Problem Session Notes

AIM Workshop on Descriptive Inner Model Theory

June 2-6, 2014

These notes were compiled by Per Stinchcombe.

1. Let λ be a singular cardinal of cof $> \omega$, and $\{\kappa < \lambda | 2^\kappa = \kappa^+\}$ is stationary, costationary. Must $\mathrm{AD}^{L(\mathbb{R})}$ or stronger forms of determinacy hold?

   (PD is known as a lower bound (Gitik, Shelah, Schindler), and a supercompact is a known upper bound (preprint - Gitik).)

2. Does $(\aleph_3, \aleph_2) \twoheadrightarrow (\aleph_2, \aleph_1)$[1] imply that there is an inner model with a Woodin cardinal?

   Upper bound: Huge (Foreman)

   Lower bound:

   - $\kappa^{+\omega}$-strong (Schindler, assuming CH)
   - Repeat point (Cox)

3. What is the consistency strength of an $\aleph_3$-saturated ideal on ω_2?

   Upper bound: almost huge (Magidor)

   Lower bound: Assume $\exists$ an ideal $I \subseteq \omega_2$ such that

   $$\{X \prec H_\theta | X \text{ is self-generic wrt } I, X \cap \omega_3 \text{ is } \omega\text{-closed below its supremum}\}$$

   is stationary.[2] (This is weaker than saturation.) Assume further that $2^{\aleph_1} \leq \aleph_3$. Then PD holds.

4. Consider the sequence

   $$\langle \aleph_n \cap \mathrm{cof}(\aleph_{i_n}) | n \geq 2 \rangle$$

   where $i_{3k+1} = 1$, $i_n = 0$ otherwise.

   What is the consistency strength of the mutual stationarity of this sequence?

[1] every structure of one type has an elementary substructure of the other type.

[2] Ralf Schindler supplied the following definition: X is self-generic with respect to I if the following holds true. Let $\sigma : H -> X \prec H_\theta$ be the inverse of the transitive collapse, let α be the critical point of σ so that $\sigma(\alpha) = \omega_2$. Write $\bar{I} = \sigma^{-1}(I)$, and write $U = \{x \in H : x \subset \alpha$ and $\alpha \in \sigma(x)\}$. Then U is the filter given by a generic w.r.t. forcing with $\bar{I}$ over H.

Upper bound: ω-many supercompacts. (Cummings, Foreman, Magidor, "Canonical Structures II")

Lower bound: $0^{\#}$? Sharps for bounded subsets of $\aleph_\omega$.[3]

5. Is it consistent that for every sequence $\langle S_n | n \in \omega \rangle$ with each $S_n \subseteq \aleph_{n+2} \cap \mathrm{cof}(\omega_1)$, each S_n, the sequence is mutually stationary?

   Lower bounds are known: inner model with infinitely many cardinals κ_n such that for all m the class of measurables $\lambda < \kappa_n$ with Mitchell order at least κ_m is stationary in V for $n > m$. (Koepke-Welch)

6. Is $MM(c^+)$ consistent with Woodin's Axiom $(*)$?

   Known: Assume MM^{++} for arbitrary partial orders, weak UBH (a proper class of Woodins, extender sequences witnessing Woodinness; then UBH holds for those extender sequences.); let Γ_∞ be the universally Baire sets. Suppose $\theta_{uB} > \aleph_1$. Then $(*)$ holds. (Schindler-Woodin)

7. Does
   $$\mathrm{Th}(L(\Gamma_{uB})) = \mathrm{Th}(L(\Gamma_{uB}))^{V^{\mathbb{P}}}$$
   (with constant symbols for each uB set) for all $\mathbb{P}$, plus a proper class of Woodin cardinals, plus MM^{++}, imply $\mathrm{cof}(\theta_{uB}) > \aleph_1$?

   Known: MM^{++}+ weak UBH + proper class of Woodins $\implies$ TFAE:

   (a) $\mathrm{cof}(\theta_{uB}) > \aleph_1$
   (b) $\exists$ semiproper $\mathbb{P}$ adding uB A such that $A >_w B$ for all uB B in V

   (conjecture: both are true)

   Remark 1. $(*)^+$*: For every $A \subseteq \mathbb{R}$ there is an* AD^+*-model $M \supseteq \mathbb{R}, g \subseteq \mathbb{P}_{\max}$ generic, $A \in M[g]$.*

   $(*)^{++}$*: $M \models \mathrm{AD}_{\mathbb{R}} + \Theta$ is regular.*

   $MM^{++} + (*)^{++} \implies \theta_{uB} = \omega_3$.

8. What is the consistency strength of $MM(c)$?

   Upper bound: $\mathrm{AD}_{\mathbb{R}} + \Theta$ is regular (Woodin: $\mathbb{P}_{\max}$ book)

   Lower bound: $\mathrm{AD}^{L(\mathbb{R})}$ is safe (Steel-Zoble), more may be known.

9. What is the consistency strength of $\neg\Box_{\omega_2} + \neg\Box(\omega_2) + 2^{\omega_1} = \omega_2$?

   Upper bound: weaker than $\mathrm{AD}_{\mathbb{R}} + \Theta$ is Mahlo.

   $\{\alpha | \mathrm{cof}(\theta_\alpha) \geq \aleph_2 + \theta_\alpha$ regular in HOD$\}$

   Lower bound: PD (maybe $\mathrm{AD}^{L(\mathbb{R})}$?)

[3] Take an elementary substructure where the cofinalities alternate. It never projects in L; get an elementary embedding $L \to L$.

10. What is the consistency strength of "$\aleph_2$ and $\aleph_3$ both have the tree property"?

    Upper bound: weakly compact above a supercompact. (Abraham)

    Lower bound: nowadays the argument in Foreman-Magidor-Schindler would give a Woodin cardinal. (2 is open)

11. Is there a unique model $L(\mathbb{R}, \mu)$ such that $L(\mathbb{R}, \mu)$ satisfies μ is a normal fine measure on $\mathcal{P}_{\omega_1}(\mathbb{R})$? What is the consistency strength of such a pair?

    Lower bound: ω^2 Woodins.

    Known: If $L(\mathbb{R}, \mu)$ and $L(\mathbb{R}, \nu)$ are two such models, then $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R}, \mu) \subseteq L(\mathbb{R}, \nu)$ or vice versa.

12. Does BMM $\implies 0^{¶}$ exists?

    Upper bound: BMM gives an inner model with a strong cardinal. (Schindler)

    Lower bound: BMM is consistent from $\omega + 1$ Woodins plus a measurable. (Woodin)

13. "Dual covering theorem" for (M, λ, δ) is the statement: For every λ, there is $f : \lambda^{<\omega} \to \lambda$ such that $\forall X \subseteq \text{Ord}$ closed under f, X is a union of δ-many sets in M.

    For reasonable inner models M, can you get the failure of dual covering for $(M, \aleph_3, \aleph_1)$ from some large cardinals?

    E.g.:

    (a) Assuming no proper class model with a Woodin cardinal, M is the one-Woodin K.

    (b) Assuming no proper class model with a strong cardinal, M is the one-Woodin K?

14. The Axiom of Strong Condensation: $\forall \kappa > \omega$ there is a bijection $h : \kappa \to H(\kappa)$ such that for all $X \prec (H(\kappa), h)$, $\pi[X \cap h] = h \upharpoonright ot(X \cap \kappa)$, for π the uncollapse.

    Suppose N is an inner model satisfying strong condensation, and covering fails relative to N. Must $N^{\#}$ exist?[4]

15. Suppose there is no inner model with a Woodin cardinal, and let κ be a singular cardinal in K. Suppose κ is a singular cardinal in V. Must κ be measurable in K?

    For K below $0^{¶}$ this is known (Cox).

[4] If N is a model of condensation there is a function which witnesses it uniformly for all κ – so indiscernibles relative to that would do.

16. Suppose there is no inner model with a Woodin cardinal, and κ is a singular strong limit of uncountable cofinality, with $2^\kappa = \lambda$, some regular $\lambda > \kappa^+$. Must $o(\kappa)^K \geq \lambda$?

    A negative answer may have applications in pcf theory.

    Known below $0^\P$ (Gitik-Mitchell).