chen.

Problem 1. Consider the three-body problem. Let E be the energy, C be the angular momentum, and $L = C^2 E$ for circular Euler motion. An exchange orbit is an unbounded solution with the property that the distance between one mass m_3 and the binary $\{m_1, m_2\}$ goes to infinity as $t \to -\infty$, m_3 exchange with m_2 and the distance between m_2 and the the binary $\{m_1, m_3\}$ goes to infinity as $t \to +\infty$. A solution has *Hill type stability* if the three bodies cannot undergo an exchange of binary. For exchange orbits, how close to 1 can $\frac{C^2 E}{L}$ be? (Christian Marchal)

Remarks.

- Known: $E < 0$, C large, then $C^2 E < L < 0$.
- The orbit comes from unstable manifold of E_3 (Euler configuration) to the stable manifold of E_2. There is a normally hyperbolic invariant 3-sphere S^3_{Euler} with W^s, W^u in dimension 5, and at infinity (P_{12} and P_{13}) of the Hill's region there are two other invariant 3-spheres $S^3_{P_{12}}$, $S^3_{P_{13}}$ with 4-dimensional W^s, W^u. Can we create chain between P_{12}, P_{13} via S^3_{Euler}? (Rick Moeckel)
- Suggestion: Consider first the intersection of W^u, W^s of the different objects. This could be tested numerically.

Problem 2. Consider the planar restricted three-body problem. For Jacobi constants such that the Hill's region looks like a barbell, do transit orbits always exist? (Rick Moeckel)

Remarks.

- Transit orbits are orbits that pass near the collinear relative equilibria of the planar three-body problem and moving from one binary configuration to another.
- Can choose the walls in figure 1 that are convex to the flow.
- Known when the neck is small.
- Suggestion: Use Maupertuis-Lagrange variational principle. Problem with that: boundary of Hill's region has 0 length. Maybe minimax could work.
- More generally, is there any situation in the n-body problem where the Maupertuis-Lagrange variational approach yields any new results?

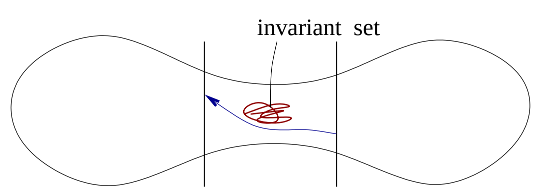

Figure 1. Transit orbit

Problem 3. Prove the existence of super-eight solution for the 4-body problem with equal masses. (Joseph Gerver)

Remarks.

- One difficulty: the square relative equilibrium is absolute minimum in that symmetry class.

- There is a computer assisted proof by interval arithmetic.

Poincaré characterize homology class in the planar three-body problem by three integers $k_{ij} = Deg(x_i - x_j, 0)$; that is, the number of oriented turns of the side $x_i - x_j$.

Problem 4. (Due to Poincaré, 1896) Can we obtain periodic solutions by minimizing while fixing the k_{ij}'s? (Alain Chenciner)

Remarks.

- Answers known when k_{ij} are all nonzero. When $(k_{12}, k_{23}, k_{31}) = (1, 1, 1)$ or $(-1, -1, -1)$, minimizers are Lagrange's equilateral solutions; otherwise, minimizers have collisions.
- In choreographies, k_{ij} are all the same.
- Question: Is the answer the same in the restricted case as well? (Richard Montgomery)

Problem 5. Is the figure-8 orbit a minimizer with homology class $(0, 0, 0)$? (Alain Chenciner)

Problem 6. Prove the existence of Broucke-Hénon orbit for the planar 3-body problem. Do Broucke-Hénon orbits exist for small + big masses? Is the Broucke-Hénon solution a minimizer in the class $(1, 0, 1)$? (Andrea Venturelli)

Remarks.

- Numerically, Broucke-Hénon solution exists for equal masses, at least.
- The solution seems to be stable numerically for equal masses.
- Solution has symmetry group D_2. Just impose D_2-symmetry, the action is not coercive, so need to impose topological constraints.
- Numerically it is a local minimum of $\mathcal{A}$.
- Shubart orbit with collision is on the closure of the homology class $(1, 0, 1)$, and it has larger action.

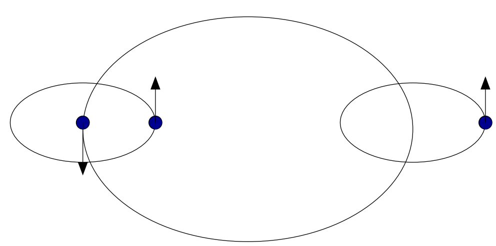

FIGURE 2. Broucke-Hénon orbit

Problem 7. Is $(1, 0, -1)$ realized by a collision-free periodic orbit (see figure 3)? If exists, is it minimizing or not? Is trivial braid realized by a collision-free periodic solution? (Andrea Venturelli)

Problem 8. Can any symbol sequence be realized in the planar three-body problem? Are they minimizing or not? Do they have zero angular momentum? (Richard Montgomery)

Remarks.

- "Symbol sequence" is given by the crossing of the equator in the shape space, count each time bodies as collinear with which body is in the middle.

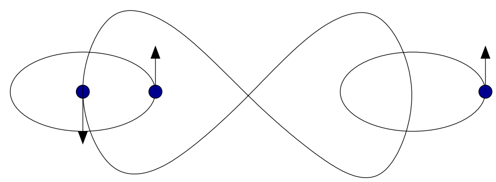

FIGURE 3. A possible solution with homology class $(1, 0, -1)$

- Could proceed numerically by systematic search and see what sequences are excluded (if any).

Problem 9. Often the minimizers in a symmetry class has more symmetry that the class asks for. Is that a coincidence? (Alain Chenciner)

Remarks. Some examples:

- Kepler problem with Italian symmetry. Minimizers are circular; that is, with $SO(2)$ symmetry.
- Maybe the figure-8 orbit is a minimizer with $\mathbb{Z}_6$ and D_3 symmetry (v.s. the D_6 symmetry the figure-8 has)
- In spatial case, can consider D_6 symmetry in which the planar figure-8 would presumably by a minimizer, then get $\mathbb{Z}_2$ symmetry for free.
- Central configurations with four bodies has a line of symmetry.

(Wu-Yi Hsiang) It is a phenomenon of optimality, not coincidence.

Problem 10. Is the central configuration for equal masses with minimal potential on the ellipsoid of moment of inertia 1 necessarily symmetric? (Rick Moeckel)

Remarks. Appears to be false for $n = 47$. Should require that the orbit is a minimizer of the normalized potential.

Problem 11. Do choreographies imply equal masses? (Alain Chenciner)

Remarks.

- Here choreography means equal time shifts, same curve.
- Alain Chenciner proved it for $n < 6$.
- Martin Celli proved it for logarithmic potential for any n.

Problem 12. Existence of choreographies with distinct time shifts.

Problem 13. What subgroups of $O(2)\times O(d)\times S_n$ can be realized as symmetries of solutions? (Davide Ferrario)
Can symmetry group arise differently? (Alain Chenciner)

Remarks. There is no upper bound to the order of the subgroup.

Problem 14. Is there a conceptual proof for Saari's conjecture? Why not fix the moment of inertia tensor and ask the same question (maybe in higher dimensions)? (Alain Chenciner)

Problem 15. Are there solutions to the n-body problem that fall in a certain affine class that are not relative equilibrium or some symmetric solutions? (c.f. Gerver's super-8) (Alain Chenciner)

Problem 16. Is the only solution satisfying (i), (ii) and having no syzygies (i.e. eclipses) the Lagrange homothety solutions?
(i) Energy < 0. (ii) Angular momentum $= 0$. (Richard Montgomery)

Problem 17. Problems on the shape space for the three-body problem: (Wu-Yi Hsiang)
– Are there solutions whose α-(or ω-)limit set is a limit cycle?
– The area of the unit shape sphere is π. Conjecture: for any $0 < A_0 \leq \pi$, there is a solution whose shape curve has its closure with area A_0.
– Can two closed shape curves with the same homotopy class be deformed to one another by closed shape curves? Conjecture: No. But then determine connected components.

Problem 18. Let Δ be the area of the triangle, I be the moment of inertia. Find periodic solutions to the (planar or spatial) three-body problem such that $\frac{\Delta}{I}$ is a constant. (Wu-Yi Hsiang)

Remarks.

- If the angular momentum is zero, then there is none.
- What motions have shapes on the meridian passing the Euler configurations?

Problem 19. What can be said about the volume of the tetrahedron in the four-body problem? Can it be nonzero forever and stay bounded away from 0 and ∞? (Joseph Gerver)

Problem 20. For circular restricted three-body problem, there is a Jacobi integral. For a complete integrability, need two more integrals. Extra integral exists when $m_1 + m_2 = 0$. Is there one more integral? (Christian Marchal)

Problem 21. Does there exist potential $U(x)$ in the plane such that if $\dot{x}(0)$ is tangential to the level curve of U, then U remains constant? (Mark Levi)

Problem 22. Assume $E = 0$ and angular momentum $= 0$. Can Jacobi's metric give new insights beyond what McGehee's coordinates give? (Richard Montgomery)

Problem 23. Among the many interesting open questions is the understanding of the natural limits of the minimization method: (Alain Chenciner)
– To what extent is it connected to symmetry constraints?
– May topological constraints be imposed without forcing the minimizers to have collisions?
– To what extent interesting results may be obtained for arbitrary masses?
– What can be said of the hyperbolic and elliptic dimensions of a minimizer?
– Is it interesting to look at minimizers with fixed energy?

Problem 24. For the n-body problem, $n \geq 4$, show that the number of central configurations is finite for all choices of masses $m_i > 0$ (or find a counterexample). (Rick Moeckel, Marshall Hampton)

Remarks.

- For $n > 4$, even generic finiteness is an open question.
- Can be posed as a purely algebraic question about the solutions of a system of polynomial equations.

Problem 25. Give a sharp upper bound for the Morse index of a nonplanar central configuration (as a critical point of the potential on the normalized configuration space). Give a sharp lower bound for the Morse index of a planar central configuration when it is viewed as part of the nonplanar configuration space. Is the Morse index related to the stability of the rigidly rotating periodic orbits? For example, does a linearly stable relative equilibrium necessarily arise from a minimum of the Newtonian potential? (Rick Moeckel)

Problem 26. Existence proof for homoclinic and heteroclinic orbits between unstable relative equilibrium solutions. (Rick Moeckel)

Problem 27. Existence proof for the "halo orbits" of the three-dimensional restricted three-body problem. (Rick Moeckel)

Remarks. Apparently these are born in a bifurcation from elliptical Lagrange orbit but as far as I know, they have only been studied numerically. (Rick Moeckel)

Problem 28. Consider negative energy three-body orbits which are unbound in both directions of time. The two Jacobi vectors of such an orbit then asymptote to Keplerian orbits in both the distant past and the distant future, and so associated to such an orbit we have pair of Kepler elements in the past (one elliptic the other hyperbolic) and another pair in the future. The "direct scattering" problem is: which pair of Kepler elements can be connected to each other in this way? (Richard Montgomery)

Problem 29. Given three different masses that are comparable in size, prove the existence of prograde orbits without assuming one ratio of mutual distances is nearly zero. (Kuo-Chang Chen)

Remarks.

- Poincaré continuation method can be applied to two cases: one mass is nearly zero, or one ratio of mutual distances is nearly zero.
- Known if two masses are equal.