Noncongruence modular forms and modularity

organized by
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Workshop Summary

The *Noncongruence modular forms and modularity* conference was held at AIM from August 17-21, 2009. The goals of the conference were to report on recent progress in the field, identify areas for future research, both for short and long term projects, and begin work on (some of) these questions. Particular emphasis was placed on

- the recent progress and examples of Atkin, Hoffman, Li, Liu, Long, Verrill, and Hoffman's former REU groups in this area. Areas of focus were Atkin-Swinnerton-Dyer Congruences, Automorphy of Scholl representations attached to noncongruence cuspforms, and the Unbounded Denominator Conjecture,
- the links between the '$R = T$' modularity theorems on the congruence subgroup side and applications to noncongruence questions,
- computational questions involving specific examples, general questions such as how to describe a noncongruence subgroup computationally and the sort of data one might wish to amass in publicly available tables.

Introductory lectures were held in the morning and we broke into working groups in the afternoons. Participants were free to join the working groups. Each afternoon each working group briefed the audience what was done in the previous day so that people could decide which working group to join that day. In fact many people did switch groups.

The working groups were (roughly)

- Modularity
- Unbounded Denominator Conjecture (UBD)
- Atkin-Swinnerton-Dyer Congruences (ASD)
- Computational Methods

Our main goals were to formulate interesting questions and to start work on some of these. Listed below are the questions proposed during the workshop.

(1) For a finite index subgroup Γ of $SL_2(\mathbb{Z})$, denote by cd_Γ its congruence defect, that is, the index of Γ in the smallest congruence subgroup containing Γ. Thus Γ is a congruence subgroup if and only if $\mathrm{cd}_\Gamma = 1$. Let K be a number field. Does there exist a constant c_K such that for any smooth projective irreducible curve C defined over K, there exists a finite index subgroup $\Gamma \subset SL_2(\mathbb{Z})$ with $\mathrm{cd}_\Gamma \leq c_K$ so that there is a surjective morphism from the modular curve $X_\Gamma \twoheadrightarrow C$? When $K = \mathbb{Q}$ and C is an elliptic curve, the Taniyama-Shimura modularity theorem says that we can choose a Γ with $cd_\Gamma = 1$.

(2) What properties do Scholl representations have that distinguish them among all l-adic representations?
(3) Let $H_5 := \left\{ \gamma \in SL_2(\mathbb{Z}) \mid \left(\frac{\eta^{12}(11_z)}{\eta^{12}(z)}\right)^{1/5} |_\gamma = \left(\frac{\eta^{12}(11_z)}{\eta^{12}(z)}\right)^{1/5} \right\}$. This is a noncongruence subgroup whose Eisenstein series are known to have algebraic Fourier coeffcients. Find a basis for weight 2 Eisenstein series for H_5 and search for ASD congruences.
(4) Can we say anything about modularity of semi-linear Galois representations in general? This is a vast generalization of the results in Atkin-Li-Liu-Long, where certain Scholl representations admitting quaternion multiplications defined over quadratic extensions of $\mathbb{Q}$ were studied.
(5) Study l-adic representations numerically associated to 2 families of abelian varieties over specific Shimura curves and compute the zeta functions.
(6) Is there any analogue of Hecke operators to decompose Scholl's representations? We need auxiliary operators to do that. That is why character groups are being considered first.
(7) Does a (restricted) 4-dimensional Scholl representation with image in GO(4) factor to reflect the fact that GO(4) is a quotient of GL(2) $\times$ GL(2)?
(8) Is it fruitful to study noncongruence modular forms via representation theory?
(9) Study the unbounded denominator (UBD) conjecture via representation theory.
(10) Study the unbounded denominator conjecture for generalized modular forms (GMF). Is there any connection between generalized modular forms and noncongruence modular forms? (During the workshop, Kohnen and Mason proved a theorem which reduced the UBD discussion of GMF to UBD of noncongruence modular forms in a special case.)
(11) Is there any good and effective method to find the conductor of Galois representations constructed from X_Γ, such as Scholl representations?
(12) Is it possible that a potentially "bad" prime on modular curves becomes a good prime for the associated Galois representation? How do we detect those primes?
(13) Study Poincare series and Kloosterman sums for noncongruence subgroups and modular forms.
(14) Study the field of definition F of X_Γ and find equations (effectively) to define the field F.
(15) Study Scholl's theory for triangle groups and their generalizations.
(16) Is it possible to define "CM" forms for noncongruence modular forms?
(17) Is it possible to recover the characteristic polynomial of the Frobenius by modular symbols? In particular, what about the case that Scholl representation is decomposable?
(18) Suppose that $\dim_{\mathbb{C}}(S_k(\Gamma)) = 1$. In this case the Scholl representations are modular. How does this relate to the Tate conjecture?
(19) Belyi's theorem says that a smooth irreducible projective curve C defined over a number field K is a modular curve X_Γ for infinitely many finite index subgroups Γ of $SL_2(\mathbb{Z})$. What are common properties among such Γ that realize C? Is there any best subgroup among them?
(20) Is there congruence relation on the Fourier coeffcients of non-cuspidal noncongruence form?
(21) Study the ASD congruence for Eisentein Series and super congruences.

(22) Compare the characteristic polynomial of Frob_p both for l-adic cohomology and p-adic cohomology over the modular curve X_Γ.
(23) Develop a software package that would compute the L-functions of Scholl's representations.
(24) Compute examples of these representations where the genus of the base curve is greater than 0, or is defined over a number field larger than $\mathbb{Q}$.
(25) Create a database of noncongruence subgroups of small index inside $SL_2(\mathbb{Z})$, including such data as: defining equation for modular curve and j-map, a basis of modular forms of low weight.
(26) Find examples of weight 1 cusp forms (do Tonghai Yang's examples coming from Fermat curves, satisfy ASwD congruences?)

For the *modularity* and *ASD* groups, various related problems were formulated and approaches discussed, but no progress was made.

The *UBD* group worked on the following problems related to the unbounded denominator (UBD) properties and more generally p-adic properties of noncongruence modular forms:

- Find the relations between noncongruence modular forms and generalized modular forms (GMF) defined by Knopp and Mason, especially their unbounded denominator (UBD) behaviors.
- Given a non-arithmetic subgroup $\Gamma of SL_2(\mathbb{Z})$, discuss to what extent a fruitful theory of p-adic modular forms on Γ can be developed. One might ask, for example, whether there are analogues for non-congruence subgroups of the well-known phenomenon that modular forms on $\Gamma_0(p)$ are p-adic limits of modular forms on $SL_2(\mathbb{Z})$. This question could be approached through a better understanding of the geometry of the rigid analytic space $X_\Gamma(C_p)$ together with its natural projection $X_\Gamma \to P_1$.

The following progress has been made:

- Computed a special case of the UBD behavior of GMFs which might be reduced to the UBD behavior of noncongruence modular forms.
- Identified a noncongruence modular form which seems to be a p-adic limit of congruence modular forms.

The core of the *computational* group consisted of Bryan Birch, David Farmer, William Hoffman, Jonas Kibelbeck, Chris Kurth, Ben Linowitz, Richard Moy, Frederick Strömberg, John Voight, and was joined on Friday by Tong Liu, Ron Livné, Ravi Ramakrishna, and Anthony Scholl, both of whom gave presentations. Each day there was a discussion about which problems to address, followed by some study of the problems. Some highlights:

(1) There were presentations by Hoffman, Strömberg, Kurth, Farmer about the computational work done so far.
(2) A list was drawn up of desiderata for software packages dealing with subgroups of finite index in $SL_2(\mathbb{Z})$. (see that attachment).
(3) John Voight gave a tutorial on his Magma package for computing fundamental domains and traces of Hecke operators of subgroups of $SL_2(\mathbb{R})$ defined by quaternion algebras.
(4) Frederick Strömberg worked on his algorithms for computing modular forms on Fuchsian subgroups of $SL_2(\mathbb{R})$.

(5) David Farmer computed the conductor and Euler factors for several of the L-functions of Scholl's representations that were computed experimentally by Richard Moy. He explained his method to the group.
(6) Tong Liu lectured on how to bound the conductors of Galois representations.
(7) A start was made on computing Atkin-Swinnerton-Dyer congruences for Eisenstein series.