Sphere Packings, Lattices, and Infinite Dimensional Algebra

organized by
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Workshop Summary

In this workshop, we examined the techniques of Cohn-Elkies and Cohn-Kumar on sphere packings and related problems. Cohn and Elkies give monotonic series of bounds on sphere packings in dimension n, which for $n = 8$ and $n = 24$ are conjectured to converge to the densities of the sphere packings coming from the E_8 and Leech lattices; this conjecture, based on extensive numerical computation and supported by some structural evidence, implies the optimality of those packings. Cohn and Kumar used these bounds, and related new techniques, to prove the optimality of E_8 and Leech among lattice packings. More recently, in order to better understand such bounds Cohn and Kumar introduced the notion of a universally optimal spherical code, and proved that many tight spherical codes — including several arising from the E_8 and Leech lattices — are universally optimal.

During the workshop, there was a new collaborative work of Cohn, Conway, Elkies and Kumar on 24-point spherical codes in R^4. They found that the D_4 configuration, though tight and recently proved optimal by Musin, is not universally optimal, which may help explain also for sphere packings why the 4-dimensional case is more resistant to their methods despite the tightness of the D_4 packing and kissing configurations.

In a different direction, we discussed the status of a number of open problems relating to the "moonshine module vertex operator algebra." These included questions concerning the structure of this moonshine module, whose automorphism group is the Monster finite simple group, and in particular, work of Frenkel, Lepowsky, Meurman and Huang indicating deep connections between codes, lattices and conformal field theories. Remarkable phenomena that take place in dimensions 8 and 24, such as those referred to above, arise in other guises in the vertex operator algebra theory associated with the moonshine module, and such phenomena indeed provided a crucial focus for this workshop as a whole.

We clarified a number of research problems, both new and old, and we developed a number of new programs of study. In workshop sessions, ideas to approach Frenkel-Lepowsky-Meurman's conjecture that their moonshine module vertex operator algebra is unique subject to conditions analogous to conditions that characterize the Leech lattice among lattices and the Golay code among codes were discussed and studied; this has been a difficult problem (since 1988) and a great deal remains to be done. We identified a program of finding tools analogous to weighted theta functions as an approach that might help prove the uniqueness of this vertex operator algebra, which is a kind of "higher-level analogue" of the Leech lattice, as the Leech lattice is itself a "higher-level analogue" of the extended binary Golay code"; suitable vertex operator algebras have generating functions that are modular functions, as the theta function of an integral lattice is automatically a modular form. We ask if there are also analogues of weighted theta functions that might be used to help prove the uniqueness of the moonshine module.

The workshop thus produced a new subject for discussion, which is a hybrid of the number theory, group theory and linear programming/combinatorics of sphere packings and the infinite-dimensional algebra of vertex operator algebras. The list of open problems that we generated can be found at http://www.aimath.org/WWN/spherepacking/.

The workshop participants continue to collaborate on and discuss these open problems. In addition, several new collaborations have come out of the workshop, for example, between infinite-dimensional algebraists (L. Carbone, H. Garland, M. Roitman) and physicists, specifically M-theorists (H. Sati) and introduction to other physicists: P. West at King's College, with meetings and collaborations planned in the near future, and also among infinite-dimensional algebraists (L. Carbone and E. Jurisich) to study generalized Kac-Moody Lie algebras and the possibility of associating groups to them.

The workshop allowed a valuable exchange of information between different research groups. Discussions among number theorists, group theorists and algebraists revealed new approaches, new angles, and new insights to difficult problems, including the uniqueness problem for the moonshine module.

Physicists communicated to algebraists the new results in physics which indicate that the physical interpretation of Lorentzian Kac-Moody groups is from dimension reduction in supergravity theories. This has allowed algebraists who attended the workshop to reformulate problems in group theory and representation theory to reflect their physical meaning. A new research proposal is in development for 'Groups and algebras in M-theory' (L. Carbone, H. Garland, H. Sati). This exchange would not have been possible without the introductions and discussion at the workshop. A summer workshop on 'Introduction to groups and algebras in M-theory', sponsored by NSF will be held at Rutgers. The announcement and materials for the workshop can be found at http://www.math.rutgers.edu/∼carbonel/. We note that the proposal for this workshop arose entirely from discussions during the AIM workshop. Furthermore, a research project for undergraduates is underway at Rutgers on 'Root systems of Lorentzian algebras', supervised by L. Carbone.

Many participants appreciated the format of the workshop, including spontaneous talks and spontaneous working-group sessions, as well as ample time for previously-unplanned discussions. It was felt that this format led to many fruitful and potentially-fruitful interactions, including those mentioned above. In other examples of such interactions, Y.-Z. Huang, J. Lepowsky and L. Zhang made progress on some work on tensor product theory for modules for a vertex operator algebra, a theory that is expected to play a crucial role in the uniqueness conjecture for the moonshine module. B. Gross mentioned ideas from two papers of Elkies-Gross concerning the Leech lattice and the exceptional Jordan algebra, and this reminded J. Lepowsky of some ideas he had been thinking about quite a while ago and which now might turn out to be related to these papers.