P-ADIC REPRESENTATIONS, MODULARITY, AND BEYOND

organized by
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Workshop Summary

The past year has seen remarkable progress towards our understanding of two-dimensional representations of the absolute Galois group of the rational numbers: two fundamental problems, Serre's modularity conjecture and (the two-dimensional case of) the Fontaine-Mazur conjecture, both seemingly far out of reach as recently as the beginning of 2005, may soon be proved in entirety. The purpose of our workshop was to take stock of recent progress, and to assess the prospects for future work (both short term and long term).

Before describing what took place at the workshop, we give a brief synopsis of these conjectures and their status at the beginning of 2005.

Let p be a prime number. By a celebrated theorem of Deligne [Deligne], to any modular form one can associate a two dimensional p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, such that the Fourier coefficients of the modular form can be recovered from the character of the representation. One can reduce this representation modulo p, to obtain a representation defined over a finite field. Serre's conjecture states that any continuous, odd, absolutely irreducible two dimensional representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ over a finite field arises in this manner from a modular form.

Similarly, the Fontaine-Mazur conjecture for $\mathrm{GL}_2(\mathbb{Q})$ states that any two-dimensional p-adic representation ρ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ satisfying a short list of necessary conditions arises as above from a modular form. The most subtle of these conditions is a hypothesis not on the whole of ρ but on the restriction ρ_p of ρ to $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$: ρ_p must be *potentially semistable*, which in geometric terms is expected to mean that there exists a finite extension $K/\mathbb{Q}_p$ and a variety V/K with semistable reduction such that ρ_p restricted to $\mathrm{Gal}(\overline{K}/K)$ is a subquotient of the cohomology of $H^{k-1}(V, \overline{K})$ for an integer $k \geq 2$. In practice one uses a more concrete description of potentially semistable representations — this is Fontaine's theory [AST223]. The integer k is the weight of the modular form from which ρ should arise.

The ideas that were discussed at the workshop were greatly influenced by two preceding developments, which we describe now. The first of these is Kisin's strengthening [KisinModularity] of the Taylor-Wiles method. To describe it, we must recall Wiles's approach to the Fontaine-Mazur conjecture [Wiles, TaylorWiles]. Beginning with a mod p representation $\overline{\rho}$ of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, Wiles proves that the space of all p-adic lifts of $\overline{\rho}$ (satisfying certain conditions) is equal to the space of all modular lift of $\overline{\rho}$ (satisfying the same conditions), and deduces that all such lifts of $\overline{\rho}$ are modular. Crucially, one must assume that $\overline{\rho}$ satisfies Serre's conjecture, to know that there are any modular lifts at all.

A key point in Wiles's method is that a certain space of lifts of $\overline{\rho}_p = \overline{\rho}\,|_{\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)}$ must be parameterized by a power series ring in one variable. Wiles was able to prove this when the prime p is odd, the field K equals $\mathbb{Q}_p$, and the weight $k = 2$; subsequently, a number of authors were able to weaken the hypothesis that $K = \mathbb{Q}_p$ for some pairs $(\overline{\rho}, K)$. Kisin's breakthrough in [KisinModularity] enabled him to circumvent this point entirely, and (when $k = 2$) to prove that the two spaces of lifts are the same by analyzing deformation spaces of finite flat group schemes. Thus Kisin was able to prove the Fontaine-Mazur conjecture for p odd and weight $k = 2$ (except for certain degenerate cases), conditionally on Serre's conjecture. The case of weight $k > 2$ remained almost entirely open.

Serre's conjecture also remained almost entirely open. For representations into $\mathrm{GL}_2(\mathbb{F}_2)$ and $\mathrm{GL}_2(\mathbb{F}_3)$, Serre's conjecture follows from work of Langlands and Tunnell [Tunnell]. This provides an entry point that allows one to prove the modularity of elliptic curves [BCDT]; in turn, these modularity theorems allowed one to prove cases of Serre's conjecture for representations into $\mathrm{GL}_2(\mathbb{F}_q)$ for other small values of q $(q = 4, 5, 7, 8, 9)$.

The second driving force behind recent progress is Breuil's p-adic Langlands philosophy. If one attempts to create a correspondence between potentially semistable p-adic representations ρ_p of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and p-adic representations of $\mathrm{GL}_2(\mathbb{Q}_p)$ that mimics the classical local Langlands correspondence, one quickly runs into the difficulty that the Weil-Deligne representatoin associated to ρ_p loses significant information about ρ_p (namely, its *Hodge filtration*). Breuil's idea is that to ρ_p one should associate to a Banach space representation; specifically, one should associate to ρ_p a completion of a certain natural construction using the Weil-Deligne representation, where the choice of the particular completion corresponds to the Hodge filtration. Using this philosophy, Breuil and Mézard [BreuilMezard] were able to give a precise conjecture for the Hilbert-Samuel multiplicity of the space of lifts of $\overline{\rho}_p$ from the previous paragraph.

We now turn to the workshop, and the astonishing progress in the past year.

Breuil opened the workshop with an overview of p-adic Langlands correspondences aimed at getting non-experts up to speed, and he outlined a preliminary conjecture, joint with P. Schneider, for a very weak form of p-adic local Langlands for $\mathrm{GL}_n(F)$.

Berger followed with a status report on p-adic Langlands correspondences. In particular he gave a detailed account of his proof [BergerReduction, APC] (joint with Breuil, and using an idea of Colmez [ColmezSemistable]) of the p-adic Langlands correspondence for representations which become crystalline over an abelian extension of $\mathbb{Q}_p$ (i.e. for which $K/\mathbb{Q}_p$ is an abelian extension; the so-called "crystabeline" case), and the compatibility of the p-adic Langlands correspondence with reduction modulo p.

Kisin spoke next, and announced a proof of the Fontaine-Mazur conjecture for $\mathrm{GL}_2(\mathbb{Q})$ in arbitrary weight for p odd, as well the Breuil-Mézard conjecture — conditionally on some claims of Colmez about the p-adic Langlands correspondence, but unconditional in the crystabeline case (and, as before, excepting some degenerate $\overline{\rho}$). Kisin also circulated a preprint [KisinFM]. The crystabeline locally irreducible case had previously been announced at a conference in Montréal in September. The conditional generalization had developed subsequently, after incorporating work of Colmez that was also announced in Montréal (but has not been written up yet). As usual, the result is conditional on Serre's conjecture for $\overline{\rho}$ (but see below).

We give a quick sketch of Kisin's ideas. Kisin observes that the Breuil-Mézard conjecture is equivalent to modularity: his approach to the Taylor-Wiles method unconditionally gives one inequality in Breuil-Mézard conjecture, and the space of all lifts is equal to the space of modular lifts precisely when the reverse inequality is true. This proves Breuil-Mézard for $k = 2$, using the modularity theorem for $k = 2$ described previously. Next, Kisin uses Colmez's "Montréal functor" to show that the Breuil-Mézard conjecture follows from the compatibility of the p-adic Langlands correspondence with reduction modulo p. By the work of Berger and Breuil, this proves the Fontaine-Mazur conjecture for crystabeline representations. Colmez may in the future be able to prove that the p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ has the necessary property in general; if this is so, the proof of Fontaine-Mazur becomes unconditional.

Emerton spoke and announced a second proof of the Fontaine-Mazur conjecture for $\mathrm{GL}_2(\mathbb{Q})$, again conditional on this anticipated work of Colmez but via an entirely different approach. Emerton [Emerton] had previously shown that if the p-adic Langlands correspondence exists and satisfies a certain local-global compatibility conjecture, then Fontaine-Mazur conjecture follows. (In fact Emerton is able to prove somewhat more: that every trianguline representation arises from an overconvergent cuspidal eigenform of finite slope.) In his talk at the workshop, Emerton sketched a proof (under certain mild hypotheses) of his local-global compatibility conjecture, using Colmez's results from Montréal. This shows in a second way that the Fontaine-Mazur conjecture would follow from the existence of the p-adic Langlands correspondence; in particular the crystabeline case follows from the work of Berger and Breuil, and the general case would follow from the possible future work of Colmez to which we have already alluded.

Colmez was invited to the workshop but did not attend; our information about Colmez's claims was limited to what Emerton was able to tell us about the Montréal functor.

The workshop heard next from Jean-Pierre Wintenberger, who in joint work with Chandrashekhar Khare has nearly completed a proof of Serre's conjecture. Khare made a significant breakthrough in early 2005, proving Serre's conjecture for representations unramified outside of p [KhareLevelOne]. The proof was novel and surprising, combining modular lifting theorems (including Kisin's theorem for $k = 2$) with a clever induction argument using a theorem of Carayol. Khare and Wintenberger now believe they can prove Serre's conjectures for all representations which are unramified at 2. Their proof requires another clever induction (contained in a preprint [KWOdd] that was circulated at the workshop) together with new modularity theorems for $p = 2$ (no preprint yet available). The workshop participants were particularly interested in grilling Wintenberger on the latter, and were eventually satisfied that their arguments seem likely to be correct. A natural question is whether the modularity theorems at $p = 2$ that would be necessary to prove the Serre's conjecture in full are within reach; Kisin believes this is so.

At this point we broke into discussion groups. There was a consensus that in the short term, it is critical that we begin to come to grips with p-adic local Langlands for $\mathrm{GL}_2(F)$ with F a finite extension of $\mathbb{Q}_p$. Our understanding is sufficiently poor that we do not even know the mod p correspondence for $\mathrm{GL}_2(F)$ when $F \neq \mathbb{Q}_p$. (For $F = \mathbb{Q}_p$ it is due to Barthel-Livné and to Breuil [BreuilGL2I].) Paskunas [Paskunas] has constructed the "right number" of irreducible mod p representations of $\mathrm{GL}_2(F)$, but they do not appear to correspond in a natural way to the irreducible two dimensional mod p representations of $\mathrm{Gal}(\overline{F}/F)$. It was agreed that an important problem on which progress could conceivably be made during the workshop was the question of the mod p local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_{p^2})$, and one group worked intently on this problem for the remainder of the week. The group eventually concluded, based on Emerton's conjectured local-global compatibility for p-adic Langlands and on the Buzzard-Diamond-Jarvis conjecture on the weights in Serre's conjectures for totally real fields, that there must be irreducible two dimensional mod p

representations of $\mathrm{GL}_2(F)$ other than those constructed by Paskunas. This is an important conceptual breakthrough, and is described in more detail in Emerton's report. Buzzard reported on this breakthrough at the workshop, and Taylor's student Michael Schein gave a talk outlining Paskunas's construction, to help the audience get up to speed.

A second discussion group quizzed Dieulefait on aspects of the proof of Serre's conjecture that were not touched on by Wintenberger. Kedlaya met with a group to explain the theory of (φ, Γ)-modules, which is crucial to Colmez's approach to p-adic Langlands [ColmezSemistable].

Another question on the minds of many participants was how Serre's conjecture should be generalized to representations of dimension greater than two. This problem has begun to garner attention in recent years, and Taylor concluded the workshop by reporting on the thesis of his student Florian Herzig, who it seems has made important conceptual progress for representations of dimension three.

Overall, we were quite satisfied with the arrangements provided by AIM for this workshop; feedback from several participants corroborates this. Allowing flexibility in scheduling made it possible for those listening to talks to make detailed inquiries of the speaker, and there was plenty of additional discussion going on over lunch, during the afternoon happy hour, and in the evenings. The AIM-provided web site provides a useful snapshot of the workshop by gathering together notes from several speakers, additional notes taken by Michael Volpato, a report by Emerton on the calculations related to Paskunas's construction, and preprints by Khare-Wintenberger and Kisin.

Our workshop came at a busy time in the theory of p-adic Galois representations, having followed closely behind the Montreal meeting discussed earlier, and having preceded closely an intensive semester at Harvard on the topic of "eigenvarieties". We are hopeful that it will be seen in retrospect to have served an important role in bringing the experts up to date on each other's work and indicating directions in which further progress will be made.