Theory and applications of total positivity

organized by
Shaun Fallat, Dominique Guillot, and Apoorva Khare

Workshop Summary

The goal of this workshop was to bring together researchers working on total positivity and allied areas within analysis, matrix theory, combinatorics, and algebra. Total positivity has been studied for over a hundred years in multiple avatars, and progress in one subfield of it has influenced that in others. The present workshop was perhaps the first comprehensive meeting of researchers in total positivity since a 1994 conference in Jaca (Spain) on the topic, and in particular the first in North America. It brought together specialists working in all of the areas above, and led to conversations between working groups who may not have interacted otherwise. One notable case in point was of matrix theorists interacting with cluster algebraists and adding techniques involving the totally non-negative Grassmannian to their toolbox.

Each morning saw two lectures – at once technical as well as expository – aimed at communicating background, results, and techniques by the speaker to the other participants. The first afternoon had a group discussion, followed by breakout problem sessions over the next four afternoons. The speakers each morning were:

- Jul 24 (Mon): Dominique Guillot/Apoorva Khare (joint talk), Olga Katkova
- Jul 25 (Tue): Prateek Kumar Vishwakarma, Lauren Williams
- Jul 26 (Wed): Alan Sokal, Chi-Kwong Li
- Jul 27 (Thu): Petter Brändén, Anna Vishnyakova
- Jul 28 (Fri): Igor Pak, Melissa Sherman-Bennett

Monday's group discussion involved a problem-proposing session, moderated by Alexander Belton and Pavlo Pylyavskyy (and scribed by Projesh Nath Choudhury and Prateek Kumar Vishwakarma), which led to the generation of 13 problems (see `http://aimpl.org/totalpos/1/` for a complete list of these problems and their corresponding descriptions). The participants worked on seven of these during the workshop, and they are briefly described below.

In conclusion, it was a productive workshop that has led not only to new collaborations, but also to the sharing and dissemination of ideas and techniques across sub-communities in total positivity. Several participants left the workshop with new insights and approaches; and it brought together teams of new researchers, while strengthening old bonds. On a separate note, the new location of the AIM is very nice and functional. There were no significant logistical issues and it was a fruitful and enjoyable workshop experience.

Transforms preserving real-rootedness

One of the groups worked on classifying the entrywise transforms of coefficients that preserve real-rootedness of polynomials: If $\sum_k a_k z^k$ is real-rooted, with all coefficients a_k

real and positive, then $\sum_k F(a_k)z^k$ is real-rooted, with $F(0) = 0$ and all coefficients $F(a_k)$ non-negative. This is parallel to the question of understanding the functions of the degree (i.e., replacing $F(a_k)$ by $f(k)$) rather than of the coefficient; in that case, the answer is given by the Pólya–Schur multiplier sequences, which connect to the Laguerre–Pólya class of entire functions.

The above question can be reformulated also as understanding the entrywise transforms preserving finite Pólya Frequency (PF) sequences, i.e., preserving semi-infinite Toeplitz totally non-negative matrices with finitely many nonzero diagonals. If one instead works with general (or one-sided) PF sequences, then the preservers are known to be very restricted. However, the class of preservers of finite PF sequences is richer by Maló's theorem: it contains all positive integer powers.

The group has begun working on the problem and has preliminary findings, which they have continued to work on post-workshop.

Inequalities involving minors of totally positive matrices

Another group considered the general problem of classifying all real linear combinations of products of minors that are non-negative for all totally non-negative (TN or TNN) / totally positive (TP) matrices. More accurately, the aim is to devise a constructive algorithmic process that certifies whether a given determinantal expression forms an inequality for all TN/TP matrices. Along these lines, it is natural to consider all available set-theoretic operations that preserve the positive Grassmannian and related objects, and study their effects on the given determinantal expression. This research builds upon a series of existing works concerning inequalities comparing two products of minors, and seems to be a setting where recent index-row operations, on certain determinantal expressions, should be applicable.

On a related note, the group also considered the problem of classifying the determinantal inequalities in which all the coefficients are forced to be non-negative. For instance, they could deduce (without much difficulty) that any real linear combination of all minors is non-negative for all TN matrices if and only if all the coefficients are non-negative. However, this is not true for combinations of products of minors, which means that additional necessary conditions are required to produce analogous results for more complicated expressions.

The group has started having weekly Zoom meetings to continue the research.

Monomial-TNN Toeplitz matrices with symmetric function entries

There are many classical theorems about operations that preserve negative-real-rootedness of polynomials: for instance, $\frac{d}{dx} + a$ or $x\frac{d}{dx} + a$ with $a \geq 0$; or Hadamard product; or Hadamard product with an extra $n!$ (sometimes known as Schur composition); or Brändén's (2011) log-concavity operation. . . . These can be reinterpreted as statements about *pointwise* total positivity for certain Toeplitz matrices involving elementary symmetric functions (e.g. $(n + a)e_n$). A group explored the question of whether these results can be upgraded to total positivity in the *monomial* basis. Computational tests suggest that the answer is yes.

Perfect matchings and infinite Toeplitz monomial-TNN matrices

Pólya frequency sequences are characterized by the total non-negativity of their (infinite) Toeplitz matrices. It is well-known that all rectangular totally non-negative (TNN) matrices arise as the path matrix of a planar acyclic directed graph, or (essentially equivalently) from dimers/perfect matchings on a planar bipartite graph. This leads to a natural question: is there a directed graph or dimer model for infinite Toeplitz TNN matrices?

One of the groups investigated this question for TNN infinite Toeplitz matrices with only finitely many nonzero diagonals. (These correspond to finite Pólya frequency sequences, which are also the coefficient sequences of real-rooted polynomials.) P. Pylyavskyy gave a construction of an infinite Toeplitz matrix whose entries are polynomials recording dimers on a bipartite graph embedded in a cylinder. He conjectured this matrix was monomial TNN and gave an argument for the 2×2 minors being non-negative. The group reframed his construction in terms of directed networks embedded on a cylinder. In this reframing, the diagonals of the matrix record weights of vertex-disjoint cycle collections with a fixed winding number. The group developed an analogue of the Lindström–Gessel–Viennot lemma for 2×2 minors in this framework, determining which pairs of cycle collections are "crossing" and thus would cancel in 2×2 minors. There is a natural generalization of this "crossing" condition to larger minors, which will be investigated in the future.

Symmetrized Fischer products for TP matrices, and majorization

It is known that if the sizes of minors of certain symmetrized Fischer products satisfy majorization monotonicity, then those products are monotonic as well. Another group considered the converse problem. More precisely, can one conclude that if the symmetrized Fischer product for one partition of an integer n is greater then that for another partition, for all totally non-negative $n \times n$ matrices, then the first partition majorizes the second.

The group was able to show that the answer is "Yes." The main idea was to construct a particular planar network with positive weights and consider path matrices which correspond to them. If the corresponding integer partitions do not satisfy majorization, then inequalities on symmetrized Fischer products can hold in both directions.

In addition, the group found a few more interesting observations regarding totally non-negative matrices.

Fisk's conjecture on real-rootedness via iterated sequences

A working group discussed various possible approaches to proving Fisk's conjecture: if $\sum_{k=0}^{n} a_k z^k$ has only negative real roots (with all $a_k > 0$), does the same hold upon replacing the coefficients by their "Toeplitz minors" of higher order? The discussion included the following:

- generalizing Brändén's proof of the 2×2 case, by evaluating the sum of Schur polynomials $\sum_{i=0}^{n} s_{k^i}(x_1, \ldots, x_n)$;
- analyzing a skew-Schur polynomial upon replacing each elementary generator e_i by s_{k^i};
- using planar networks or cylindrical networks;
- using induction with an interlacing argument;
- using induction with the Desnanot–Jacobi identity.

The group also discussed proving a q-analogue of the result of Yoshida (2013) that the polynomial $(1+x)^n$ satisfies Fisk's conjecture.

Local sufficient conditions to upgrade TP_k to TP

In 2006, Olga Katkova and Anna Vishnyakova (*Linear Algebra Appl.*, Vol. 416) proved a fascinating result concerning a subclass of totally non-negative (TN) matrices. They verified that any $n \times n$ matrix $A = [a_{ij}]$ with positive entries that also satisfies $a_{ij}a_{i+1,j+1} \geq ca_{i+1,j}a_{i,j+1}$ for all $i, j < n$, and $c \geq 4\cos^2\left(\frac{\pi}{n+1}\right)$, then A is TN, and if the inequality above is strict, then A is totally positive (TP). We denote this subclass by $\mathrm{TN}_2(c)$ or $\mathrm{TP}_2(c)$.

During the talks of Olga and Anna in the workshop, the question was raised by Steven Karp, and by Charles Johnson whether there were higher order, similar conditions for a TP_{k-1} matrix to be TP? This attracted considerable attention and became the focus of a working group. This group had success showing that there are such results, though it remains to be seen if the results found are optimal. The original result above is best possible for 2×2 minors. This group focused upon the existence of an analogue of the titled type. The k-th compound $C_k(A)$ of a matrix A is an array of all the $k \times k$ minors, usually, but not necessarily, ordered lexicographically.

The key lemma is

Lemma. *Suppose that A is TP_{k-1} and that $C_k(A)$ is TP. Then A is TP.*

An application of the original result by Olga and Anna to $C_k(A)$ in a natural way then gives the higher order result.

Theorem. *Suppose that $k \geq 1$, that A is TP_k, and that $C_k(A)$ has entries b_{ij}. Then for $c_k \geq 4\cos^2\left(\frac{\pi}{\binom{n}{k}+1}\right)$, if whenever*

$$\frac{b_{ij}b_{i+1,j+1}}{b_{i+1,j}b_{i,j+1}} > c_k \quad \forall i, j < n,$$

it follows that A is TP.

The result is quite analogous to the one by Katkova–Vishnyakova. But questions remain: is this constant best possible? Will fewer, or other sets of minors suffice? Are there other such conditions that ensure when a compound of a TP matrix is TP, etc. The group intends to continue working on this problem and will soon begin meeting via Zoom.