Open Problems and Questions:
Stochastic Methods for Non-Equilibrium Dynamical Systems

Notes by Ben Webb

Problem 1. *Suppose the deterministic system of coupled ODEs*

$$\dot{x} = f(x, y)$$
$$\dot{y} = g(x, y)$$

has a stable fixed point, at which the associated linear system has two complex eigenvalues. Then the linearized version of this system about this fixed point with a stochastic term

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} dt + \begin{bmatrix} \sigma_1 dW_1 \\ \sigma_2 dW_2 \end{bmatrix}$$

can be approximated by a constant times a rotation times pair of standard independent Ornstein-Uhlenbeck processes.

What is the analogue of this result in the context of a discrete time reaction diffusion equation?

Problem 2. *Given a Saini billard on a torus with a finite horizon and two masses m_1 and m_2, is there a way to find the system's viscosity, i.e. transfer of momentum. Additionally, is there a way to define* heat *in this model and can one check the Einstein relation? Lastly, if $m_1 \neq m_2$ then, in the limit, do the masses have the same kinetic energy?*

Problem 3. *Suppose we are given two expanding maps $T_{\beta_i} : X \to X$ for $i = 1, 2$ on $X = [0, 1]$. If these are randomly composed then, in this setting, there is a central limit theorem.*

Given an observable $\phi : X \to \mathbb{R}$, can one show in the annealed dynamics that, for almost every $P \in X \times \Omega$, the quenched system has a central limit theorem?

Problem 4. *For a discrete-time Lorentz gas with a finite horizon, is there always a measure of maximal entropy?*

Problem 5. *For a time dependent non-stationary process, is it possible to use coupling to study the statistical properties of this process?*

Problem 6. *Can we find Lasota-York type inequalities for a composition of operators?*

Problem 7. *Do stable laws exist for a non-stationary process? Also, is it possible to find a diffusion rate if there are is no limiting law?*

Problem 8. *Is there an annealed almost sure invariant principle (asip) with rate $n^{1/4} \log n$ instead of $n^{1/4+\delta}$?*
Suppose μ is an invariant measure of the system $y_{n+1} = Ty_n$. For $\epsilon > 0$, how does the dynamics of the system

$$x_{n+1}^{\epsilon,y} = x_n^{\epsilon,y} + \epsilon f(x_n^{\epsilon,y}, y_n)$$

compare with the dynamics of the system

$$\bar{x}_{n+1} = \bar{x}_n + \epsilon \int f(\bar{x}_n, y_n) d\mu(y)?$$

Moreover, what can be said about the quantity $\sup_{n<1/\epsilon} |\bar{x}_n - x_n^{\epsilon,y}|$*, specifically with respect to limit theorems?*

Problem 9. *Can one either determine or improve any estimates on the rate of convergence to the Poisson limit law in slowly mixing systems? For instance, from say a decay rate of* $1/n^4$ *to* $1/n$*.*

Problem 10. *What can be said about return times for coupled map lattices in the infinite case? Similarly, what can be said about almost sure invariant principles for such maps? That is, how fast does the signal go to infinity in such systems? Also, what the case of finite verses infinite coupling?*

Problem 11. *Is there some limiting distribution and/or escape rate in the case that can be found in the case of eclipsing scatterers?*

Open Problems and Questions: Tuesday

Problem 12. *Calculate transport coefficients or related physical quantities for a variety of billard systems.*

Problem 13. *Suppose the deterministic system of coupled ODEs*

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has a stable fixed point, at which the associated linear system has two complex eigenvalues. Then the linearized version of this system about this fixed point with a stochastic term

$$\begin{bmatrix} dx \\ dy \end{bmatrix} = A \begin{bmatrix} x \\ y \end{bmatrix} dt + \begin{bmatrix} \sigma_1 dW_1 \\ \sigma_2 dW_2 \end{bmatrix}$$

can be approximated by a constant times a rotation times pair of standard independent Ornstein-Uhlenbeck processes.

What is the analogue of this result in the context of a discrete time reaction diffusion equation? Also, what is the connection of this system to more general fast/slow systems?

Problem 14. *Quenched central limit theorem (CTL): Are the normalizing constants and variance almost surely the same?*

Problem 15. *Can one use generalized function spaces to get martingales for concatenations of random hyperbolic maps?*

Problem 16. *What kind of limit theorems can be found for random billards with moving/deforming scatterers?*

Problem 17. *What new statistical properties can be found for slowly mixing systems.*

Problem 18. *What can be said about spectral properties/Lyapunov spectrum in sequential or nonstationary settings?*

Problem 19. *What can be said about return times in coupled map lattices (CML) in the infinite case? What about related problems regarding almost sure invariance principles?*

Open Problems and Questions: Wednesday

Problem 20. *Calculate transport coefficients or related physical quantities for a variety of billard systems.*

Problem 21. *Can one use function spaces to get martingale decompositions and concatenations for some class of hyperbolic maps?*

Problem 22. *What kind of limit theorems can be found for random billards with moving/deforming scatterers?*

Problem 23. *What type of limiting theorems/statistical properties can be found for slowly mixing systems.*

Problem 24. *What can be said about spectral properties/Lyapunov spectrum in sequential or nonstationary settings?*

Problem 25. *What can be said about hitting times/EVT to shrinking targets in billards where the target is not a ball but a strip induced by a set in configuration space?*

Open Problems and Questions: Thursday

Problem 26. *Calculate transport coefficients or related physical quantities for a variety of billard systems.*

Problem 27. *Can one use function spaces to get martingale decompositions and concatenations for some class of hyperbolic maps?*

Problem 28. *Can one prove existence and/or physical properties of a measure of maximal entropy for dispersing billards? Connections to periodic orbits?*

Problem 29. *How can one bound rates of convergence, using the Wasserstein distance, for Markov generators of the type,*

$$LA(x_1, \dots, x_n) = \sum_i \int \omega_i(x, dy)[A(y) - A(x)],$$

which involve only a finite number of $x's$ and $y's$ near i.

Problem 30. *What type of limiting theorems/statistical properties can be found for slowly mixing systems?*

Problem 31. *What can be said about spectral properties/Lyapunov spectrum in sequential or nonstationary settings?*

Problem 32. *Consider the following system of equations with constraint*

$$\begin{aligned}
\dot{x} &= a(x, y) + \frac{1}{\epsilon} b(x) v(y) \\
\dot{y} &= \frac{1}{\epsilon^2} g(y), \\
\int & v(y) du(y) = 0.
\end{aligned}$$

Assuming multiple correlations, at some given rate for μ, can one show

$$x_\epsilon \to_\omega X \text{ where } dX = a(x)dt + b(x) * d\omega?$$