AIM OPEN PROBLEM SESSION

Problem 1. Browning: Can we develop a version of the circle method over $\mathbb{Q}(t)$?

Wooley: The major arcs are difficult to understand.

Browning: For example, consider the diagonal quadratic form

$$A_1 x_1^2 + \cdots + A_s x_s^2 = 0 \tag{1}$$

with $A_j \in \mathbb{Q}(t)$. There are "obvious" local conditions, e.g., arising from discrete valuation rings for $\mathbb{Q}(t)$. Does the Hasse principle hold?

Wooley: If A_j are linear, say $A_j = c_j + td_j$ with $c_j, d_j \in \mathbb{Q}$, then (1) is equivalent to the system of equations

$$\sum_{j=1}^{s} c_j x_j^2 = \sum_{j=1}^{s} d_j x_j^2 = 0 \tag{2}$$

by Amer–Brumer, which defines a quartic del Pezzo surface over $\mathbb{Q}$. Do the "obvious" local obstructions over $\mathbb{Q}(t)$ for (1) capture the Brauer–Manin obstruction over $\mathbb{Q}$ for (2)?

Harari: We may ask these questions over $\mathbb{Q}_p(t)$; see work of Harari–Szamuely.

Problem 2. Cheltsov: What is the "right" assumption on a variety V for considering height zeta functions? Definitely smooth V with ample anticanonical sheaf ω_V^{-1} should be allowed. How about klt (i.e., Kawamata log terminal) V? Or V of Fano type (i.e., there is an effective $\mathbb{Q}$-divisor Δ such that (V, Δ) is ample and $-(K_V + \Delta)$ is ample)?

An example for the latter: A quasi-smooth hypersurface $V \subset \mathbb{P}(a_0, \ldots, a_n)$ in weighted projective space of degree $\deg(V) < a_0 + \cdots + a_n$.

Problem 3. Skorobogatov: Does Bhargava's machinery have implications for the Hasse principle for special surfaces?

For example, let $F, G \in \mathbb{Q}[x, y]$ be homogeneous polynomials of degree 3. Consider the cubic surface

$$S = \{F(x, y) = G(z, w)\} \subset \mathbb{P}^3_{\mathbb{Q}}.$$

The defining equation is equivalent to the system of equations

$$\{u^3 = tF(x, y),\ v^3 = tG(z, w)\}.$$

This defines a family of cubic twists of curves of genus 1 over the t-line. Swinnerton-Dyer has discussed how to search for t such that this system is solvable over $\mathbb{Q}$ in the diagonal case [Ann. Sci. ENS]. Can we extend his work beyond the diagonal case?

Similarly, consider Kummer K3 surfaces defined by

$$z^2 = f(x)g(y),$$

where f, g are quartic separable polynomials. This is equivalent to the family of quadratic twists of curves of genus 1 defined by

$$u^2 = tf(x),\ v^2 = tg(y).$$

The goal is to eliminate the condition in Swinnerton-Dyer's work that (the 2-primary part of) Ш has finite order for quadratic twists, using the recent work presented in Bhargava's talk.

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Problem 4. Viray: Let $\phi : X \to E$ be a fibration over an elliptic curve of positive rank over $\mathbb{Q}$ whose generic fiber is smooth and geometrically irreducible. Let

$$Z = \{p \in E(\mathbb{Q}) \mid X_p = \phi^{-1}(p) \text{ has points everywhere locally}\}.$$

What can we say about Z? Is $|Z| < \infty$ with $Z \neq \emptyset$ possible?

The motivation is that work of Poonen, Skorobogatov–Harpaz and Colliot-Thélène–Pal–Skorobogatov constructs X failing the Hasse principle such that none of the known obstructions apply. All of these use a map $X \to C$ to a curve with $0 < |C(\mathbb{Q})| < \infty$.

Browning: The case where ϕ is a conic bundle may already be interesting.

Problem 5. Harari: The following question is due to Borovoi: Consider weak approximation for $X = \mathrm{SL}_n / G$ over $\mathbb{Q}$, where G is a finite group scheme that is not necessarily constant. For example, is $X(\mathbb{Q})$ dense in $X(\mathbb{R})$? If G is constant, this is known to be true.

A variant is the following. Given $X = \mathrm{SL}_n / G$ with a constant finite group scheme G over a number field K with $r \geq 2$ real places $v_1, \dots, v_r$. Is $X(K)$ dense in $\prod_{j=1}^{r} X(K_{v_r})$?

A formulation via non-abelian Galois cohomology is given in the case $K = \mathbb{Q}$ as follows: Is

$$H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), G(\overline{\mathbb{Q}})) \to H^1(\mathrm{Gal}(\mathbb{C}/\mathbb{R}), G(\mathbb{C}))$$

surjective?

Lucchini Arteche: The algebraic Brauer–Manin obstruction says nothing for this problem.

Skorobogatov: The result is known for abelian G, due to Borovoi.

Problem 6. Wittenberg: Let X be a smooth variety over a number field K, let S be a finite set of places. Assume that X satisfies strong approximation outside S. Take a closed subvariety $Z \subset X$ of codimension two. Does $X \setminus Z$ satisfy strong approximation outside S?

Tschinkel: We can also ask for Zariski density (of S-integral points).

Wittenberg: The result is known for $X = \mathbb{A}^n$ and arbitrary Z of codimension 2. Interesting cases are:

- Wittenberg: affine quadric hypersurfaces $X \subset \mathbb{A}^4$, for example, defined by $q(x_1, x_2, x_3, x_4) = c$ for a quadratic form q and a constant c.
- Harari: X a simply connected linear algebraic group
- Wooley: is this true by the circle method for hypersurfaces of fixed degree as soon as the dimension is large enough? Heath-Brown: for example, is this true for quadrics $X \subset \mathbb{A}^5$?

Harari: If X satisfies strong approximation, then X is algebraically simply connected. If X is algebraically simply connected, then $X \setminus Z$ is also simply connected. Hence considering π_1 should not be helpful to get a counterexample to the problem. Also Brauer groups are not expected to be helpful.

Heath-Brown: Can we drop the condition that $Z \subset X$ has codimension ≥ 2? Check the topology.

Colliot-Thélène: Can the circle method be used to prove that $\pi_1(X)$ is trivial? For example, the circle method handles

$$x_1^{r_1} - x_2^{r_2} + x_3^{r_3} - \cdots \pm x_n^{r_n} = c \in \mathbb{Z}$$

in sufficiently many variables. Can we show that $\pi_1(X)$ is trivial without the circle method?

Problem 7. Heath-Brown: Can you construct a sequence of smooth projective varieties $X_k \subset \mathbb{P}^k_{\mathbb{Q}}$ with $X_k(\mathbb{Q}_p) \neq \emptyset$ for all places p but $X_k(\mathbb{Q}) = \emptyset$ such that $\frac{\dim(X_k)}{\deg(X_k)}$

is unbounded? Browning–Heath-Brown have given a sequence where $\frac{\dim(X_k)}{\deg(X_k)}$ tends to $\frac{1}{3}$ and $\dim(X_k)$ is tends to ∞.

Wooley: How about removing the requirement of smoothness and considering the singular norm forms

$$N_{K/\mathbb{Q}}(x_1\alpha_1 + \cdots + x_d\alpha_d) = ct^d$$

where $d = [K : \mathbb{Q}]$?

Heath-Brown: What happens when $X_k \subset \mathbb{P}^k$ is a hypersurface? Is there any example of a smooth hypersurface of dimension ≥ 3 failing the Hasse principle?

Colliot-Thélène: Sarnak–Wang have shown that the Bombieri–Lang conjecture would imply that there are many such examples of general type.

Wooley: An analytic attack to show that there exist some such varieties could be as follows. Choose a locally soluble smooth hypersurface $Y \subset \mathbb{P}^N$ of degree $d \gg N$. The determinant method implies that the number of points in a large box grows slowly. Intersect with linear subspaces to maintain local solubility. Use a counting argument to find a linear section without rational points.

Colliot-Thélène: Won't this just force the coefficients to be large?

Harari: Does $\frac{\dim(X_k)}{\deg(X_k)} \to \infty$ imply that X_k is geometrically rationally connected? Note that if X over $\mathbb{Q}$ is a geometrically rationally connected complete intersection, then the Hasse principle is hard to obstruct cohomologically.

Browning: A conjecture of Hartshorne implies that if $Y \subset \mathbb{P}^N$ is smooth, non-degenerate, with $\dim(Y) \geq 2\deg(Y) + 1$, then Y is a complete intersection, hence rationally connected. Therefore, it might be easier to look for examples with

$$\frac{1}{3} < \frac{\dim(X_k)}{\deg(X_k)} \leq 2$$

in Heath-Brown's original question.

Tschinkel: Let X be a Fano variety over $\mathbb{C}$. Can we have $\mathrm{Br}(X) = H^3(X, \mathbb{Z})_{\mathrm{tors}} \neq 0$ in all dimensions ≥ 4?

Problem 8. Várilly-Alvarado: Skorobogatov has asked whether a K3 surface X over $\mathbb{Q}$ can have odd order torsion in $\mathrm{Br}(X)$ obstructing the Hasse principle? Even in $\mathrm{Br}_1(X)$?

Skorobogatov: For example, for quartics $X \subset \mathbb{P}^3$ and $\alpha \in \mathrm{Br}(X)[u]$ for u odd: For each place v, there exists a zero cycle Z_v over $\mathbb{Q}_v$ of degree one such that α is orthogonal to Z_v. Then a conjecture of Colliot-Thélène implies that X has zero cycles of degree one over $\mathbb{Q}$. Will there be a rational point? So given a quartic surface $X \subset \mathbb{P}^3$ with $(\mathrm{Br}(X)/\mathrm{Br}(\mathbb{Q}))[2] = 0$, does the Hasse principle hold?

Tschinkel: What about weak approximation? Skorobogatov: This will probably fail.

Hassett: How about $X \subset \mathbb{P}^4$ of degree six?

Problem 9. Wooley: Consider the set

$$Q_k := \{Q(y_1^k, \ldots, y_s^k) \in \mathbb{Q}[y_1, \ldots, y_s] \mid Q \in \mathbb{Q}[x_1, \ldots, x_s] \text{ quadratic form}\}$$

of certain forms of degree $2k$. Fixing k, how large must s be in order for the Hasse principle to hold? Let

$$h(k) := \inf_{s \in \mathbb{N}} \{s \mid \text{the Hasse principle holds for all } Q \in Q_k \text{ in } s \text{ variables}\}.$$

Challenge: prove that $\log h(k) = o(k)$ as $k \to \infty$.

By Birch's result on forms in many variables, we know that $h(k) \leq 2k \cdot 2^{2k}$. On the other hand, for diagonal forms of degree k (i.e., $Q(y_1^k, \ldots, y_s^k)$ with linear $Q \in \mathbb{Q}[x_1, \ldots, x_s]$, we have the much stronger bound $h(k) \sim 2k(\log k)$.

Browning: Replace k-th powers by norm forms from fixed extensions of degree k.

Problem 10. Peyre: Back to quartic surfaces $X \subset \mathbb{P}^3$. Let U be the complement of all rational curves over $\mathbb{Q}$. Assume $U(\mathbb{Q}) \neq \emptyset$. There is numerical evidence that

$$\#\{u \in U(\mathbb{Q}) \mid H(u) \leq B\} \sim c(\log B)^{\rho(X)}$$

where $\rho(X)$ is the rank of the Picard group of X and c is the product of local densities. Can such a formula hold without weak approximation being valid?

Tschinkel: Given a quartic K3 surface $X \subset \mathbb{P}^3$, with $x \in X(\mathbb{Q})$. Is there a procedure for deciding whether x lies on a rational curve over $\mathbb{Q}$?

Skorobogatov: What about removing elliptic curves?

Browning: Is there any numerical evidence? (See van Luijk's work.) What is the Peyre freedom of rational curves of small degree on K3 surfaces?

Colliot-Thélène: Given a K3 surface X over $\mathbb{Q}$ and $x \in X(\mathbb{Q}) \neq \emptyset$, does there exist a rational curve $R \subset X$ over $\mathbb{Q}$? Does there exist a rational curve $R \subset X$ over $\mathbb{Q}$ containing x?

Tschinkel: How about finite fields? Let X be a K3 surface over $\mathbb{F}_q$ and $x \in X(\mathbb{F}_q)$. Does there exist a rational curve $R \subset X$ defined over $\mathbb{F}_q$ containing x? Are there any rational curves $R \subset X$ defined over $\mathbb{F}_q$? Bogomolov–Tschinkel show for Kummer surfaces X that we can find rational curves over $\overline{\mathbb{F}}_q$ for most $x \in X$.

Skorobogatov / Testa: Are there K3 surfaces X over $\mathbb{Q}$ with infinitely may rational curves over $\mathbb{Q}$ and $\mathrm{Pic}(X_{\mathbb{C}}) \cong \mathbb{Z}$?