Emerging applications of complexity for CR mappings

organized by
John P. D'Angelo and Peter Ebenfelt

Workshop Summary

The workshop began Monday with introductory talks by the two organizers, John P. D'Angelo and Peter Ebenfelt. Each of them described simple ideas, known results, and potential generalizations of ideas concerning CR complexity. This broad notion encompasses several ideas. One of the aims of the workshop was to help clarify the basic concepts.

Two of the fundamental questions about CR complexity follow. Let M and M' be CR manifolds, and suppose $f : M \to M'$ is a nonconstant CR mapping. How complicated can f be? If M' is rather simple, such as a sphere or hyperquadric, how complicated can M be? The first question is already non-trivial when both and M and M' are spheres (of different dimensions). When the map is rational, its degree serves as a measure of its complexity, and degree estimates were discussed at the workshop. The case of rational mappings between spheres has many applications and these ideas also recurred throughout the meeting. Consider the second question in the case when there is a transversal CR map from a strictly pseudoconvex hypersurface M in $\mathbf{C}^{n+1}$ to a sphere in $\mathbf{C}^{N+1}$. The complexity of M is then defined to be the minimum possible value of the difference $N - n$. These ideas were then generalized to allow hyperquadrics of different signatures as targets, and partial rigidity results hold even in this setting when the complexity is under control. This set of ideas involves the Gauss equation, which relates CR curvature to the second fundamental form, and hence makes a nice connection (pun intended) with differential geometric aspects of complexity.

After these lectures, the workshop divided into several working groups. These groups were fluid, with people moving among groups and new groups developing. The groups for the first day considered the following five topics: degree estimates, Menger curvature, bumping, Gauss equation, complex vector fields.

The degree estimate group actually switched topics and focused on the following related topic: Assume $n \geq 2$, and let r be a real-analytic function on $\mathbf{C}^n$. For each complex hyperplane H, consider the restriction $r|H$. Suppose that there is a constant k such that the rank of all $r|H$ is bounded by k. Is the rank of r itself bounded? If so, what is the best constant $c(n, k)$? The group proved some special cases during the week.

The Menger curvature group made considerable progress. They considered an identity in one complex variable that arose in harmonic analysis, and was described there as a miracle. Some background and details follow.

In 1995 Melnikov and Verdera provided a new proof of $L^2(bD, \sigma)$-regularity of the Cauchy integral for a Lipschitz planar domain D in $\mathbf{C}$: their proof is based on the notion of *Menger curvature* defined for any three distinct points $z_1, z_2, z_3 \in \mathbf{C}$ as the reciprocal $\frac{1}{R}$

of the radius of the circle going through the three points) and in particular on the following identity:

$$\sum_{\sigma} 1/[(z_{\sigma(1)} - z_{\sigma(2)})(\bar{z}_{\sigma(3)} - \bar{z}_{\sigma(1)})] = \frac{1}{R}$$

Analogues in higher dimensions do not work, and so far no one has been able to truly explain the one-dimensional "miracle". The working group set out to do so and succeeded in making several interesting observations that point to a prominent role that a complex analysis-based approach to the original problem could play in harmonic analysis.

1. The group found a Moebius-invariant version of the identity, which itself is not Moebius invariant.

2. The group found an alternate proof using the classical Liouville Theorem for meromorphic functions.

3. The group found a promising new version of the identity.

4. The group observed new connections among the identity, the Kerzman-Stein operator, and Calderon's first commutator.

On Tuesday morning two senior mathematicians, Fornaess and Zaitsev, gave talks on more advanced topics where complexity issues arise. Fornaess considered embedding problems for Levi flat CR manifolds, attempting to develop a function theory for them. He connected these ideas to suspensions and laminations. He also provided a careful analysis of a real variables version of the problem wherein solving $\overline{\partial}$ was replaced by solving ordinary differential equations. Zaitsev discussed homogeneous CR manifolds. He posed and illuminated several of the problems appearing on the problem list.

On Tuesday afternoon, the same groups met, although a third group formed concerning the following problem. Suppose that $f(z, \overline{z})$ is a polynomial that is positive on an algebraic set M. What are the obstructions to finding a holomorphic vector-valued polynomial h such that $f = ||h||^2$ on X. Based on this discussion, during the week D'Angelo and Putinar formulated and proved a new result in this direction, which is likely to lead to a fruitful interaction between CR geometry and positivity conditions in operator theory. It is also worth mentioning that Putinar and Scheiderer have recently given a negative answer to a related question posed by D'Angelo at an AIM workshop in 2006. Based on that work, D'Angelo has reformulated the question, and the work with Putinar done at the workshop makes a crucial first step in solving it.

Several participants regarded the first hour of Wednesday AM as the highlight of the workshop. Three graduate students, each expected to finish his degree in spring 2011, gave 20 minute talks on the results in their theses. Shroff and Minor, students of Ebenfelt, discussed theorems that clarified some of the issues from Ebenfelt's talk. Shroff discussed a partial rigidity result for side-preserving CR transversal mappings to hyperquadrics. Grundmeier, a student of D'Angelo, spoke about CR mappings invariant under finite subgroups of the unitary group and connections of this topic to representation theory. Minor spoke about Fefferman's circle bundle over a strictly pseudoconvex hypersurface and the possibility of lifting CR immersions to conformal isometries between the circle bundles.

Also on Wednesday morning the workshop held the first of two problem sessions. Mike Bolt served as moderator and Ravi Shroff took careful notes. The second problem session was held Friday morning with Bernhard Lamel serving as the moderator and Shroff again taking notes. On Friday afternoon D'Angelo and Shroff spent some time trying to make the problems more succinct and to express them in consistent notation.

On Wednesday afternoon the groups continued their meetings, although there was considerable movement of people among the groups.

On Thursday AM, participants shuffled among the groups. In one session, Sorin Dragomir discussed the Gauss equation from the point of view of differential geometry. This equation arose in Ebenfelt's talk and plays a key role in Minor's thesis. Graduate students reported that this presentation was considerably valuable. Also on Thursday morning Zaitsev met with D'Angelo and Huang to discuss an aspect of subelliptic estimates. He gave a counter-example to a lemma in a recent paper on that topic.

Thursday afternoon began with a talk by Huang on rigidity results. After his talk the workshop held some progress reports. Lebl spoke about the progress his group made on the question of the rank of a mapping when the ranks of its restrictions to hyperplanes are known. He presented a positive result in a specific simple situation. D'Angelo gave a short presentation of the recent work of Putinar-Scheiderer and the related advance made by D'Angelo-Putinar at the workshop. Lanzani and Bolt then presented the work on Menger curvature. Group discussions continued after these presentations.

On Friday the problem session continued, and the list of problems posed and disc ussed grew to 21. Friday afternoon saw considerable continued discussion of these problems. We anticipate that additional results including all the working groups will be obtained in the near future.