Towards Relative Symplectic Field Theory

organized by
Kai Cieliebak, Tobias Ekholm, Yakov Eliashberg, Kenji Fukaya, Dennis Sullivan, and Michael Sullivan

Workshop Summary

1. The goal

The goal of this workshop was to understand the structure of relative Symplectic Field Theory (SFT), and to discuss its analytical foundations as well as possible applications. To achieve this goal, it brought together leading researchers in the areas considered relevant to relative SFT: Floer homology, absolute SFT, Legendrian contact homology, and string topology.

2. Organization

The morning lectures provided introductions to various topics that are expected to play a role in relative SFT: string topology (D. and M. Sullivan), A_∞-algebras (Fukaya, Oh, Ohta, Ono), cluster homology (Cornea, Lalonde), Lagrangian immersions (Akaho), and homological algebra (Terilla). Moreover, three approaches to relative SFT were suggested: one based on clusters (Ng), one based on string topology (Cieliebak, Latschev, Mohnke), and one based on parallel copies (Ekholm).

In the afternoon sessions the participants broke up into two groups: the first one concentrating on 1-dimensional Legendrian knots, the second one pursuing the general case. Moreover, on the first and on the last day we had discussion sessions on open problems.

Here is a more detailed schedule of the workshop.

Monday, September 24

9.00 am	Tobias Ekholm
	Introduction to the problem of relative SFT
11.00 am	Dennis Sullivan
	Homotopy theory package for relative SFT
2.00 pm	Lenny Ng
	Discussion of failed atempts to relative SFT
4.00 pm	Dusa McDuff
	Discussion of open problems

Tuesday, September 25

9.00 am	Kenji Fukaya, Hiroshi Ohta and Kaoru Ono
	Homotopy theory of A_∞-algebras
2.00 pm	Breakout
	group 1: the 1-dimensional case

group 2: the general case
6.45 pm Workshop Banquet

Wednesday, September 26

9.00 am Janko Latschev
A proposed setup for relative SFT
11.00 am Octav Cornea
Cluster homology
2.00 pm Breakout
group 1: the 1-dimensional case
group 2: the general case

Thursday, September 27

9.00 am Dennis Sullivan
String topology
11.00 am Tobias Ekholm
Parallel copy relative SFT
2.00 pm Breakout
group 1: the 1-dimensional case
group 2: algebraic structures and transversality issues

Friday, September 28

9.00 am Eleny Ionel
Discussion of progress and open problems
11.00 am John Terilla
Processing algebraic information in SFT
12.00 am Manabu Akaho
Floer homology for Lagrangian immersions

3. The results

The results of the workshop can be summarized as follows:

(1) The string topology approach to relative SFT reduces to a purely combinatorial theory in dimension one. The existence of the theory for as given Legendrian knot can be shown combinatorially; a combinatorial proof of invariance appears more difficult but within reach. First computations suggest that the theory is highly nontrivial; in particular, in contrast to Legendrian contact homology, it might distinguish stabilized knots. After symmetrization, the theory coincides with the one based on clusters; the latter is not invariant, at least not with its algebra structure. The resolution of the string-transversality problems, see (3) below, in this case was obtained by imposing conditions on the asymptotics of the topological strings which are naturally satisfied by holomorphic curves.

(2) String topology plays a central role in relative SFT. This appears already in the definition in terms of string topology operations. At the next stage, relative SFT for Lagrange cobordisms seems to require a string topology for manifolds with boundary; the outline of

such a theory was developed in the course of this workshop. Conversely, ideas from SFT have also fed back into string topology. Most notably, D. Sullivan's formulation of string topology in terms of operations of chains in the moduli space of Riemann surfaces on chains in string space brings the two subjects even closer together.

(3) Transversality has proved notoriously difficult in SFT and is still not rigorously established even in the absolute case. However, several groups are exploring different approaches, and it is likely that these programs will be completed soon. In particular, the polyfold program by Hofer, Wysocki and Zehnder is close to being completed. Another approach, based on the theory of Kuranishi structures, which was used by Fukaya, Ohta, Ono and Oh in the Lagrangian Floer homology theory, will also likely to generalize to the relative SFT case.

Note that relative SFT encounters an additional problem of transversality in string topology. Discussions during the workshop indicate that this problem may be circumvented by passing to "diffuse" string topology operations, at the expense of replacing strict algebraic structures by infinity structures.

(4) Homological algebra will be another important ingredient in relative SFT. This has already become apparent in the work of Fukaya, Oh, Ohta and Ono on Floer homology, as well as the work of D. Sullivan on string topology. In particular, the formalism of relative SFT should be formulated in terms of the homotopy theory of algebraic infinity structures. The relevant homological algebra is already being developed in several ongoing projects. Furthermore, it seems likely that the parallel copies version of SFT gives successive approximations to the string topology version.

4. Conclusion

The workshop had a vibrant atmosphere, with lively discussions and a free flow of ideas between participants. It benefited from the presence of researchers from diverse fields, which led to a great cross-fertilization between different groups. It provided a stimulating environment for existing collaborations as well as new partnerships. The list of problems compiled on the last day of the conference provides a guidance for completion of the relative SFT project.

5. Appendix by Dennis Sullivan

This workshop was of a high quality. The AIM arrangements were helpful, the participants were of the highest calibre and the mathematics was fecund and exciting. The AIM encouraged structure of exposition followed by group efforts with discussion is a useful model with one caveat described below.

In retrospect, I feel more significant progress could have been achieved during the week. If the structure "exposition followed by group discussion" would have been employed on the scale of the entire week instead of on the scale of each day the precise nature of our problem and available rersources would have interacted differently. By this 'week scaling" of the AIM encouraged structure I mean during the first part of the week everyone who wants to explains where they are in the problem, all the cards put on the table; and then with all of this data in everyone's mind during the second part of the week we go to the second stage and jointly hit the the problem with group discussions, blackboard work etc.

For my part by the end of the actual workshop exposition, say on thursday plus one more day of thinking , I understood much better where everyone was, what their difficulties and strengths were and what part of the algebra that I understood would be worthwhile to share at the blackboard and relate to the problem of the workshop.

This precise thing didn't happen. In particular, the group working on Legendrian knots went on working without enough relief from cross fertilization which was in some sense not available yet because the time scale being used was too short.

In summary I feel that while the model of exposition followed by discussion encouraged by the administrators of the workshop is of course a good idea, I think that the scale of the splitting up of exposition and discussion should be influenced by the organizers with specific knowledge if they have a preference. I also feel that in this instance that if the organizers were allowed, as they originally wanted to have a longer scale with more expositions up front, then our workshop would have been even more efficient, though it was a splendid workshop anyway by normal standards.