ABSTRACT

ZEROES OF DERIVATIVES OF RIEMANN'S XI FUNCTION ON THE CRITICAL LINE

by

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We show that the entire function $\xi^{(m)}(s)$ has asymptotically

$$X/(2\pi)\log(X/(2\pi))$$

zeroes with $0 \leqslant t \leqslant X$ and $0<\sigma<1$. The zeroes of

$$\xi(s)=(1/2)s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$$

are coincident with those of $\zeta(s)$ in the critical strip $0<\sigma<1$. Since $\xi(1/2+it)$ is real, the proportion of zeroes of $\xi^{(m)}(s)$ with real part 1/2 increases with m.

We show that the proportion of zeroes of $\xi^{(m)}(s)$ with real part 1/2 is at least

$$1 - [\log F_m(R)]/R$$

for any $R>0$, where

$$F_m(R) = \frac{1}{2} - \frac{\Gamma_1}{4}(R+\phi_1'(0)) - \int_0^1 e^{2Rx}\Psi(x)\,dx.$$

Here

$$\phi_1(x)=\phi(x)(1-2x)^m$$

where ϕ is an entire function which satisfies

$$\phi(x)+\phi(1-x)=1$$

and some other less important conditions. Also

$$\Psi(x)=\frac{\Gamma_1}{4}[\phi(x)\phi''(x)-\phi'(x)\phi'(x)]-2\Gamma_3\phi(x)\phi(x)$$

and

$$\Gamma_1=\int_0^1[P(x)]^2dx\ ,\ \Gamma_3=\int_0^1[P'(x)]^2dx$$

where P is any real polynomial which satisfies

$$P(0)=0\ ,\ P(1)=1.$$

As a consequence of this theorem, we show that the proportion of zeroes with real part 1/2 of

$$\left\{\begin{matrix}\zeta(s) \text{ exceeds } .3585\\ \xi'(s) \text{ exceeds } .7186\\ \xi''(s) \text{ exceeds } .8209\end{matrix}\right\}.$$

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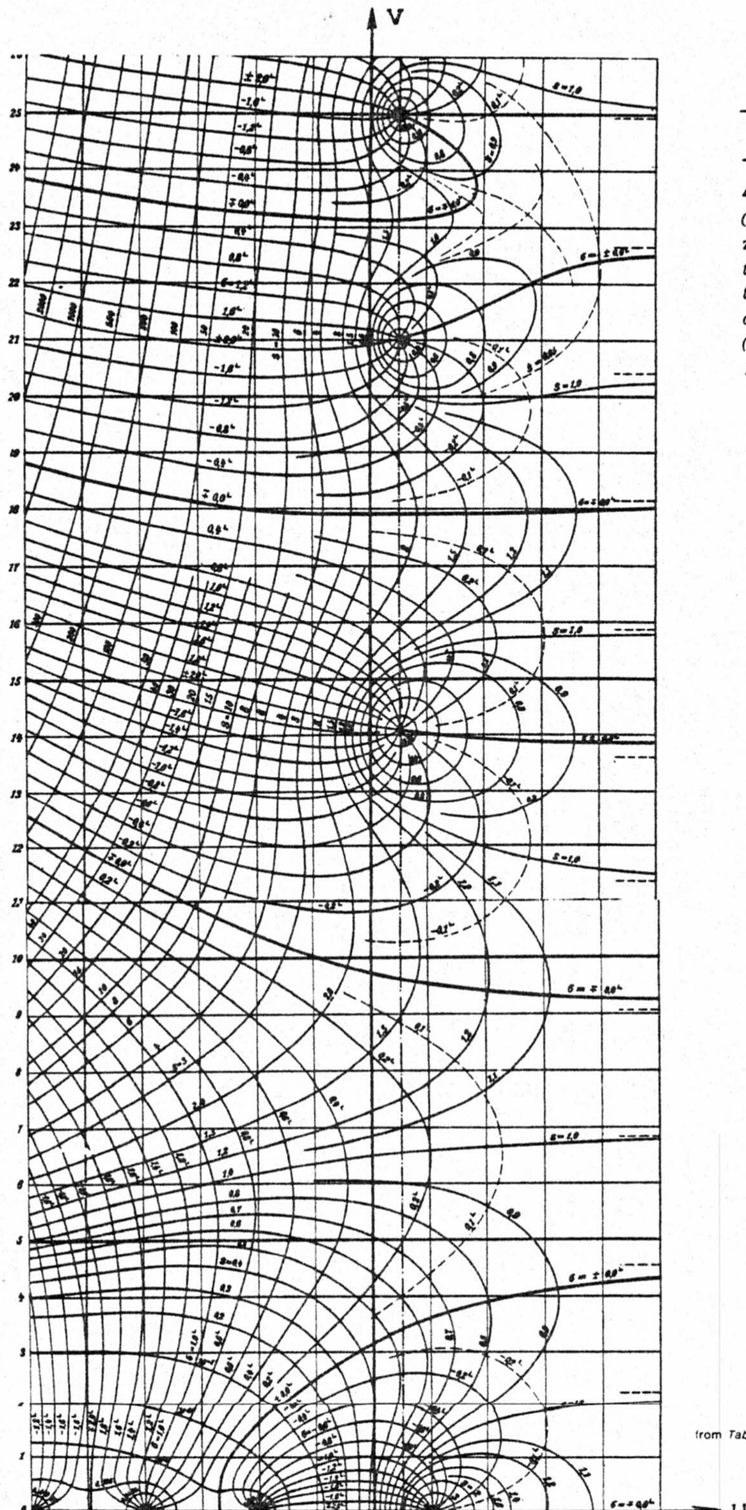

The Riemann Zeta Function

Graph of the contour curves of $z = |\zeta(s)| = |\zeta(u + iv)|$ exhibiting the first three zeros ρ_1, ρ_2, ρ_3 on the line $u = \frac{1}{2}$, the zeros at the negative even integers ($u = -2k$, $v = 0$) and the pole at 1 ($u = 1$, $v = 0$).

from *Tables of Higher Functions*, by E. Jahnke and F. Emde, 1952

To Jan

PREFACE

The topic of this dissertation is a generalization of a method which N. Levinson used to show that the Riemann zeta-function has more than 34.7% of its zeroes on the vertical line passing through 1/2, the "critical" line. We improve this result a bit and extend the technique to show that $\xi^{(m)}(s)$ has a large proportion of zeroes on the critical line for all m. (Levinson also considered m=1; we improve his result in this case and consider the case m=2.)

The first two chapters contain introductory material and results needed later. In Chapter III we establish that the zeroes of $\xi^{(m)}(s)$ are located in a vertical strip and give an asymptotic formula for the number of zeroes up to a given height. Hence it makes sense to speak of the "proportion" of zeroes of $\xi^{(m)}(s)$ with real part 1/2. In Chapters IV, V, VI, and VII we prove the main theorem and Chapter VIII contains computations based on the theorem.

The method used here largely follows Levinson's technique as presented in his papers listed in the bibliography. We also incorporate a simplification due to Pan. The new features in this paper include a simple way to derive the basic identity in §4.1, the use of an entire function rather than a polynomial in the identity, the use of a more general mollifier in Chapter VIII and the treatment throughout of all the derivatives $\xi^{(m)}(s)$ in a unified manner which leads to Theorem 7.6.1. Of course, the computations and ensuing theorems of Chapter VIII are new, as well.

The derivatives $\xi^{(m)}(s)$ with $m=3,4$ were considered using the mollifier with $P(x)=x$ and the identity with $\phi(x)=1-x$ but led to poorer results than the case $m=2$, though Rolle's theorem and the results of Chapter II imply that the proportion of zeroes of $\xi^{(m)}(s)$ with real part 1/2 cannot decrease with m. There is good reason to believe that the use of a second degree polynomial $P(x)$ in the mollifier would improve the result for $m=3$. Computations with $P(x)$ different from x have not yet been carried out.

It should be pointed out that with the exception of Chapter IV, the first section of each chapter contains results that are well-known, easy to prove, or obvious generalizations of Levinson's results.

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NOTATION

Let m be a fixed non-negative integer. Many of the constants in the O-estimates depend on m, and results stated for "a bounded range" of k are usually for $k \leqslant m$. We let T be a large fixed number,

$$T > T_0(m).$$

Also, we use

$$\begin{aligned} L &= \log(T/(2\pi)), \\ U &= T/L^{10}, \\ y &= T^{1/2}/L^{20}, \\ \tau &= (T/(2\pi))^{1/2}, \\ \tau_1 &= ((T+U)/(2\pi))^{1/2}, \\ \eta &= (t/(2\pi))^{1/2}, \\ 1/2 - a &= R/L, \quad R>0. \end{aligned}$$

Usually the above notation occurs in the text without further reference. Also s and w are complex variables with

$$s = \sigma + it$$

$$w = u + iv.$$

We normally reserve s for use in the particular range

$$0 \leqslant \sigma \leqslant 4 \log L, \quad T \leqslant t \leqslant T+U.$$

CHAPTER I

THE GAMMA AND RELATED FUNCTIONS

1.1 Well-known properties of the gamma-function.

The formulae in this section are derived in Rademacher [14, pp. 28-44].

As usual we take

(1.1.1) $\Gamma(w) = w^{-1} \prod_{n=1}^{\infty} \left((1 + n^{-1})^{w}(1 + wn^{-1})^{-1}\right).$

Then Γ is a meromorphic function throughout the finite complex w-plane. It has no zeroes and has poles only at 0 and the negative integers; these poles are simple. It satisfies the functional equation

(1.1.2) $$\Gamma(w)\Gamma(1 - w) = \pi \csc \pi w.$$

It satisfies another functional equation known as the duplication formula,

(1.1.3) $$\Gamma(w)\Gamma(w + 1/2) = \sqrt{\pi}\, 2^{1-2w}\Gamma(2w).$$

Also

(1.1.4) $$\Gamma(w + 1) = w\Gamma(w),$$

and for n a non-negative integer

(1.1.5) $$\Gamma(n + 1) = n!$$

It is well-known that from the Euler-Maclaurin summation formula one can show that

(1.1.6) $$\log\Gamma(w) = (1/2)\log(2\pi) + (w-1/2)\log w - w - \Omega(w)$$

is valid for w not on the negative real axis, where

$$(1.1.7) \qquad \Omega(w) = (1/2)\int_0^\infty \frac{(x-[x])^2-(x-[x])}{(x+w)^2}\,dx.$$

The logarithm on the right side of (1.1.6) is real for real $w>0$, and the logarithm on the left side is real for real $w\geq 1$. It is easily shown that

$$(1.1.8) \qquad |\Omega(w)| \leq \begin{cases} (8u)^{-1} & \text{for } v=0,\ u>0 \\ (8|v|)^{-1}\arctan(|v|/u) & \text{for } v\neq 0, \end{cases}$$

where

$$0 \leq \arctan(|v|/u) = |\arg w| < \pi.$$

For $u>0$ the Γ-function is represented by the integral

$$(1.1.9) \qquad \Gamma(w) = \int_0^\infty \exp(-x)\,x^{w-1}\,dx.$$

Using (1.1.5) and (1.1.6) one can derive Stirling's formula. However, we only need the simpler estimate

$$(1.1.10) \qquad n! \ll (n/e)^n(\sqrt{n})$$

which can be derived by comparing the sum

$$\sum_1^n \log i$$

with the integral

$$\int_1^{n+1} \log t\,dt.$$

1.2 Three functions. In this section, we use results from §1.1 to deduce some properties of three functions related to the Γ-function. Let

$$(1.2.1) \qquad H(w) = (1/2)w(w-1)\pi^{-w/2}\Gamma(w/2);$$

let

(1.2.2) $$\chi(s) = H(1-s)/H(s);$$

and let

(1.2.3) $$F(w) = H'(w)/H(w).$$

Then, H is regular and non-zero in the region $u \geqslant 0$ excluding the segment of the real axis between 0 and 1. $H(u)>0$ if $u>1$. Define arg $H(w)$ in the region by starting from the value arg $H(3) = 0$ and varying continuously along a path from 3 to w which does not leave the region. Then arg $H(w)$ is well-defined, and it is clear that for w in this region

(1.2.4) $$\arg H(w) = \text{Im} \log H(w)$$

where $\log H(u)$ is real for $u>1$.

Lemma 1.2.1 For $|v| \geqslant 1$

$$\arg H(1/2+iv) = v/2 \log(|v|/(2\pi)) - v/2 + O(1).$$

Proof. By (1.2.1) and (1.2.4) we have

(1.2.5) $$\arg H(1/2+iv) = \text{Im}(\log(-1/4-v^2) - (1/4+iv/2)\log\pi +$$
$$+\log\Gamma(1/4+iv/2)) = -v/2 \log\pi + O(1) + \text{Im}(\log\Gamma(1/4+iv/2)).$$

By (1.1.6) and (1.1.8) we have

(1.2.6) $$\text{Im}(\log\Gamma(1/4+iv/2))$$
$$= -1/4 \arg(1/4+iv/2) + v/2 \log|1/4+iv/2| - v/2 + O(|v|^{-1})$$
$$= v/2 \log(|v|/2) - v/2 + O(1).$$

The Lemma follows from (1.2.5) and (1.2.6).∎

Lemma 1.2.2 For $0 \leqslant \sigma \leqslant 4 \log L$ and $T \leqslant t \leqslant T+U$ we have

$$\chi(s) = (t/(2\pi))^{1/2-\sigma} \exp(\pi i/4 - it \log(t/(2\pi e)))\left(1+O(\log^2 L/T)\right).$$

Proof. By (1.2.1) and (1.2.2) and then by (1.1.2) with $w = (1-s)/2$, we have

$$(1.2.7) \qquad \chi(s) = \pi^{s-1/2}\Gamma(1/2-s/2)/\Gamma(s/2)$$

$$= \pi^{s+1/2}\csc \pi(1/2-s/2)/(\Gamma(s/2)\Gamma(1/2+s/2)).$$

By (1.1.3), with $w = s/2$, and (1.2.7) we have

$$(1.2.8) \qquad \chi(s)=(2^{s-1}\pi^{s}\sec \pi s/2)/\Gamma(s).$$

It is clear that

$$(1.2.9) \qquad \log|s|=\log t+\log|1-i\sigma/t|$$

$$= \log t + (1/2)\log(1+\sigma^2/t^2)=\log t + O(\log^2 L/T^2),$$

and

$$(1.2.10) \quad \arg s=\arctan(t/\sigma)=\pi/2-\arctan(\sigma/t) \quad , \sigma\neq 0$$

$$= \pi/2-\sigma/t+O(\log^3 L/T^3) \quad , \sigma\geqslant 0.$$

Also, by (1.1.8) we have

$$(1.2.11) \qquad |\Omega(s)| = O(T^{-1}).$$

By (1.1.6), (1.2.9), (1.2.10), and (1.2.11) we have

$$(1.2.12) \qquad \log \Gamma(s)$$

$$= (1/2)\log(2\pi)+(\sigma-1/2)\log t-(\pi/2)t+i(t \log t-t+(\sigma-1/2)\pi/2)+$$

$$+ O(\log^2 L/T).$$

We have

$$(1.2.13) \qquad \sec \pi s/2=2e^{\pi is/2}/(1+e^{\pi is})$$

$$= 2e^{-\pi t/2}e^{i\pi\sigma/2}(1+O(e^{-\pi t})).$$

The Lemma follows from (1.2.8), (1.2.12), and (1.2.13).■

We now estimate F and its derivatives. First, observe that for $v\neq 0$ and $k\geqslant 1$ we have

$$(1.2.14) \quad \int_0^\infty [(x+u)^2+v^2]^{-k}dx = |v|^{-2k}\int_u^\infty [1+(y/v)^2]^{-k}dy =$$

$$= |v|^{1-2k} \int_{u/|v|}^{\infty} [1+x^2]^{-k} dx$$

$$\leqslant |v|^{1-2k} \int_{-\infty}^{\infty} (1+x^2)^{-1} dx \ll |v|^{1-2k}.$$

<u>Lemma 1.2.3</u> For $|v| \geqslant 1$ we have

$$F(w) = (1/2)\log(w/(2\pi)) + O(|v|^{-1}),$$

and for a bounded range of $k \geqslant 1$ we have

$$F^{(k)}(w) = O(|v|^{-k}).$$

<u>Proof.</u> By (1.2.1) and (1.2.3) we have

(1.2.15)
$$F(w) = 1/w + 1/(w-1) - (1/2)\log\pi + (1/2)\Gamma'(w/2)/\Gamma(w/2).$$

By (1.1.6) we have

(1.2.16)
$$\Gamma'(w)/\Gamma(w) = \log w - (2w)^{-1} - \Omega'(w)$$

where by (1.1.7) and (1.2.14) it is clear that

(1.2.17)
$$\Omega'(w) = -\int_0^{\infty} \frac{(x-[x])^2 - (x-[x])}{(x+w)^3} dx$$

$$\ll \int_0^{\infty} [(x+u)^2 + v^2]^{-3/2} dx \ll |v|^{-2}.$$

The estimate for $F(w)$ follows from (1.2.15), (1.2.16), and (1.2.17). By (1.2.16), for $k \geqslant 1$ we have

(1.2.18)
$$\left(\frac{d}{dw}\right)^k \frac{\Gamma'(w)}{\Gamma(w)}$$

$$= (-1)^{k-1}(k-1)!w^{-k} + (1/2)(-1)^{k-1}k!w^{-k-1} - \Omega^{(k+1)}(w)$$

where, by (1.1.17) and (1.2.14),

(1.2.19)
$$\Omega^{(k+1)}(w)$$
$$= (-1)^{k+1}(1/2)(k+2)!\int_0^\infty \frac{(x-[x])^2-(x-[x])}{(x+w)^{k+3}}\,dx$$

$$<< \int_0^\infty [(x+u)^2+v^2]^{-(k+3)/2}dx << |v|^{-k-2}.$$

By (1.2.15), for $k \geqslant 1$ we have

(1.2.20)
$$F^{(k)}(w)$$
$$= (-1)^k k!\,[w^{-k-1}+(w-1)^{-k-1}]+(1/2)\left(\frac{d}{dw}\right)^k \frac{\Gamma'(w/2)}{\Gamma(w/2)}.$$

Since $|v| \leqslant |w|$ and $|v| \leqslant |w-1|$, (1.2.18), (1.2.19), and (1.2.20) imply the estimate for $F^{(k)}(w)$.■

1.3 A combinatorial lemma. We will need some information about $H^{(k)}(w)$. Because of (1.2.3) we can express any derivative of H in terms of H and derivatives of F. For example, by (1.2.3) we have

$$H''(w) = \frac{d}{dw} H(w)F(w) = H(w)[(F(w))^2 + F'(w)].$$

Let p_n be a partition (unordered) of n into positive integers. Let $p_n(k)$ be the number of times k occurs in the partition p_n. For any p_n let $P(p_n)$ be the number of ways to partition a set S of n elements into a union of $p_n(1)$ sets with one element in each, $p_n(2)$ sets with two elements each and so on up to $p_n(n)$ sets with n elements in each; that is, $P(p_n)$ enumerates the ways to partition S "via" p_n.

Lemma 1.3.1 Let $H'(w) = H(w)F(w)$. Then

$$H^{(n)}(w)=H(w)\sum_{p_n} P(p_n)\prod_{k=1}^{n}[F^{(k-1)}(w)]^{p_n(k)} \tag{1.3.1}$$

where the sum is over all partitions of n.

Proof. The proof is by induction on n. Since there is only one partition of 1 the Lemma is true when n=1. Assume, as the inductive hypothesis, that the statement (1.3.1) holds. Under this assumption, we will show that (1.3.1) with n replaced by n+1 is valid. We differentiate (1.3.1) and use $H'(w)=H(w)F(w)$ to see that

$$H^{(n+1)}(w)=H(w)\sum_{p_n} P(p_n)\left(\prod_{k=1}^{n}[F^{(k-1)}(w)]^{p_n(k)}\right) \cdot \left(F(w)+\sum_{i=1}^{n} p_n(i)F^{(i)}(w)/F^{(i-1)}(w)\right). \tag{1.3.2}$$

Let p_{n+1} be an arbitrary partition of (n+1). The coefficient of

$$H(w)\prod_{k=1}^{n+1}[F^{(k-1)}(w)]^{p_{n+1}(k)}$$

in $H^{(n+1)}(w)$ is, by (1.3.2), equal to

$$\sum_{p_n}{}' P(p_n)+\sum_{i=1}^{n}\sum_{p_n}{}'' P(p_n)p_n(i) \tag{1.3.3}$$

where Σ' is for all partitions p_n of n with $1+p_n(1)=p_{n+1}(1)$

and $p_n(k)=p_{n+1}(k)$ for $2\leqslant k\leqslant n$, and Σ'' is for all partitions p_n of n for which $p_{n+1}(i+1)=1+p_n(i+1)$, $p_{n+1}(i)=p_n(i)-1$, and $p_{n+1}(k)=p_n(k)$ for $k\neq i$, $i+1$. It remains to show that the expression (1.3.3) is $P(p_{n+1})$.

To evaluate $P(p_{n+1})$, let S be a set with (n+1) elements and suppose $\alpha \in S$ is a distinguished element. The partitions of S via p_{n+1} may be formed from partitions of $S-\{\alpha\}$ via various partitions p_n of n, by (i) including α as a singleton with p_n that satisfy the conditions of Σ_1 or by (ii) including α in a set of $i\geqslant 1$ elements with p_n that satisfy Σ_2. In case (i), there are $P(p_n)$ partitions of $S-\{\alpha\}$ each giving rise to a distinct partition of S. In case (ii), for each of the $P(p_n)$ partitions of $S-\{\alpha\}$, α can be added to any one of the $p_n(i)$ sets of i elements. Thus, $P(p_{n+1})$ is equal to the expression in (1.3.3).■

We can now replace $H^{(n)}(w)$ by a simple expression which is a good approximation to it.

<u>Lemma 1.3.2</u> Let

$$H_n(w)=H^{(n)}(w)/H(w)-[F(w)]^n. \tag{1.3.4}$$

For $0 \leqslant \sigma \leqslant 4 \log L$, $T \leqslant t \leqslant T+U$ and a bounded range of n,

$$H_n(s)=O(L^{n-1}T^{-1}),\ H_n(1-s)=O(L^{n-1}T^{-1}).$$

<u>Proof</u>. Let p_n be the partition of n for which $p_n(1)=n$, $p_n(k)=0$ for $k\neq 1$. Then $P(p_n)=1$ and this partition gives rise to the term

$$H(w)[F(w)]^n$$

on the right side of (1.3.1). For any other partition a first or higher derivative of F must occur in the product on the right side of (1.3.1). By Lemma 1.2.3, the next largest term of (1.3.1) is a constant times

$$H(w)[F(w)]^{n-2}F'(w).$$

Therefore, by (1.3.4) and Lemma 1.2.3 if $|v| \geqslant 1$ we have

$$H_n(w) << |\log(w/(2\pi))|^{n-2}|v|^{-1}. \tag{1.3.5}$$

For s in the indicated range we have

$$|\log(s/(2\pi))| << L \ , \quad |\log(1-s)/(2\pi))| << L$$

so that the Lemma follows from (1.3.5).■

In the range of s we are considering, namely $0 \leqslant \sigma \leqslant 4 \log L$, $T \leqslant t \leqslant T+U$, we have another estimate for F(s) and F(1-s). Let

$$\ell = \ell(t) = \log(t/(2\pi)). \tag{1.3.6}$$

Lemma 1.3.3 For $0 \leqslant \sigma \leqslant 4 \log L$, $T \leqslant t \leqslant T+U$ we have

$$F(s) = \ell/2+\pi i/4+O((\log L)/T),$$
$$F(1-s) = \ell/2-\pi i/4+O((\log L)/T).$$

Proof. For s in the indicated range, we have

$$\begin{aligned} \log(|s/(2\pi)|) &= \log(t/(2\pi))+\log|1+i\sigma/t| \\ &= \ell+(1/2)\log(1+\sigma^2/t^2) \\ &= \ell+O(T^{-2}\log^2 L). \end{aligned} \tag{1.3.7}$$

Similarly,

$$\log(|1-s|/(2\pi)) = \ell+O(T^{-2}\log^2 L). \tag{1.3.8}$$

Also,

$$\begin{aligned} \arg(s/(2\pi)) &= \arctan(t/\sigma) = \pi/2-\arctan(\sigma/t) \\ &= \pi/2+O(T^{-1}\log L) \end{aligned} \tag{1.3.9}$$

and

(1.3.10) $\arg((1-s)/(2\pi))=-\pi/2+O(T^{-1}\log L)$.

The Lemma follows from (1.3.7), (1.3.8), (1.3.9), and (1.3.10).∎

CHAPTER II

THE RIEMANN ZETA-FUNCTION

2.1 Well-known properties of the Riemann zeta-function

For $u>1$ the Riemann zeta-function is defined by

(2.1.1) $$\zeta(w) = \sum_{n=1}^{\infty} n^{-w}.$$

Because of uniqueness of factorization of positive integers, $\zeta(w)$ has an Euler product absolutely convergent for $u>1$,

(2.1.2) $$\zeta(w) = \prod_{p} (1-p^{-w})^{-1}$$

where the product is over all primes. By (2.1.2) we have

(2.1.3) $$\zeta(w) \neq 0, \quad u>1.$$

It is well-known that $\zeta(w)$ is meromorphic in $u>0$ with a simple pole at $w=1$, residue 1, and no other singularities. (See Ingham [5, Theorem 8]). Hence for $|w-1|$ small we have

(2.1.4) $$1/\zeta(w) = (w-1) + O(|w-1|^2)$$

and

(2.1.5) $$\zeta'(w)/\zeta^2(w) = -1 + O(|w-1|).$$

It is shown in Ingham [5, theorem 10] that

(2.1.6) $$\zeta(1+iv) \neq 0.$$

Titchmarsh [18, equations 3.11.8 and 3.11.10] shows that there is a constant $A>0$ such that for $v>v_0$, (2.1.9) $$1/\zeta(1+iv)=O(\log v).$$

The θ-function, defined by

(2.1.10) $$\theta(x)=\sum_{p\leq x}\log p$$

where the sum is on primes, is somewhat related to the logarithmic derivative of the zeta-function. A well-known elementary estimate due to Chebyshev is

(2.1.11) $$x\ll\theta(x)\ll x.$$

2.2 The Riemann-Siegel formula. Because of its importance to this paper, we sketch a proof of a formula found by Riemann and reconstructed by Siegel [16,§1 and §3] from Riemann's notes.

As a notational convenience let $\nearrow 1/2$ signify a path that is a straight line of slope + 1 passing through 1/2 with Im w increasing. Similarly, $\searrow -1/2$ is a straight line of slope -1 through -1/2 with Im w decreasing, and so on.

Let x be a complex variable. Let

(2.2.1) $$\Phi(x)=\int_{\nwarrow 1/2}\exp(-\pi iw^2+2\pi ixw)(-i/2)\csc\pi w\,dw.$$

We can evaluate $\Phi(x)$ by using Cauchy's theorem in two different ways. First

(2.2.2) $$\Phi(x+1)-\Phi(x)$$

$$=\int_{\nwarrow 1/2}\exp(-\pi iw^2+2\pi ixw)(\exp 2\pi iw-1)(-i/2)\csc\pi w\,dw$$

$$=\int_{1/2}\exp(-\pi iw^2+2\pi ixw+\pi iw)\,dw$$

$$=\exp(\pi i(x+1/2)^2)\int_{1/2}\exp(-\pi i(w-x-1/2)^2)\,dw$$

$$= \exp(\pi i(x+1/2)^2) \int_{\nwarrow \mathrm{Re}(-x)} \exp(-\pi i w^2)\,dw$$

$$= \exp(\pi i(x+1/2)^2) \int_{\nwarrow 1/2} \exp(-\pi i w^2)\,dw.$$

The integrand of (2.2.1) has a simple pole at $w=0$ with residue $(2\pi i)^{-1}$. Therefore, we have

$$(2.2.3) \quad \Phi(x)-1= \int_{\nwarrow -1/2} \exp(-\pi i w^2+2\pi i x w)(-i/2)\csc \pi w\,dw$$

$$= \int_{\nwarrow 1/2} \exp(-\pi i(w-1)^2+2\pi i x(w-1))(-i/2)\csc\pi(w-1)\,dw$$

$$= \exp(-2\pi i x) \int_{\nwarrow 1/2} \exp(-\pi i w^2+2\pi i w(x+1))(-i/2)\csc\pi w\,dw$$

$$= \exp(-2\pi i x)\Phi(x+1).$$

Put $x=0$ in (2.2.2) and (2.2.3). It follows that

$$(2.2.4) \qquad \int_{\nwarrow 1/2} \exp(-\pi i w^2)\,dw=\exp(3\pi i/4)$$

as is well-known. We eliminate $\Phi(x+1)$ from (2.2.2) and (2.2.3) and use (2.2.4) to show that

$$(2.2.5) \quad \Phi(x)=(1-\exp(-2\pi i x))^{-1}-\exp(\pi i x^2)(-i/2)\csc\pi x.$$

Let $\sigma<0$ and let x^{-s} have its principal value in the x-plane without the negative real axis. Let

$$\varepsilon=\exp(\pi i/4).$$

Multiply (2.2.5) by x^{-s} and integrate from 0 to $\varepsilon\infty$ along a straight line path. Now by (2.1.1), (1.1.9), and Cauchy's theorem we have

$$(2.2.6)\quad \int_0^{\varepsilon\infty} x^{-s}(1-\exp(-2\pi i x))^{-1}dx = -\int_0^{\varepsilon\infty} x^{-s}\sum_{n=1}^{\infty}\exp(2\pi i n x)\,dx$$

$$= -\sum_{n=1}^{\infty}\int_0^{\varepsilon\infty} x^{-s}\exp(2\pi i n x)\,dx = -\sum_{n=1}^{\infty}(-2\pi i n)^{s-1}\int_0^{\bar{\varepsilon}\infty} y^{-s}e^{-y}\,dy$$

$$= -(2\pi)^{s-1}\exp(\pi i(1-s)/2)\zeta(1-s)\Gamma(1-s).$$

By (1.1.9) and Cauchy's theorem we have

$$(2.2.7)\quad \int_0^{\varepsilon\infty} x^{-s}\left(\int_{\nwarrow 1/2}\exp(-\pi i w^2+2\pi i x w)(-i/2)\csc\pi w\,dw\right)dx$$

$$= \int_{\nwarrow 1/2}\exp(-\pi i w^2)(-i/2)\csc\pi w\left(\int_0^{\varepsilon\infty} x^{-s}\exp(2\pi i x w)\,dx\right)dw$$

$$= (-2\pi i)^{s-1}\int_{\nwarrow 1/2}\exp(-\pi i w^2)(-i/2)(\csc\pi w)w^{s-1}\left(\int_0^{\bar{\varepsilon}\infty} y^{-s}e^{-y}dy\right)dw$$

$$= (2\pi)^{s-1}\exp(\pi i(1-s)/2)\Gamma(1-s)\cdot$$

$$\cdot\int_{\nwarrow 1/2}\exp(-\pi i w^2)(-i/2)(\csc\pi w)w^{s-1}dw.$$

By (2.2.5), (2.2.6), and (2.2.7) we have

$$(2.2.8)\quad (2\pi)^{s-1}\exp(\pi i(1-s)/2)\Gamma(1-s)\cdot$$

$$\cdot\left\{\zeta(1-s)+\int_{\nwarrow 1/2}\frac{w^{s-1}\exp(-\pi i w^2)}{2i\sin\pi w}dw\right\} = -\int_0^{\varepsilon\infty}\frac{x^{-s}\exp(\pi i x^2)}{2i\sin\pi x}dx.$$

The integral on the right side of (2.2.8) is easily seen to be

$$(2.2.9)\quad -(\exp(\pi is)-1)^{-1}\int_{\swarrow 1/2} w^{-s}\exp(\pi iw^2)(-i/2)\csc\pi w\,dw.$$

Multiply both sides of (2.2.8) by $-(\exp(\pi is)-1)$. Then by (1.2.8), (2.2.8), and (2.2.9) we have

$$(2.2.10)\quad (1/\chi(1-s))\{\zeta(1-s)+\int_{\nwarrow 1/2}\frac{w^{s-1}\exp(-\pi iw^2)}{2i\sin\pi w}\,dw\}$$

$$=\int_{\swarrow 1/2}\frac{w^{-s}\exp(\pi iw^2)}{2i\sin\pi w}\,dw.$$

The integrals in (2.2.10) are convergent for all s. Therefore, by (1.2.2) we have

<u>Lemma 2.2.1</u> For any s

$$H(1-s)\zeta(1-s)$$

$$=H(s)\int_{\swarrow 1/2}\frac{w^{-s}\exp(\pi iw^2)}{2i\sin\pi w}\,dw+H(1-s)\int_{\searrow 1/2}\frac{w^{s-1}\exp(-\pi iw^2)}{2i\sin\pi w}\,dw.$$

<u>Remark 1</u>. Suppose that

$$I=\int_C f(w)\,dw$$

where C is a path in the complex w-plane. Then

$$\bar{I}=\int_{\bar{C}}\overline{f(\bar{w})}\,dw$$

where $\bar{C}$ is the path conjugate to C.

Remark 2. If f(w) is a regular function for w in some region R of the w-plane, then $\overline{f(\bar{w})}$ is a regular function for w in $\bar{R}$ where

$$\bar{R} = \{w : \bar{w} \varepsilon R\}.$$

We define the function $\bar{f}$ by

$$\bar{f}(w) = \overline{f(\bar{w})}$$

so that $\bar{f}$ is an analytic function when f is.

Let

$$(2.2.11) \qquad f(s) = \int_{\swarrow 1/2} \frac{w^{-s}\exp(\pi i w^2)}{2i \sin \pi w} dw.$$

By the remarks above we see that

$$(2.2.12) \qquad \bar{f}(1-s) = \int_{\searrow 1/2} \frac{w^{s-1}\exp(-\pi i w^2)}{2i \sin \pi w} dw.$$

Thus, from Lemma 2.2.1 and Equations (2.2.11) and (2.2.12) we have

Corollary 2.2.2 For any s we have the identity

$$(2.2.13) \qquad H(1-s)\zeta(1-s) = H(s)f(s)+H(1-s)\bar{f}(1-s).$$

When $s = 1/2+it$, the above becomes

$$(2.2.14) \qquad H(1/2-it)\zeta(1/2-it) =$$

$$= H(1/2+it)f(1/2+it)+\overline{H(1/2+it)f(1/2+it)}.$$

We take complex congugates in (2.2.14) and see that

(2.2.15) $$H(1/2+it)\zeta(1/2+it)=H(1/2-it)\zeta(1/2-it)$$

holds for all real t; hence by analytic continuation (2.2.15) holds for all complex t. In particular, for all s we have

(2.2.16) $$H(s)\zeta(s)=H(1-s)\zeta(1-s);$$

hence, by Corollary 2.2.2, we have

Corollary 2.2.3 For any s

(2.2.17) $$H(s)\zeta(s)=H(s)f(s)+H(1-s)\bar{f}(1-s).$$

The above is the form of the Riemann-Siegel formula we will use.

2.3 An estimation by the saddle-point method

Suppose that $h(w)$ is a function that is regular in the w-plane slit along the negative real axis, and that for $|w|>10$ we have

(2.3.1) $$|h(w)|<<|w|^{\delta}|\log w|^{j}$$

where

(2.3.2) $$0<\delta<<L^{-1}$$

and j is a fixed positive integer. For $t>0$ let

(2.3.3) $$\eta = \sqrt{t/(2\pi)}.$$

Let

(2.3.4)
$$f_1(s)=\int_{\swarrow 1/2} \frac{w^{-s}\exp(\pi i w^2)}{2i\,\sin \pi w}\,h(w)\,dw.$$

Lemma 2.3.1 For $0\leqslant|\sigma|\leqslant 4\log L$, and $T\leqslant t\leqslant T+U$ we have

(2.3.5)
$$f_1(s)=\sum_{n\leqslant\eta} h(n)n^{-s}+O(\eta^{-\sigma}L^{j+12}).$$

Proof. Let $k=[\eta]$, and let $r=k+3/2-\eta$. Move the path of integration so that by Cauchy's theorem

(2.3.6)
$$f_1(s)=C_0+C_1+C_2+C_3+C_4$$

where

(2.3.7)
$$C_0=\sum_{n\leqslant k+1} h(n)n^{-s}$$

is the residue from the poles passed over at the positive integers $\leqslant k+1$, C_1 is the integral on $L_1=\{w=\eta+\varepsilon\rho:\ -\sqrt{2}\eta\geqslant\rho>-\infty\}$, with $\varepsilon=\exp(\pi i/4)$, C_2 is the integral on $L_2=\{w=\eta\exp(i\alpha):\ 0\geqslant\alpha_0\geqslant\alpha\geqslant-\pi/2\}$, C_3 is the integral on $L_3=\{w=\eta+r\exp(i\beta):\ \pi/4\geqslant\beta\geqslant\beta_0\geqslant-\pi/4\}$, and C_4 is the integral on $L_4=\{w=\eta+\varepsilon\rho:\ \infty>\rho\geqslant r\}$. (See figure 1).

The exact values of α_0 and β_0 are not important. Let

(2.3.8)
$$\gamma(w)=\pi i w^2-s\log w.$$

Let

(2.3.9)
$$\lambda(w)=t\arg w-2\pi uv.$$

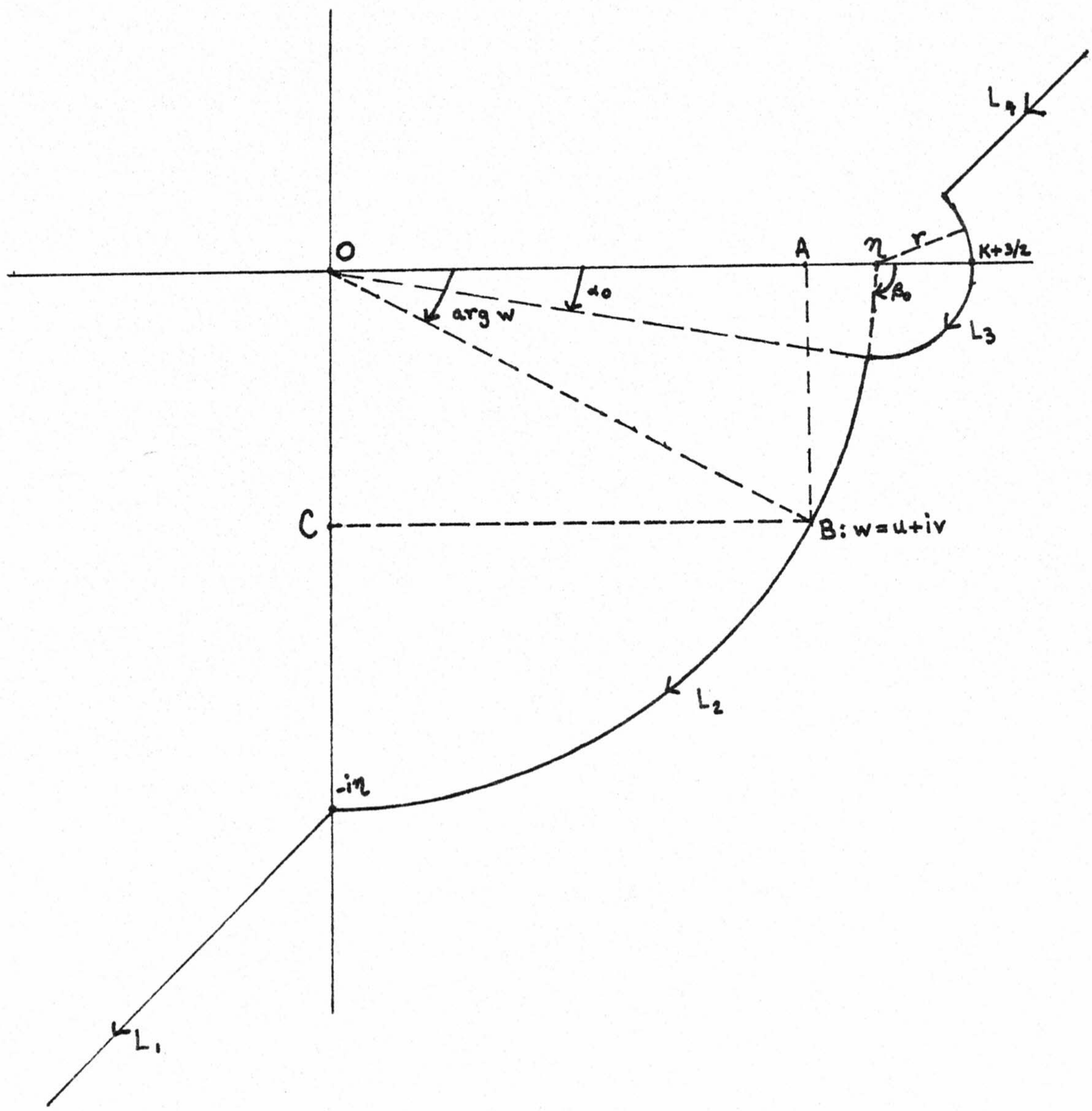

Fig. 1.--New Path of Integration for $f_1(s)$.

Then by (2.3.8) and (2.3.9) we have

$$\text{(2.3.10)} \qquad \operatorname{Re}\{\gamma(w)-\gamma(\eta)\}=\sigma \log|\eta/w|+\lambda(w).$$

Since

$$\operatorname{Re} \gamma(\eta) = -\sigma \log \eta ,$$

it follows from (2.3.4), (2.3.6), (2.3.8), and (2.3.10) that

(2.3.11)

$$f_1(s)=C_0+O\left(\eta^{-\sigma} \int_{L_1,L_2,L_3,L_4} \frac{\exp(\sigma \log|\eta/w|+\lambda(w))\,|h(w)|}{|\sin\pi w|}\,dw\right).$$

We will first show that

$$\text{(2.3.12)} \qquad \exp\ \lambda(w)=O(1) \quad , w\varepsilon L_1, L_2, L_3, L_4 .$$

On L_1 it is clear that $\lambda(w)<0$. On L_2 (see figure 1) we have

(2.3.13)

$$-uv=\text{area } (\square OABC)\leqslant 2 \text{ area } (\triangleleft OAB)=-\eta^2 \arg w=-(t \arg w)/(2\pi).$$

By (2.3.9) and (2.3.13), $\lambda(w)\leqslant 0$ on L_2. On L_3 we have

(2.3.14)

$$\arg w=\arctan(r \sin\beta/(\eta+r \cos\beta))=(r \sin\beta)/\eta+O(T^{-1}),$$

and

$$\text{(2.3.15)} \qquad 2\pi uv=2\pi r(\sin\beta)(\eta+r \cos\beta)=2\pi r\eta\sin\beta+O(1).$$

By (2.3.3), (2.3.9), (2.3.14), and (2.3.15), $\lambda(w)=O(1)$ on L_3.

Let $\mu=\rho/\eta$. On L_4 we have

$$\text{(2.3.16)} \qquad \lambda(w)=t[\arctan(\mu/(\mu+\sqrt{2}))-\mu/\sqrt{2}-\mu^2/2].$$

It is an easy matter to check that for $\mu > 0$ the quantity on the right of (2.3.16) is negative. Hence (2.3.12) is established.

On L_2 we have $|w| = \eta$, so that $\sigma \log|\eta/w| = 0$. Thus, by (2.3.1),

(2.3.17) $$|h(w)| \ll \eta^{\delta}(\log\eta)^{j} \ll L^{j}.$$

Since $|\sin\pi w| \geq (e^{\pi|v|}-1)/2$, it follows from (2.3.11), (2.3.12), and (2.3.17) that

(2.3.18) $$C_2 \ll \eta^{-\sigma}L^{j}.$$

On L_3 equation (2.3.17) is valid, $|\sin\pi w| \gg 1$, and

$$\log(\eta/(\eta+2)) \leq \log|\eta/w| \leq \log(\eta/(\eta-2)).$$

Therefore, $\exp(\sigma\log|\eta/w|)$ is between

(2.3.19) $$(\eta/(\eta+2))^{\sigma} \text{ and } (\eta/(\eta-2))^{\sigma}.$$

Both bounds in (2.3.19) are $O(1)$ as $T \to \infty$ for s in the range considered. Hence it follows that

(2.3.20) $$C_3 \ll \eta^{-\sigma}L^{j}.$$

Also $\lambda(w)$ is larger and $|\sin\pi w|$ smaller on L_4 than on L_1, while $\log|\eta/w|$ and $h(w)$ are the same on L_1 and L_4. Thus C_1 is majorized by C_4.

On L_4, with $w = \eta + \varepsilon\rho$, we have

(2.3.21) $$u = \eta + \rho/\sqrt{2} \text{ and } v = \rho/\sqrt{2}.$$

Therefore,

(2.3.22) $$|\sin\pi w| \geq (e^{\pi v}-1)/2 \gg e^{\pi\rho/\sqrt{2}}, \quad w \in L_4.$$

We split up the path L_4 so that L_4' is for $r\leqslant\rho\leqslant 2\eta$, and L_4'' is for $2\eta\leqslant\rho<\infty$. On L_4' the estimate (2.3.17) is valid, as are (2.3.12) and (2.3.22). Also, on L_4' we have

$$1\leqslant\log|w/\eta|\leqslant 3 ,$$

so $\exp(\sigma\log|\eta/w|)$ is between $e^{-\sigma}$ and $e^{-3\sigma}$. Because of the range of σ under consideration, in all cases we have

(2.3.23) $$\exp(\sigma\log|\eta/w|)\leqslant L^{12} \qquad (w\varepsilon L_4').$$

Hence

(2.3.24) $$C_4'<<\eta^{-\sigma}L^{j+12}.$$

Finally on L_4'' by (2.3.21) we have

(2.3.25) $$2\pi uv=\pi\rho^2+\sqrt{2}\pi\rho\eta \ , \ t \arg w<t\pi/4.$$

Thus, (2.3.8) and (2.2.25) with $\rho\geqslant 2\eta$ imply that

(2.3.26) $$\lambda(w)<-\pi\rho^2 \qquad (w\varepsilon L_4'').$$

Also, if T is sufficiently large, $T>T_0=T_0(j)$, we have

(2.3.27) $$|h(w)|/|\sin\pi w|<<1 \qquad (w\varepsilon L_4'').$$

Further, by (2.3.21) we have

(2.3.28)
$$\sigma\log|\eta/w|=-(\sigma/2)\log|w/\eta|^2=-(\sigma/2)\log(1+\sqrt{2}\rho/\eta+\rho^2/\eta^2),$$

and $\rho\geqslant 2\eta$ implies that

(2.3.29) $$\rho^2/\eta^2\leqslant 1+\sqrt{2}\rho/\eta+\rho^2/\eta^2\leqslant 2\rho^2/\eta^2 .$$

Thus, by (2.2.28) and (2.2.29), $\sigma\log|\eta/w|$ lies between

(2.3.30) $$-\sigma\log(\rho/\eta) \text{ and } -\sigma\log(\sqrt{2}\rho/\eta).$$

By (2.2.30), for any σ in the considered range ($|\sigma|\leqslant 4\log L$) we have

$$(2.3.31)\qquad \exp(\sigma\log|\eta/w|)\ll L^2(\rho/\eta)^{-\sigma} \qquad (w\in L''_4).$$

Thus, by (2.3.11), (2.3.26), (2.3.27), and (2.3.31) we have

$$(2.3.32)\qquad C''_4\ll L^2\int_{2\eta}^{\infty}\exp(-\pi\rho^2)\rho^{-\sigma}d\rho.$$

Let

$$(2.3.33)\qquad I=\int_{2\eta}^{\infty}\exp(-\pi\rho^2)\rho^{-\sigma}d\rho.$$

Suppose $\sigma\geqslant 0$. Then

$$(2.3.34)\qquad I<\int_{2\eta}^{\infty}\exp(-\pi\rho^2)d\rho<\int_{2\eta}^{\infty}\exp(-\pi\rho^2)2\pi\rho d\rho$$

$$=\exp(-4\pi\eta^2)=\exp(-2t).$$

If $\sigma<0$, then we integrate by parts to see that

(2.3.35)

$$I=(\exp(-\pi\rho^2)\rho^{1-\sigma})/(1-\sigma)\Big|_{2\eta}^{\infty}+(2\pi/(1-\sigma))\int_{2\eta}^{\infty}\exp(-\pi\rho^2)\rho^{2-\sigma}d\rho$$

$$\geqslant -\exp(-4\pi\eta^2)(2\eta)^{1-\sigma}/(1-\sigma)+(2\pi/(1-\sigma))4\eta^2 I.$$

By (2.3.35) we have

(2.3.36)

$$I\leqslant[\exp(-2t)(2\eta)^{1-\sigma}]/[4t-(1-\sigma)]\ll\exp(-2t)(2\eta)^{-\sigma}\ll\exp(-t).$$

By (2.3.32), (2.3.33), (2.3.34), and (2.3.36) we have

(2.3.37) $$C_4'' \ll L^2 \exp(-t).$$

Hence by (2.3.11), (2.3.18), (2.3.20), (2.3.24), and (2.3.37) we have

(2.3.38) $$f_1(s) = C_0 + O(\eta^{-\sigma} L^{j+12}).$$

By (2.3.7) and (2.3.1) we have

(2.3.39) $$C_0 - \sum_{n \leqslant \eta} h(n) n^{-s} = h(k+1)(k+1)^{-s} \ll L^j (k+1)^{-\sigma}.$$

Since $\eta < k+1 \leqslant \eta+1$, it follows that $(k+1)^{-\sigma}$ lies between $\eta^{-\sigma}$ and $(\eta+1)^{-\sigma}$.

Hence, we have

(2.3.40) $$(k+1)^{-\sigma} \ll \eta^{-\sigma}.$$

The Lemma follows from (2.3.38), (2.3.39), and (2.3.40).■

<u>Corollary 2.3.2</u> For $0 \leqslant \sigma \leqslant 4 \log L$, $T \leqslant t \leqslant T+U$, f_1 as in (2.3.4), we have

$$\bar{f}_1(1-s) = \sum_{n \leqslant \eta} h(n) n^{s-1} + O(\eta^{\sigma-1} L^{j+12}).$$

<u>Proof</u>. For s in this range, $1-\bar{s} = 1-\sigma+it$ is in the range of Lemma 2.3.1. Therefore, we replace σ by $1-\sigma$ in Lemma 2.3.1 and find that

(2.3.41) $$f_1(1-\bar{s}) = \sum_{n \leqslant \eta} h(n) n^{\bar{s}-1} + O(\eta^{\sigma-1} L^{j+12}).$$

The Corollary follows by conjugating (2.3.41).■

<u>2.4 Estimates for $\zeta^{(k)}(w)$.</u> For $u>1$, by (2.1.1) we have

$$\zeta^{(k)}(w)=\sum_{n=1}^{\infty}(-1)^k(\log^k n)n^{-w}. \tag{2.4.1}$$

For a bounded range of k and $u\geqslant 3$, by (2.4.1) we have

$$|\zeta^{(k)}(w)|<<1. \tag{2.4.2}$$

By (2.4.1) and Stieltjes integration, for $u>1$ we have

$$(-1)^k\zeta^{(k)}(w) \tag{2.4.3}$$

$$=\sum_{n=1}^{N}(\log^k n)n^{-w}+\int_N^{\infty}(\log^k x)x^{-w}d([x]-x)+\int_N^{\infty}(\log^k x)x^{-w}dx$$

$$=\sum_{n=1}^{N}(\log^k n)n^{-w}+\int_N^{\infty}(k-w\log x)(\log^{k-1}x)x^{-w-1}([x]-x)dx+$$

$$+\int_N^{\infty}(\log^k x)x^{-w}dx.$$

Let $P_k(x)$ stand for a k^{th} degree polynomial in x, not necessarily the same at each occurrence. We integrate by parts in (2.4.3) and see that for $u>1$ we have

$$(-1)^k\zeta^{(k)}(w) \tag{2.4.4}$$

$$=\sum_{n=1}^{N}(\log^k n)n^{-w}+\int_N^{\infty}(k-w\log x)(\log^{k-1}x)x^{-w-1}([x]-x)dx+$$

$$+\frac{N^{1-w}}{(1-w)^{k+1}}P_k(\log N)/(1-w)).$$

The expression (2.4.4) determines $\zeta^{(k)}(w)$ for $u>0$ since the integrals are convergent for $u>0$. Hence, for $1/4\leqslant u$, and a bounded range of k,

$$|\zeta^{(k)}(w)| << (\log^k N)\sum_{n=1}^{N} n^{-u}+u^{-k}N^{-u}P_{k-1}((\log N)/u)$$

$$+\ |w|u^{-k-1}N^{-u}P_k((\log N)/u)+\frac{N^{1-u}}{|1-w|^{k+1}}|P_k((\log N)/(1-w))|$$

$$<<(\log^k N)(N^{3/4}+N^{-1/4}+|w|N^{-1/4}+N^{3/4}/|w|^{k+1}).$$

Take $N=[|v|]$. Then for $|v|>v_0$, k in a bounded range, and $1/4\leqslant u$, we have

(2.4.5) $$|\zeta^{(k)}(w)|<<|v|$$

For estimations such as $|\zeta(w)-1|$ it is useful to observe that for $u>1$,

(2.4.6)
$$\sum_{n=2}^{\infty} n^{-u}\leqslant 2^{-u}+\int_2^{\infty} x^{-u}dx=2^{-u}+2^{1-u}/(u-1)=((u+1)/(u-1))2^{-u}.$$

CHAPTER III

GENERAL LOCATION OF ZEROES OF $\xi^{(m)}(w)$

3.1 Well-known properties of entire functions.

Suppose that f is an entire function, and $f(0)\neq 0$. Let the zeroes of f be arranged in order of increasing modulus

$$0<|a_1|\leqslant|a_2|\leqslant|a_3|\leqslant \dots . \tag{3.1.1}$$

Let the exponent of convergence τ of the sequence $\{|a_n|\}$ be defined by

$$\tau=\text{g.l.b.}\{\alpha:\sum_n|a_n|^{-\alpha}<\infty\}.$$

Let k be the least non-negative integer for which

$$\sum_n|a_n|^{-k-1}<\infty. \tag{3.1.2}$$

For D a subset of the complex w-plane let

$$M(f,D)=\max_{w\varepsilon D}|f(w)| \tag{3.1.3}$$

and let M(f,r) be an abbreviation for $M(f,\{w:|w|\leqslant r\})$. The order ω of f is defined by

$$\omega=\limsup_{r\to\infty}(\log\log M(f,r))/\log r. \tag{3.1.4}$$

Suppose f is of finite order. Then by Hadamard's factorization theorem (see Ingham [5, Chap. III, §7]) we have

(3.1.5) $f(w)=\exp(g(w))\prod_n\{(1-w/a_n)\exp(\sum_{i=1}^{k} i^{-1}(w/a_n)^i)\}$

where k is as in (3.1.2), g is a polynomial of degree h, and

(3.1.6) $$\omega=\max\{\tau,h\}.$$

The product is absolutely convergent for all w.

If f is an entire function of order ω, then f' is entire and also has order ω.

Let D be a region of the w-plane, and suppose f is regular throughout D. Let

(3.1.7) $$N(f;D)$$

be the number of zeroes of f inside D, counting multiplicity. For discs D we can bound N(f;D) in terms of M(f,D) by Jensen's theorem. We need only the following simpler theorem which is proved in Ingham [5, theorem D].

Lemma3.1.1 Let f be regular in $|w-w_0|\leqslant r_2$, let $r_1<r_2$ and let

$$M=\max_{|w-w_0|\leqslant r_2}|f(w)|.$$

Suppose $f(w_0)\neq 0$. Then,

(3.1.8) $N(f,|w-w_0|\leqslant r_1)\leqslant(\log M-\log|f(w_0)|)/\log(r_2/r_1)$.

3.2 The strip of zeroes of $\xi^{(m)}(w)$. We define

(3.2.1) $$\xi(w)=\zeta(w)H(w)$$

where H(w) is defined in (1.2.1) and ζ is Riemann's function of §2.1. The simple pole of $\zeta(w)$ at $w=1$ is cancelled by the factor $(w-1)$ of $H(w)$. Since $H(w)$ is regular for $u>0$, $\xi(w)$ is regular for $u>0$. Equation (2.2.16) implies that

$$(3.2.2) \qquad \xi(w)=\xi(1-w).$$

Hence ξ is an entire function. Since $(w-1)\zeta(w)$ has no zeroes for $u\geq 1$, and $(w\Gamma(w/2))\pi^{-w/2}$ has no zeroes anywhere, $\xi(w)$ has no zeroes for $u\geq 1$. Hence, by (3.2.2), the zeroes of $\xi(w)$ are in $0<u<1$.

It follows easily from (1.1.6), (2.4.5), (3.1.4), and (3.2.2) that ξ has order 1; this result is well-known (see Ingham [5, Theorem 17]). Hence $\xi^{(m)}$, the m^{th} derivative of ξ, also has order 1.

<u>Lemma 3.2.1</u> For $m\geq 0$, if $\xi^{(m)}(w)=0$ then the real part u of w satisfies

$$(3.2.3) \qquad 0<u<1.$$

<u>Proof</u>. The proof is by induction on m. The case $m=0$ is demonstrated below (3.2.2). Suppose, as the inductive hypothesis, that equation (3.2.3) is true. We will show that (3.2.3) with $(m+1)$ replacing m is also true. By (3.2.2) we have

$$(3.2.4) \qquad \xi^{(m)}(w)=(-1)^m\xi^{(m)}(1-w).$$

Therefore, for any m,

$$(3.2.5) \qquad \xi^{(2m+1)}(1/2)=0.$$

Suppose $\xi^{(m)}(w)$ has a zero at $w=1/2$ of order n. Then

$$0=\xi^{(m)}(1/2)=\xi^{(m+1)}(1/2)= \ldots =\xi^{(m+n-1)}(1/2)$$

but

(3.2.6) $$\xi^{(m+n)}(1/2)\neq 0.$$

By (3.2.5) and (3.2.6), $(m+n)$ is even. Let

(3.2.7) $$\Xi(w)=\xi^{(m)}(1/2+iw)/w^n.$$

Then $\Xi(0)\neq 0$ and by (3.2.6) and (3.2.7) we have

(3.2.8) $$\Xi(w)=\Xi(-w),$$

that is, Ξ is an even function of w. Therefore, there exists an entire function Λ, $\Lambda(0)\neq 0$, such that

(3.2.9) $$\Lambda(w^2)=\Xi(w).$$

It is clear from (3.2.9) that

(3.2.10) $$M(\Lambda,r^2)=M(\Xi,r).$$

By (3.2.10), the order of Λ is 1/2 of the order of Ξ. By (3.1.4) and (3.2.7), Ξ has order 1 since $\xi^{(m)}(1/2+iw)$ does. Thus Λ has order 1/2. By (3.1.5) and (3.1.6) we have

(3.2.11) $$\Lambda(w)=\Lambda(0)\prod_{\rho}(1-w/\rho)$$

where the product is over the zeroes ρ of Λ. By (3.2.11),

(3.2.12) $$\Lambda'(w)/\Lambda(w)=\sum_{\rho}(w-\rho)^{-1}.$$

By (3.2.7) and (3.2.9) we have

(3.2.13) $$\xi^{(m)}(1/2+iw)=w^n\Lambda(w^2).$$

By (3.2.12) and (3.2.13) we have

$$(3.2.14) \qquad i\xi^{(m+1)}(1/2+iw)/\xi^{(m)}(1/2+iw)$$

$$= n/w+2w\Lambda'(w^2)/\Lambda(w^2)=w(2\sum_{\rho}(w^2-\rho)^{-1}+n/w^2).$$

Let w_1 and w_2 be complex variables,

$$(3.2.15) \qquad w_1=1/2+iw \ , \ w_2=w^2.$$

Then

$$(3.2.16) \qquad w_2=-(w_1-1/2)^2.$$

By (3.2.13) and (3.2.15) we have

$$(3.2.17) \qquad \xi^{(m)}(w_1)=w^n\Lambda(w_2).$$

Let a typical zero of $\Lambda(w_2)$ be denoted by $\rho_2=\beta_2+i\gamma_2$ and denote the corresponding zero of $\xi^{(m)}(w_1)$ by $\rho_1=\beta_1+i\gamma_1$. Let

$$(3.2.18) \qquad R_1=\{w_1:0<u_1<1\}.$$

By the inductive hypothesis each $\rho_1\epsilon R_1$. Under the mapping (3.2.16) the region R_1 of the w_1-plane corresponds to the region

$$(3.2.19) \qquad R_2=\{w_2:u_2>v_2^2-1/4\}$$

of the w_2 plane. Therefore, each $\rho_2\epsilon R_2$. It is sufficient to show that the zeroes of

$$(3.2.20) \qquad 2\sum_{\rho_2}(w_2-\rho_2)^{-1}+nw_2^{-1}$$

are in R_2. For then by (3.2.14), the zeroes of $\xi^{(m+1)}(w_1)$ are in R_1, which is what was to be proved.

The region R_2 is a convex, (parabolic) region which includes the point $w_2=0$. Therefore, (3.2.20) is of the form

$$\sum_\rho (w_2-\rho)^{-1}$$

where all the ρ lie in $\mathcal{R}_2$ (some are repeated). Let $w\notin\mathcal{R}_2$. We will show that

$$\sum_\rho (w-\rho)^{-1}\neq 0 \tag{3.2.21}$$

and that will conclude the proof. Let $\mathcal{L}$ be a line passing through w but not intersecting $\mathcal{R}_2$. Such a line exists since $\mathcal{R}_2$ is convex and $w\notin\mathcal{R}_2$. Let $\mathcal{L}'$ be the line passing through the origin which is parallel to $\mathcal{L}$. Then all the vectors

$$(w-\rho)^{-1},$$

when attached at the origin, have their heads on the same side of, but not on, $\mathcal{L}'$. Therefore, their vector sum is not 0 and (3.2.21) is established.∎

<u>3.3 The quantity of zeroes of $\xi^{(m)}(w)$.</u> For $X>0$ let

$$N^{(m)}(X)$$

be the number of zeroes $\rho_m=\beta_m+i\gamma_m$ of $\xi^{(m)}(w)$ which satisfy

$$0\leqslant\gamma_m\leqslant X.$$

<u>Lemma 3.3.1</u> For any m, and for $X>3$ we have

$$N^{(m)}(X)=(X/(2\pi))\log(X/(2\pi))-X/(2\pi)+O(\log X). \tag{3.3.1}$$

<u>Proof.</u> We apply the argument principle to $\xi^{(m)}(w)$ on the rectangle E which has vertices $3\pm iX$, $-2\pm iX$. Then, since

$$\xi^{(m)}(\bar{s})=\overline{\xi^{(m)}(s)}, \tag{3.3.2}$$

the argument principle implies that

$$[\Delta \arg \xi^{(m)}(w)]_E = 4\pi N^{(m)}(X)+O(1). \tag{3.3.3}$$

The O(1) is to account for zeroes of $\xi^{(m)}(w)$ on the real axis. Let E_1 be the path which consists of two line segments: $3+iv$, $0\leqslant v\leqslant X$, followed by $u+iX$, $3\geqslant u\geqslant 1/2$. Then, because of (3.2.4) and (3.3.2), it follows from (3.3.3) that

$$[\Delta \arg \xi^{(m)}(w)]_{E_1} = \pi N^{(m)}(X)+O(1). \tag{3.3.4}$$

We differentiate m times in (3.2.1) and have

$$\xi^{(m)}(w)=\sum_{k=0}^{m}\binom{m}{k}\zeta^{(k)}(w)H^{(m-k)}(w). \tag{3.3.5}$$

By (1.3.4) and (3.3.5) we have

$$\xi^{(m)}(w)=H(w)\sum_{k=0}^{m}\binom{m}{k}\zeta^{(k)}(w)\left([F(w)]^{m-k}+H_{m-k}(w)\right). \tag{3.3.6}$$

It follows from

$$H(3)=3\pi^{-3/2}\Gamma(3/2)\zeta(3)>0,$$

that $\arg H(3)=0$. We let $\arg H(w)$ vary continuously along E_1 from 3 to $1/2+iX$. Then by Lemma 1.2.1 we have

$$[\Delta \arg H(w)]_{E_1} = (X/2)\log(X/(2\pi))-X/2+O(1). \tag{3.3.7}$$

It is easily checked, (see (1.2.15), (1.2.16) and (1.2.17)) that $F(3)>0$, so that

$$\arg F(3)=0.$$

Then by Lemma 1.2.3 we have

$$[\Delta \arg(F(w))^m]_{E_1} = m \arg F(1/2+iX) \tag{3.3.8}$$

$$= m \arg(\log(1/2+iX)+O(X^{-1}))$$

$$= m \arctan[(\pi/2+O(X^{-1}))/(\log|1/2+iX|+O(X^{-1}))]$$

$$\ll 1/\log X.$$

It remains to estimate the change in $\arg G(w)$ along E_1, where

$$G(w)=\sum_{k=0}^{m}\binom{m}{k}\left(\zeta^{(k)}(w)/[F(w)]^k+H_{m-k}(w)/[F(w)]^m\right). \tag{3.3.9}$$

By (1.3.5), for $v>v_0$ we have

$$H_n(3+iv)\ll(\log v)^{n-2}/v \tag{3.3.10}$$

and by Lemma 1.2.3 for $v>v_0$ we have

$$F(3+iv)\gg\log v\ ,\quad v>v_0.$$

Here v_0 is an absolute constant. From (2.4.2), (3.3.9), and (3.3.10) for $v>v_0$, it follows that

$$|G(3+iv)-1|\leqslant\sum_{n=2}^{\infty}n^{-3}+O(1/\log v). \tag{3.3.11}$$

Thus, by (2.4.6) and (3.3.11) we have

$$|G(3+iv)-1|<1/2 \qquad (v>v_1) \tag{3.3.12}$$

where v_1 is an absolute constant. By (3.3.12) we have

(3.3.13)

$$[\Delta\arg G(3+iv)]_{v=0}^{v=X} = [\Delta\arg G(3+iv)]_{v=0}^{v=v_1} + [\Delta\arg G(3+iv)]_{v=v_1}^{v=X}$$

$$\leqslant O(1)+\pi=O(1).$$

We shall apply Lemma 1.3.2 to estimate the change in arg $G(w)$ on the second part of E_1. For $X>X_0$, it follows from (1.3.5) that

$$H_n(u+iX)<<(\log X)^{n-2}/X \qquad (0\le u\le 6,\ n\le m) \tag{3.3.14}$$

and Lemma 1.2.3 implies that

$$F(u+iX)>>\log X \qquad (0\le u\le 6). \tag{3.3.15}$$

Equation (2.4.5) implies that for $X>X_0$ and $k\le m$ we have

$$\zeta^{(k)}(u+iX)<<X \qquad (1/4\le u\le 6). \tag{3.3.16}$$

By (3.3.9), (3.3.14), (3.3.15), and (3.3.16) we have

$$G(u+iX)<<X \qquad (X>X_0,\ 1/4\le u\le 6). \tag{3.3.17}$$

Let

$$M=\max_{|w-3|\leqslant 11/4}|G(w+iX)+G(w-iX)|. \tag{3.3.18}$$

Then by (3.3.17) we have

(3.3.19) $$M \ll X.$$

In (3.3.12) with v=x we take real parts. Since $G(\bar{w})=\overline{G(w)}$, we have

(3.3.20) $$|G(3+iX)+G(3-iX)|=|2\mathrm{Re}\ G(3+iX)|\geq 1\ .$$

Therefore, by (3.3.18), (3.3.19), (3.3.20), and (3.1.8) with $r_2=11/4$, $r_1=5/2$ we have

(3.3.21)
$$[\Delta\arg G(u+iX)]_{u=1/2}^{u=3} \leq \pi(1+N(\mathrm{Re}\ G(u+iX)\ ,\ 1/2\leq u\leq 3))$$

$$\leq \pi(1+N(G(w+iX)+G(w-iX)\ ,\ |w-3|\leq 5/2))$$

$$\leq \pi(1+(\log M-\log|G(3+iX)+G(3-iX)|)/\log(11/10))$$

$$\ll \log X.$$

Thus, by (3.3.13) and (3.3.21) we have

(3.3.22) $$[\Delta\arg G(w)]_{E_1} = O(\log X).$$

Hence, by (3.3.6), (3.3.7), (3.3.8), (3.3.9), and (3.3.22) we have

(3.3.23)
$$[\Delta\arg\xi^{(m)}(w)]_{E_1} = (X/2)\log(X/(2\pi))-X/2+O(\log X)\ .$$

The Lemma follows from (3.3.4) and (3.3.23).■

It follows easily from Lemma 3.3.1 that

(3.3.24) $$N^{(m)}(T+U)-N^{(m)}(T)=UL/(2\pi)+O(UL^{-10}).$$

Let

$$N_m(X)$$

be the number of zeroes of $\xi^{(m)}(1/2+iv)$ with $0 \leqslant v \leqslant X$. The Riemann hypothesis is

(3.3.25) $$N^{(0)}(X)=N_0(X)$$

for all $X>0$. Now (3.2.4) and (3.3.2) imply that $\xi^{(m)}(1/2+iv)$ is real if m is even and purely imaginary if m is odd. Therefore, Lemma 3.3.1, equation (3.3.25), and Rolle's theorem imply that

$$N_m(X)=N^{(m)}(X)+O(\log X)$$

<u>if Riemann's hypothesis is true</u>. What we intend to show is that

(3.3.26) $$N_m(T+U)-N_m(T)>\beta_m UL/(2\pi)$$

for a reasonably good value of β_m. In light of (3.3.24), it follows from (3.3.26) that

(3.3.27) $$N_m(X)>\beta_m N^{(m)}(X) \quad ,$$

if X is sufficiently large.

Levinson [8] proved that one can take $\beta_0=.3420$. He also sketched a proof that $\beta_0=.3470$ is admissible [11], and gave an indication [9] of how one could obtain $\beta_1=.7172$. The method used here is a generalization of Levinson's method. (See also [10]).

CHAPTER IV

THE BASIC IDENTITY

4.1 The identity. We begin with the Riemann-Siegel formula of (2.2.17), which by (3.2.1) is

(4.1.1) $$\xi(s)=H(s)f(s)+H(1-s)\bar{f}(1-s).$$

Here f is given by (2.2.11), and $\bar{f}$ by (2.2.12). We differentiate the above m times, and since the integral for f converges uniformly in any compact range of s, we may differentiate under the integral sign. Thus we have

(4.1.2)
$$\xi^{(m)}(s)=\sum_{k=0}^{m}\binom{m}{k}H^{(k)}(s)\int_{\swarrow 1/2}\frac{w^{-s}\exp(\pi i w^2)}{2i\sin\pi w}(-\log w)^{m-k}dw$$

$$+\sum_{k=0}^{m}\binom{m}{k}(-1)^k H^{(k)}(1-s)\int_{\searrow 1/2}\frac{w^{s-1}\exp(-\pi i w^2)}{2i\sin\pi w}(\log w)^{m-k}dw.$$

Let $C\neq 0$, and let ϕ be an entire function of w which satisfies the conditions

(4.1.3)
$$\phi(\bar{w})=\overline{\phi(w)},\quad \phi(w)+\phi(1-w)=C,\quad \phi(0)=1,\quad |\phi(w)|<<\exp(d|w|)$$

where

$$d<\min\{R,1\}.$$

For w not on the negative real axis, let

(4.1.4) $$c(w)=\phi(L^{-1}\log w)\ ,\ d(w)=\phi(1-L^{-1}\log w).$$

By (4.1.3), for $|w|>3$ we have

(4.1.5) $$c(w)\ll\exp((d/L)\log|w|+(d/L)\arg w)\ll|w|^{1/L}.$$

Similarly, we see that

(4.1.6) $$d(w)\ll\exp(d+dL^{-1}|\log w|)\ll|w|^{1/L}.$$

Define the function Q by

(4.1.7)
$$Q(s)=\sum_{k=0}^{m}\binom{m}{k}H^{(k)}(s)\int_{\swarrow 1/2}\frac{w^{-s}\exp(\pi iw^2)}{2i\sin\pi w}(-\log w)^{m-k}c(w)\,dw$$

$$+\sum_{k=0}^{m}\binom{m}{k}(-1)^kH^{(k)}(1-s)\int_{\searrow 1/2}\frac{w^{s-1}\exp(-\pi iw^2)}{2i\sin\pi w}(\log w)^{m-k}d(w)\,dw.$$

Then by (4.1.2), (4.1.3), (4.1.4), (4.1.7), and Remarks 1 and 2 of §2.2 we have pg 15

(4.1.8) $$c\xi^{(m)}(s)=Q(s)+(-1)^m\bar{Q}(1-s)$$

which is the basic identity. Its significance is for $s=1/2+it$ when

$$(4.1.9) \qquad C\xi^{(m)}(1/2+it)=Q(1/2+it)+(-1)^m\,\overline{Q(1/2+it)}$$

which expresses $C\xi^{(m)}(1/2+it)$ as a sum, when m is even, and a difference, when m is odd, of complex conjugates.

4.2 The argument of Q(s). The function Q(s) is regular for s not on the negative real axis. Since $C\neq 0$, it is clear that the zeroes of $\xi^{(m)}(1/2+it)$ occur precisely when

$$(4.2.1) \qquad \text{Re}\, Q(1/2+it)=0$$

or $$\arg Q(1/2+it)\equiv\begin{cases}\pi/2 \bmod \pi & \text{if } m \text{ is even}\\ 0 \bmod \pi & \text{if } m \text{ is odd.}\end{cases}$$

Since arg $Q(1/2+it)$ is well defined mod 2π (if $Q(1/2+it)\neq 0$), the conditions (4.2.1) are independent of how one defines arg $Q(s)$. We are interested in how often (4.2.1) holds with $T\leq t\leq T+U$.

Suppose that T is such that $Q(1/2+iT)\neq 0$, $Q(1/2+i(T+U))\neq 0$. Starting with some value of arg $Q(1/2+iT)$, we determine arg $Q(1/2+i(T+U))$ by letting arg $Q(s)$ vary continuously on the path $s=1/2+it$, $T\leq t\leq T+U$. If $Q(1/2+it_0)=0$ for some $T<t_0<T+U$ we detour around $1/2+it_0$ on the semicircle

$$|s-(1/2+it_0)|=\varepsilon\ , \quad \text{Re } s\geq 1/2$$

where $\varepsilon>0$ is so small that

(i) no zeroes of $Q(s)$, except those at $1/2+it_0$, are in $|s-(1/2+it_0)|\leq\varepsilon$,

(ii) no points $1/2+it$, $t \neq t_0$, for which (4.2.1) holds are missed because of the detour.

With this definition we claim that

(4.2.2)

$$N_m(T+U)-N_m(T) \geqslant (1/\pi)(\arg Q(1/2+i(T+U))-\arg Q(1/2+iT))-2.$$

The only difficulty in showing (4.2.2) arises from the detours, so we deal with them. Suppose $1/2+it_0$ is a zero of $Q(s)$ of multiplicity n_0. Then by (i) and the argument principle we have

$$[\Delta \arg Q(1/2+it_0+\varepsilon \exp(i\theta))]_{\theta=-\pi/2}^{\theta=\pi/2} \to \pi n_0$$

as $\varepsilon \to 0$. Thus the detour contributes n_0 to the right side of (4.2.2). But $1/2+it_0$ is a zero of $\bar{Q}(1-s)$ of multiplicity n_0 and by (4.1.8) it is a zero of $\xi^{(m)}(s)$ of multiplicity at least n_0. Thus, the zero of $\xi^{(m)}(s)$ at $1/2+it_0$ contributes n_0 to the left side of (4.2.2), and (4.2.2) is valid.

Let

(4.2.3)
$$Q_1(s)=$$

$$=\sum_{k=0}^{m}\binom{m}{k}(F^k(s)+H_k(s))\int_{\swarrow 1/2}\frac{w^{-s}e^{\pi i w^2}}{2i\,\sin\pi w}(-\log w)^{m-k}c(w)\,dw+\chi(s)\;\cdot$$

$$\cdot\sum_{k=0}^{m}\binom{m}{k}(-1)^k(F^k(1-s)+H_k(1-s))\int_{\searrow 1/2}\frac{w^{s-1}e^{-\pi i w^2}}{2i\,\sin\pi w}(\log w)^{m-k}d(w)\,dw.$$

Then by (1.2.2), (1.3.4), and (4.1.7) we have

(4.2.4)
$$Q(s)=H(s)Q_1(s).$$

Then Q_1 is regular for s not on the negative real axis. By (4.2.4) we have

(4.2.5)
$$\arg Q(s)=\arg H(s)+\arg Q_1(s).$$

Since H(s) has no zeroes or poles for $\sigma>0$, $t>0$, we define arg H(3)=0, and arg H(s) may be obtained by continuous variation as in Equation (1.2.4). Then by Lemma 1.2.1,

(4.2.6)
$$[\Delta\arg H(1/2+it)]_T^{T+U}=UL/2+O(UL^{-10}).$$

Thus, by (4.2.2), (4.2.5), and (4.2.6) we have

(4.2.7)
$$N_m(T+U)-N_m(T)\geqslant UL/(2\pi)+(1/\pi)\;[\Delta\arg Q_1(1/2+it)]_{t=T}^{t=T+U}+O(UL^{-10}),$$

where arg $Q_1(1/2+it)$ is determined as above (4.2.2).

4.3 A more tractable function. We can use Lemma 2.3.1 to simplify $Q_1(s)$ considerably. Let $c_n=c(n)$ and $d_n=d(n)$. With $h(w)=(\log w)^{m-k}c(w)$, it follows from (4.1.5) that (2.3.1) is satisfied with $\delta=L^{-1}$ and j replaced by m-k. Thus, by Lemma 2.3.1 we have

(4.3.1)
$$\int_{\swarrow 1/2} \frac{w^{-s}\exp(\pi i w^2)}{2i\sin\pi w}(\log w)^{m-k}c(w)\,dw$$

$$= \sum_{n\leqslant\eta} c_n(\log n)^{m-k}n^{-s}+O(\eta^{-\sigma}L^{m-k+12})$$

for $0\leqslant\sigma\leqslant 4\log L$, $T\leqslant t\leqslant T+U$. Similarly, by Corollary 2.3.2,

(4.3.2)
$$\int_{\searrow 1/2} \frac{w^{s-1}\exp(-\pi i w^2)}{2i\sin\pi w}(\log w)^{m-k}d(w)\,dw$$

$$= \sum_{n\leqslant\eta} d_n(\log n)^{m-k}n^{s-1}+O(\eta^{\sigma-1}L^{m-k+12})$$

also for $0\leqslant\sigma\leqslant 4\log L$, $T\leqslant t\leqslant T+U$. Let

(4.3.3)
$$f_1(s)=\sum_{n\leqslant\eta} c_n(F(s)-\log n)^m n^{-s} ,$$

(4.3.4)
$$f_2(s)=\sum_{n\leqslant\eta} d_n(\log n-F(1-s))^m n^{s-1}.$$

Lemma 4.3.1 For $0\leqslant\sigma\leqslant 4\log L$, $T\leqslant t\leqslant T+U$ we have

$$Q_1(s)=f_1(s)+\chi(s)f_2(s)+O(T^{-\sigma/2}L^{m+12}+T^{-1/2}L^m).$$

Proof. By Lemma 1.2.2, for $0\leqslant\sigma\leqslant 4\log L$, $T\leqslant t\leqslant T+U$,

$$|\chi(s)|<<(t/2\pi)^{1/2-\sigma}=\eta^{1-2\sigma}. \tag{4.3.5}$$

Then by Lemmas 1.2.3 and 1.3.2, and (4.3.12) the error in replacing the integrals of (4.2.3) by the sums in (4.3.1) and (4.3.2) is

$$<<\eta^{-\sigma}L^{m+12}+\eta^{1-2\sigma}\eta^{\sigma-1}L^{m+12}=\eta^{-\sigma}L^{m+12}. \tag{4.3.6}$$

By (4.1.5) and (4.1.6) we have

$$c_n<<1\ ,\ d_n<<1 \text{ for } 1\leqslant n<<T. \tag{4.3.7}$$

Thus for $0\leqslant\sigma\leqslant 4\log L$ we have

$$\sum_{n\leqslant\eta} c_n(\log n)^{m-k}n^{-s}<<L^{m-k}\sum_{n\leqslant\eta} n^{-\sigma}<<L^{m-k}\eta \tag{4.3.8}$$

and

$$\sum_{n\leqslant\eta} d_n(\log n)^{m-k}n^{s-1}<<L^{m-k}\sum_{n\leqslant\eta} n^{\sigma-1}<<L^{m-k+1}\eta^{\sigma}. \tag{4.3.9}$$

Thus, by Lemma 1.3.2 and Equations (4.3.5), (4.3.8), and (4.3.9), the error in ignoring $H_k(s)$ and $H_k(1-s)$ in Equation (4.2.3) is

$$<<\sum_{k=1}^{m} L^{k-1}T^{-1}(L^{m-k}\eta+\eta^{1-2\sigma}L^{m-k+1}\eta^{\sigma})<<L^m\eta^{-1} \tag{4.3.10}$$

for $0\leq\sigma\leq 4\log L$. The Lemma follows from Equations (4.3.6) and (4.3.10).∎

Corollary 4.3.2 For $0\leqslant\sigma\leqslant 4\log L$, $T\leqslant t\leqslant T+U$ we have

$$Q_1(s)<<T^{1/2}L^{m+1}<<T.$$

Proof. For s in this region

(4.3.11) $$F(s)<<L\ ,\ F(1-s)<<L.$$

By (4.3.3), (4.3.7), and (4.3.11) we have

(4.3.12) $$f_1(s)<<L^m\eta.$$

By (4.3.4), (4.3.5), (4.3.7), and (4.3.11) we have

(4.3.13) $$\chi(s)f_2(s)<<\eta^{1-2\sigma}L^m\eta^{\sigma}L<<L^{m+1}\eta.$$

The Corollary follows from (4.3.12) and (4.3.13).■

Corollary 4.3.3 For $3\leqslant\sigma\leqslant 4\log L$, $T\leqslant t\leqslant T+U$ we have

$$Q_1(s)=[F(s)]^m+O(2^{-\sigma}L^m).$$

Proof. It follows from (4.1.3) and (4.1.4) that $C_1=1$. Therefore, by (4.3.7) and (4.3.11) we have

(4.3.14) $$f_1(s)=[F(s)]^m+O(L^m\sum_{2\leqslant n\leqslant\eta}n^{-\sigma}).$$

By (4.3.4), (4.3.5), (4.3.7), and (4.3.11) we have

(4.3.15) $$\chi(s)f_2(s)<<\eta^{1-2\sigma}L^m\eta^{\sigma}<<L^m\eta^{-2}.$$

The Corollary follows from (2.4.6), (4.3.14), and (4.3.15).■

<u>4.4 The argument principle</u>. Let D be the indented rectangle with vertices $1/2+iT$, σ_0+iT, $\sigma_0+i(T+U)$, and $1/2+i(T+U)$ and with small semicircular indents on the left side centered at zeroes of $Q_1(1/2+it)$ as described in (i) and (ii) above (4.2.2). Here we have

(4.4.1) $$\sigma_0=2\log L.$$

Then by the argument principle we have

(4.4.2) $$[\Delta\arg Q_1(s)]_D=2\pi N(Q_1(s),D).$$

By Lemma 1.3.3, Corollary 4.3.3, and Equation (4.4.1), for $T\leqslant t\leqslant T+U$ we have

(4.4.3) $$Q_1(\sigma_0+it)=(\ell/2+\pi i/4+O(T^{-1}\log L))^m+O(L^{m-2\log 2})$$

$$=(\ell/2)^m+O(\ell^{m-1})+O(L^{m-1})=(\ell/2)^m+O(L^{m-1}).$$

By (4.4.3) we have

(4.4.4) $$[\Delta\arg Q_1(\sigma_0+it)]_{t=T}^{t=T+U}<\pi.$$

We use Lemma 3.1.1 on the upper and lower sides of D. Let

(4.4.5) $$M=\max_{|s-\sigma_0|\leqslant\sigma_0}|Q_1(s+iT)+\overline{Q_1(\bar{s}+iT)}|.$$

By Corollary 4.3.2 and Equation (4.4.1) we have

(4.4.6) $$M<<T.$$

By (4.4.3) we have

(4.4.7) $$|Q_1(\sigma_0+iT)+\overline{Q_1(\sigma_0+iT)}|=2|\mathrm{Re}\ Q_1(\sigma_0+iT)|>>L^m.$$

Then using (3.1.8) it follows from (4.4.5), (4.4.6), and (4.4.7) that

(4.4.8)

$$[\Delta \arg Q_1(\sigma+iT)]_{\sigma=1/2}^{\sigma=\sigma_0} \leqslant \pi(1+N(\mathrm{Re}\ Q_1(\sigma+iT), 1/2\leqslant\sigma\leqslant\sigma_0))$$

$$\leqslant \pi(1+N(Q_1(s+iT)+\overline{Q_1(\bar{s}+iT)},\ |s-\sigma_0|\leqslant\sigma_0-1/2))$$

$$\leqslant \pi(1+(\log M-m\log L)/\log(\sigma_0/(\sigma_0-1/2))) \ll L\log L.$$

The same argument works to show that

(4.4.9) $$[\Delta \arg Q_1(\sigma+i(T+U))]_{\sigma=1/2}^{\sigma=\sigma_0} = O(L\log L).$$

By (4.4.2), (4.4.4), (4.4.8), and (4.4.9) we have

(4.4.10) $$[\Delta \arg Q_1(1/2+it)]_{t=T}^{t=T+U} = -2\pi N(Q_1,D)+O(L\log L)$$

where the expression on the left of (4.4.10) signifies

$$\arg Q_1(1/2+i(T+U))-\arg Q_1(1/2+iT)$$

where $\arg Q_1(1/2+it)$ is defined as above (4.2.2).

By (4.2.7) and (4.4.10) we have

<u>Lemma 4.4.1</u>

$$N_m(T+U)-N_m(T) \geqslant UL/(2\pi)-2N(Q_1,D)+O(UL^{-10}).$$

4.5 Littlewood's lemma We apply Littlewood's lemma (see Titchmarsh [17,§3.8]) on the rectangle D_1 which has vertices $a+iT$, σ_0+iT, $\sigma_0+i(T+U)$, $a+i(T+U)$ to the function

$$2^{m_L-m_{Q_1}}(s)\psi(s).$$

Here we define the mollifier

(4.5.1) $$\psi(s)=\sum_{j\leq y} b_j/j^s$$

where

(4.5.2) $$b_1=1,\quad |b_j|\leq 1\ .$$

We may think of ψ as an approximation to $1/Q_1(s)$.

The result is

(4.5.3) $$2\pi\Sigma\,\text{dist} =$$

$$= \int_T^{T+U} \log|(2/L)^m\psi Q_1(a+it)|\,dt - \int_T^{T+U} \log|(2/L)^m\psi Q_1(\sigma_0+it)|\,dt$$

$$+ \int_a^{\sigma_0} \arg((2/L)^m\psi Q_1(\sigma+i(T+U)))\,d\sigma - \int_a^{\sigma_0} \arg((2/L)^m\psi Q_1(\sigma+iT))\,d\sigma,$$

where Σ dist is the sum of the distances of the zeroes of $2^{m_L-m}\psi(s)Q_1(s)$ from the left side of D_1.

We estimate the last two integrals of (4.5.3). Clearly,

(4.5.4) $$\arg 2^{m_L-m}=0\ .$$

Recall, that arg $HQ_1(1/2+iT)$ was arbitrary mod 2π. We have determined arg $H(3+it)$ below (4.2.5). We now determine arg $Q_1(s)$ by first specifying (after Corollary 4.3.3) that

(4.5.5) $$|\arg Q_1(\sigma_0+iT)|\leqslant\pi/2.$$

Then by (4.5.5) and a slightly more general version of (4.4.8) we have

(4.5.6) $$|\arg Q_1(\sigma+it)|=O(L\log L)$$

uniformly for $\sigma\geqslant a$, $t\geqslant T$. Therefore,

(4.5.7) $$\int_a^{\sigma_0}\arg(Q_1(\sigma+iT))d\sigma=O(L\log^2 L)$$

and

(4.5.8) $$\int_a^{\sigma_0}\arg(Q_1(\sigma+i(T+U))d\sigma=O(L\log^2 L).$$

By (4.5.1), for $\sigma\geqslant 0$ and for any t we have

(4.5.9) $$|\psi(s)|<T^{1/2}.$$

Also by (4.5.1) and (2.4.6), for $\sigma\geqslant 2$ and any t we have

(4.5.10) $$|\psi(s)-1|\leqslant 3\cdot 2^{-\sigma}.$$

Therefore, for $\sigma\geqslant 2$, it follows from (4.5.10) that

(4.5.11) $$\mathrm{Re}\,\psi(\sigma+it)\geqslant 1/4.$$

By (4.5.10), for $\sigma \geqslant 2$ and any t we have

$$(4.5.12) \qquad |\arg(\psi(s))| < \pi/2.$$

Now suppose $a \leqslant \sigma_1 < 2$. Then for any t, by (4.5.9), (4.5.11), and (3.1.8), we have

$$(4.5.13) \quad [\Delta \arg \psi(\sigma+it)]_{\sigma=\sigma_1}^{\sigma=2} < \pi(1+N(\mathrm{Re}\ \psi(\sigma+it),\ \sigma_1 \leqslant \sigma \leqslant 2))$$

$$\leqslant \pi(1+N(\psi(s+it)+\psi(s-it),\ |s-2| \leqslant 7/4))$$

$$<< L.$$

By (4.5.12) and (4.5.13) we have

$$(4.5.14) \qquad \arg\psi(s) = O(L) \quad , \ \sigma \geqslant a.$$

By (4.5.4), (4.5.7), (4.5.8), and (4.5.14) the last two integrals of (4.5.3) are $O(L \log^2 L)$.

We now deal with the second integral of (4.5.3). By Lemma 1.3.3, Corollary 4.3.3, and (4.4.1), for $T \leqslant t \leqslant T+U$,

(4.5.15)
$$(2/L)^m Q_1(\sigma_0+it) = [\ell/L + \pi i/(2L0 + O(T^{-1})]^m + O(L^{-2\log 2})$$

$$= (\ell/L)^m + O(L^{-1}) = (1+L^{-1}\log(t/T))^m + O(L^{-1})$$

$$= 1 + O(L^{-1}).$$

It follows from (4.5.14) that for $T \leqslant t \leqslant T+U$ we have

$$(4.5.16) \qquad \log|(2/L)^m Q_1(\sigma_0+it)| << L^{-1}.$$

By (4.5.10), for $s=\sigma_0+it$ we have

(4.5.17) $$\log|\psi(s)|=\log|1+(\psi(s)-1)|$$

$$= \log(1+O(2^{-\sigma_0}))$$

$$= \log(1+O(L^{-2\log 2}))<<L^{-2\log 2}<<L^{-1}.$$

By (4.5.16) and (4.5.17) we have

(4.5.18) $$\int_T^{T+U} \log|(2/L)^m\psi Q_1(\sigma_0+it)|dt=O(U/L).$$

The zeroes of $(2/L)^m\psi Q_1(s)$ inside D_1 include the zeroes of $Q_1(s)$ inside D. The zeroes of $Q_1(s)$ inside D are a distance at least $(1/2-a)$ from the left side of D_1. Therefore, it follows that

(4.5.19) $$\Sigma\ \text{dist}\geqslant(1/2-a)N(Q_1,D).$$

Since log is a concave function, it follows that

(4.5.20) $$\int_T^{T+U} \log|(2/L)^m\psi Q_1(a+it)|dt$$

$$\leqslant U\log(U^{-1}\int_T^{T+U} |(2/L)^m\psi Q_1(a+it)|dt).$$

By Equations (4.5.3), (4.5.18), (4.5.19), (4.5.20), and the assertion below (4.5.14) we have

(4.5.21)
$$2\pi N(Q_1,D)$$
$$\leq (1/2-a)^{-1}U\log(U^{-1}\int_T^{T+U}|(2/L)^m\psi Q_1(a+it)|dt)+O(U).$$

Thus our concern is now for evaluating the integral

$$\int_T^{T+U}|(2/L)^m\psi Q_1(a+it)|dt.$$

CHAPTER V

THE SIMPLIFICATION OF THE INTEGRAND

5.1 Useful tools. A typical assertion in this chapter will be that

$$(5.1.1) \qquad I = \left| \int_{T}^{T+U} |\psi Q_{\alpha}(a+it)|\,dt - \int_{T}^{T+U} |\psi Q_{\beta}(a+it)|\,dt \right|$$

is small, since we shall replace Q_1 by a string of simpler functions. The ideas for these simplifications are due to Levinson [12] and Pan [13].

The main device we use involves the triangle inequality and the Cauchy-Schwarz inequality. By the triangle inequality and (5.1.1) we have

$$(5.1.2) \qquad I \leq \int_{T}^{T+U} |\psi(a+it)| \Big| |Q_{\alpha}(a+it)| - |Q_{\beta}(a+it)| \Big|\,dt$$

$$\leqslant \int_{T}^{T+U} |\psi(a+it)|\,|Q_{\alpha}(a+it) - Q_{\beta}(a+it)|\,dt.$$

If we let

$$Q_{\alpha}(a+it) = f_{1\alpha}(t) + \chi_{\alpha}(t) f_{2\alpha}(t)$$

and similarly for $Q_\beta(a+it)$, then by (5.1.2) and the triangle inequality we have

(5.1.3)
$$I \leqslant \int_T^{T+U} |\psi(a+it)|\,|f_{1\alpha}(t)-f_{1\beta}(t)|\,dt + $$

$$+ \int_T^{T+U} |\psi(a+it)|\,|\chi_\alpha f_{2\alpha}(t)-\chi_\beta f_{2\beta}(t)|\,dt.$$

Then the Cauchy-Schwarz inequality may be applied to (5.1.2) or to (5.1.3). For example by (5.1.2) we have

(5.1.4)
$$I \leqslant \left(\int_T^{T+U} |\psi(a+it)|^2 dt\right)^{1/2} \left(\int_T^{T+U} |Q_\alpha(a+it)-Q_\beta(a+it)|^2 dt\right)^{1/2}.$$

To estimate the integrals of squares of Dirichlet polynomials, we have the following lemma due to Levinson [8, Lemma 3.2].

<u>Lemma 5.1.1</u> Let $1 \leqslant A_1,\ A_2 \leqslant T^{1/2}$ and suppose $(A_1, A_2)=1$. Suppose that a_1 satisfies

(5.1.5)
$$|1/2-a_1| << L^{-1}.$$

Then we have

(5.1.6)

$$\sum_{\substack{j_1,j_2\leqslant Y\\ j_1A_1\neq j_2A_2}}\sum j_1^{-a}\, j_2^{-a}\,/|\log(j_2A_2/(j_1A_1))| = O(T^{1/2}L).$$

We apply Lemma 5.1.1 to estimate the integral of $|\psi|^2$. We have

Lemma 5.1.2 For a and ψ as usual we have

$$\int_T^{T+U} |\psi(a+it)|^2 dt = O(UL).$$

Proof. We use Equation (4.5.1) and interchange summation and integration to see that

(5.1.7)
$$\int_T^{T+U} |\psi(a+it)|^2 dt$$

$$= \sum_{j_1,j_2\leqslant Y}\sum j_1^{-a}\, j_2^{-a}\, b_{j_1}\overline{b_{j_2}} \int_T^{T+U} \exp(it\log(j_2 j_1^{-1}))\,dt$$

$$= U\sum_{j\leqslant Y} |b_j|^2 j^{-2a} + O\Big(\sum_{\substack{j_1,j_2\leqslant Y\\ j_1\neq j_2}}\sum |b_{j_1} b_{j_2}|\, j_1^{-a}\, j_2^{-a}\,/|\log(j_2/j_1)|\Big).$$

The Lemma now follows from (4.5.2), Lemma 5.1.1 with $A_1=A_2=1$ and $a_1=a$, and the fact that

(5.1.8) $$j^{-a}=j^{1/2-a}j^{-1/2}\ll j^{-1/2}$$

for $j\ll T$. ■

5.2 Simplifying the integrand. We first replace $(2/L)^m Q_1(s)$ by

(5.2.1) $$Q_2(s)=(2/L)^m(f_1(s)+\chi(s)f_2(s))$$

where f_1 and f_2 are defined in (4.3.3) and (4.3.4). Then by Lemma 4.3.1 and the fact that

(5.2.2) $$1\ll t^{1/2-a}\ll 1 \qquad (t\ll T)$$

we have for $T\le t\le T+U$ that

(5.2.3) $$|(2/L)^m Q_1(a+it)-Q_2(a+it)|\ll T^{-1/4}L^{12}.$$

Thus, by (5.1.1), (5.1.4), Lemma 5.1.2, and (5.2.3),

(5.2.4) $$(2/L)^m\int_T^{T+U}|\psi Q_1(a+it)|dt$$

$$=\int_T^{T+U}|\psi Q_2(a+it)|dt+O(U^{7/8}L^8).$$

Next, let us define

(5.2.5) $$f_1^*(s)=\sum_{n\leq\eta} c_n(\ell/L+\pi i/(2L)-2L^{-1}\log n)^m n^{-s}$$

and

(5.2.6) $$f_2^*(s)=\sum_{n\leq\eta} d_n(2L^{-1}\log n+\pi i/(2L)-\ell/L)^m n^{s-1}.$$

Then by Lemma 1.3.3, (4.3.3), (4.3.4), (5.2.2), (5.2.5), (5.2.6), and (4.3.7), we have for $T\leq t\leq T+U$ that

(5.2.7)
$$|f_1(a+it)-f_1^*(a+it)<<\sum_{n\leq\eta} n^{-a}(\ell/L)^{m-1}T^{-1}L^{-1}\log L<<U^{-1/2}L^5.$$

By (4.3.5) we have for $T\leq t\leq T+U$ that

(5.2.8) $$|\chi(a+it)|<<1.$$

Thus, as in Equation (5.2.7) we find that

(5.2.9) $$|\chi(a+it)||f_2(a+it)-f_2^*(a+it)|<<U^{-1/2}L^5.$$

Therefore, if we define

(5.2.10) $$Q_3(s)=f_1^*(s)+\chi(s)f_2^*(s),$$

then by (5.1.4), Lemma 5.1.2, (5.2.1), (5.2.7), and (5.2.9),

(5.2.11)
$$\int_T^{T+U}|\psi Q_2(a+it)|dt=\int_T^{T+U}|\psi Q_3(a+it)|dt+O(U^{3/4}L^3).$$

Now let

$$(5.2.12)\qquad g_1(s) = \sum_{n\leq\eta} c_n(1+\pi i/(2L)-2L^{-1}\log n)^m n^{-s}$$

and

$$(5.2.13)\qquad g_2(s) = \sum_{n\leq\eta} d_n(2L^{-1}\log n+\pi i/(2L)-1)^m n^{s-1}.$$

Further, we define

$$(5.2.14)\qquad Q_4(s)=g_1(s)+\chi(s)g_2(s)\ .$$

This time we use (5.1.3). First observe that by the binomial theorem we have

(5.2.15)

$$(\ell/L+\pi i/(2L)-2L^{-1}\log n)^m-(1+\pi i/(2L)-2L^{-1}\log n)^m$$

$$= \sum_{k=0}^{m}\binom{m}{k}(\pi i/(2L)-2L^{-1}\log n)^{m-k}[(\ell/L)^k-1].$$

Therefore, by (5.2.5), (5.2.12), and (5.2.15) we have

$$(5.2.15)\qquad f_1^*(a+it)-g_1(a+it)$$

$$= \sum_{k-0}^{m}\binom{m}{k}[(\ell/L)^k-1]\sum_{n\leq\eta} c_n(\pi i/(2L)-2L^{-1}\log n)^{m-k}n^{-s}.$$

By (5.2.15), the Cauchy-Schwarz inequality, and Lemma 5.1.2 we have

(5.2.16)
$$\int_T^{T+U} |\psi(a+it)|\,|f_1^*(a+it)-g_1(a+it)|\,dt$$

$$= \sum_{k=0}^{m}\binom{m}{k}\left[\left(\frac{\ell}{L}\right)^k-1\right]\left(\int_T^{T+U} |\psi|\left|\sum_{n\leqslant\eta} c_n\left[\frac{\pi i}{2L}-\frac{2\log n}{L}\right]^{m-k} n^{-a-it}\right|dt\right)$$

$$\leqslant \sum_{k=0}^{m}\binom{m}{k}\left[\left(\frac{\ell}{L}\right)^k-1\right](UL)^{1/2}\left(\int_T^{T+U} \left|\sum_{n\leqslant\eta} c_n\left(\frac{\pi i}{2L}-\frac{2\log n}{L}\right)^{m-k} n^{-a-it}\right|^2 dt\right)^{1/2}.$$

For the moment, let

(5.2.17) $$a_n=c_n(\pi i/(2L)-2L^{-1}\log n)^{m-k}.$$

Then for $1\leqslant n\leqslant\eta$, it follows from (4.3.7) that

(5.2.18) $$a_n \ll 1.$$

We define

(5.2.19) $$T_1=\max\{T,2\pi j_1^2,\ 2\pi j_2^2\},\quad \tau_1=((T+U)/(2\pi))^{1/2}.$$

Then by Lemma 5.1.1 with $A_1=A_2=1$, (5.2.16), and (5.2.19) we have

$$\int_T^{T+U} \Big| \sum_{j\leqslant\eta} a_j j^{-a-it} \Big|^2 dt$$

$$= \sum\sum_{j_1, j_2 \leqslant \tau_1} a_{j_1} \overline{a_{j_2}} j_1^{-a} j_2^{-a} \int_{T_1}^{T+U} \exp(it \log(j_2/j_1))\, dt$$

$$= (T+U-T_1) \sum_{j\leqslant\tau_1} |a_j|^2 j^{-2a} + O\Big(\sum\sum_{\substack{j_1, j_1 \leqslant \tau_1 \\ j_1 \neq j_2}} j_1^{-a} j_2^{-a} / |\log(j_2/j_1)| \Big)$$

$$= O(UL).$$

For $T\leqslant t\leqslant T+U$, $k\leqslant m$ we have

(5.2.21) $$(\ell/L)^k - 1 = (1+L^{-1}\log(t/T))^k - 1$$

$$<< L^{-1}\log(t/T) \leqslant L^{-1}\log((T+U)/T) << L^{-1}UT^{-1} = L^{-11}.$$

Therefore, by (5.2.16), (5.2.17), (5.2.20), and (5.2.21),

(5.2.22) $$\int_T^{T+U} |\psi(a+it)|\,|f_1^*(a+it) - g_1(a+it)|\, dt = O(UL^{-10}).$$

Because of (5.2.8), the same argument shows that

(5.2.23) $$\int_T^{T+U} |\psi\chi(a+it)|\,\Big|f_2^*(a+it) - g_2(a+it)\Big|\, dt = O(UL^{-10}).$$

Hence, by (5.2.10), (5.2.14), (5.2.22),(5.2.23), and (5.1.3),

(5.2.24)

$$\int_T^{T+U} |\psi Q_3(a+it)|dt = \int_T^{T+U} |\psi Q_4(a+it)|dt + O(UL^{-10}).$$

Let

(5.2.25)
$$\chi_1(t) = (t/2\pi)^{1/2-a} \exp(\pi i/4 - it \log(t/(2\pi e))).$$

By Lemma 1.2.2, for $T \leqslant t \leqslant T+U$ we have

(5.2.26)
$$\chi(a+it) = \chi_1(t)(1+O(T^{-1}\log^2 L)) = \chi_1(t) + O(T^{-1}\log^2 L).$$

Now let

(5.2.27) $$Q_5(a+it) = g_1(a+it) + \chi_1(t) g_2(a+it).$$

By (5.2.13) we have

(5.2.28) $$g_2(a+it) \ll \eta^{1/2} \ll T^{1/2}.$$

Then by (5.2.14), (5.2.26), (5.2.27), and (5.2.28) we have

(5.2.29)

$$Q_4(a+it) - Q_5(a+it) = g_2(a+it)(\chi(a+it) - \chi_1(t)) \ll T^{-1/2}\log^2 L.$$

Thus by (5.1.4), Lemma 5.1.2, and (5.2.29) we have

(5.2.30) $$\int_T^{T+U} |\psi Q_4(a+it)|dt = \int_T^{T+U} |\psi Q_5(a+it)|dt + O(U^{3/4}L^4).$$

Now let

(5.2.31)
$$\chi^*(t)=(T/(2\pi))^{1/2-a}\exp[\pi i/4-it\log(t/(2\pi e))].$$

For $T\leqslant t\leqslant T+U$, it follows from the mean value theorem that

(5.2.32)
$$\left|\left(\frac{t}{2\pi}\right)^{1/2-a}-\left(\frac{T}{2\pi}\right)^{1/2-a}\right|\leqslant\frac{(t-T)}{2\pi}\max_{T\leqslant\xi\leqslant t}\left|(1/2-a)\left(\frac{\xi}{2\pi}\right)^{-1/2-a}\right|$$

$$<<UL^{-1}T^{-1}=L^{-11}.$$

It follows from (5.2.25), (5.2.31), and (5.2.32) that for $T\leqslant t\leqslant T+U$ we have

(5.2.33.) $$|\chi^*(t)-\chi_1(t)|=O(L^{-11}).$$

Let

(5.2.34) $$Q_6(a+it)=g_1(a+it)+\chi^*(t)g_2(a+it).$$

By (5.2.27), (5.2.33), and (5.2.34), for $T\leqslant t\leqslant T+U$ we have

(5.2.35) $$|Q_5(a+it)-Q_6(a+it)|<<L^{-11}|g_2(a+it)|.$$

For $0\leqslant x\leqslant 1$ let

(5.2.36) $$\phi^*(x)=\phi(x)(1+\pi i/(2L)-2x)^m$$

where ϕ is described in (4.1.3). Let

(5.2.37) $$c_n^*=\phi^*(L^{-1}\log n)\ ,\ d_n^*=\phi^*(1-L^{-1}\log n)$$
so that by (4.1.4) and (5.2.36),

(5.2.38)
$$c_n^*=c_n(1+\pi i/(2L)-2L^{-1}\log n)^m\ ,\ d_n^*=d_n(2L^{-1}\log n+\pi i/(2L)-1)^m.$$

For $n \ll T$, by (4.3.7) and (5.2.38) we have

(5.2.39) $$c_n^* \ll 1 \ , \ d_n^* \ll 1.$$

By (5.2.12) and (5.2.13) we have

(5.2.40) $$g_1(s) = \sum_{n \leqslant \eta} c_n^* n^{-s} \ , \ g_2(s) = \sum_{n \leqslant \eta} d_n^* n^{s-1}.$$

By Lemma 5.1.1, and Equations (5.2.39), (5.2.40), (just as in (5.2.20)) we have

(5.2.41) $$\int_T^{T+U} |g_2(a+it)|^2 dt = O(UL).$$

Therefore by (5.1.4), Lemma 5.1.2, (5.2.35), and (5.2.41) we have

(5.2.42) $$\int_T^{T+U} |\psi Q_5(a+it)| dt = \int_T^{T+U} |\psi Q_6(a+it)| dt + O(UL^{-10}).$$

In our final simplification, we define

(4.2.43) $$\tau = (T/(2\pi))^{1/2}$$

and

(5.2.44) $$g_1^*(s) = \sum_{n \leqslant \tau} c_n^* n^{-s} \ , \ g_2^*(s) = \sum_{n \leqslant \tau} d_n^* n^{s-1}.$$

Let

(5.2.45) $$Q^*(a+it) = g_1^*(a+it) + \chi^*(t) g_2^*(a+it).$$

For i=1,2 let

(5.2.46) $$G_i(t)=g_i(a+it)-g_i^*(a+it).$$

Then by (5.2.19), (5.2.40), (5.2.41), (5.2.39) and Lemma 5.1.1 we have

(5.2.47) $$\int_T^{T+U} |G_1(t)|^2 dt$$

$$=\sum\sum_{\tau<j_1,j_2\leqslant\tau_1} c_{j_1}^* \overline{c_{j_2}^*} j_1^{-a} j_2^{-a} \int_T^{T+U} \exp(it\log(j_1/j_2))dt$$

$$=O(U\sum_{\tau<j\leqslant\tau_1} j^{-1})+O(T^{1/2}L)$$

$$=O(U\log(\tau_1/\tau))+O(T^{1/2}L)=O(U\log(1+U/T))=O(UL^{-10}).$$

The same argument works to show that

(5.2.48) $$\int_T^{T+U} |G_2(t)|^2 dt=O(UL^{-10}).$$

By Lemma 5.1.2 and the fact that

(5.2.49) $$\chi^*(t)<<1,$$

it follows from (5.1.5), (5.2.34), (5.2.45), (5.2.46), (5.2.47), and (5.2.48) that

(5.2.50)

$$\int_T^{T+U} |\psi Q_6(a+it)|dt = \int_T^{T+U} |\psi Q^*(a+it)|dt + O(UL^{-9/2}).$$

Hence by (5.2.4), (5.2.11), (5.2.24), (5.2.30), (5.2.42), and (5.2.50) we have

(5.2.51)

$$\int_T^{T+U} |(2/L)^m \psi Q_1(a+it)|dt = \int_T^{T+U} |\psi Q^*(a+it)|dt + O(UL^{-9/2}).$$

By the Cauchy-Schwarz inequality we have

(5.2.52)

$$\int_T^{T+U} |\psi Q^*(a+it)|dt \leq U^{1/2} \left(\int_T^{T+U} |\psi Q^*(a+it)|^2 dt \right)^{1/2}.$$

Let

(5.2.53) $$J = \int_T^{T+U} |\psi Q^*(a+it)|^2 dt.$$

We assume for now, as we will later show, that

(5.2.54) $$J = O(U).$$

It follows from Lemma 4.4.1, (4.5.21), (5.2.51), (5.2.52), and (5.2.53) that

(5.2.55) $N_m(T+U)-N_m(T) \geqslant (UL/2\pi)(1-R^{-1}\log(J/U))+O(U)$.

In the next two chapters, we will evaluate J explicitly.

CHAPTER VI

THE TREATMENT OF THE INTEGRAL

6.1 Some lemmas. We state some lemmas which are of use in calculating J. The first is the special case $a=1/2$ of Lemmas 3.3 and 3.4 of Levinson [8].

Lemma 6.1.1 Let

$$I(r)=\int_T^{T+U} \exp(it\log(t/(re)))\,dt. \tag{6.1.1}$$

Then for $T \le r \le T+U$ we have

$$I(r)=(2\pi r)^{1/2}\exp(-ir+\pi i/4)+E(r)$$

while for $r<T$ or $r>T+U$ we have

$$I(r)=E(r),$$

where in any case it is true that

$$E(r)=O(1)+O(T/(|T-r|+\sqrt{T}))+O((T+U)/(|T+U-r|+\sqrt{T+U})).$$

Next we have summation by parts.

Lemma 6.1.2 Suppose $\{\gamma_n\}$ and $\{\gamma'_n\}$ are two sequences and

$$\sum_{i\leqslant n}\gamma'_i = C_n \quad .$$

Then

$$\sum_{n=M}^{N} \gamma_n \gamma_n' = \sum_{n=M}^{N} \gamma_n (C_{n+1}-C_n) + \sum_{n=M+1}^{N+1} \gamma_{n-1} C_n - \sum_{n=M}^{N} \gamma_n C_n$$

$$= \sum_{n=M}^{N} C_n(\gamma_{n-1}-\gamma_n) + \gamma_N C_{N+1} - \gamma_M C_{M+1}.$$

Third, we have a lemma about a geometric series. With

(6.1.2) $$e(x)=\exp(2\pi i x)$$

we have

<u>Lemma 6.1.3</u> If $1 \leqslant p<q$ then

$$\sum_{j \leqslant n} e(-jp/q) \ll q/p+q/(q-p)$$

uniformly in p, q, and n.

<u>Proof</u>. We have

$$\sum_{j \leqslant n} e(-jp/q)=e(-p/q)(e(-np/q)-1)/(e(-p/q)-1)$$

$$\leqslant 2/|e(-p/q)-1| = \csc(\pi p/q) \leqslant (2/\pi)\max\{q/p, 1/(1-p/q)\}$$

from which the Lemma follows. ∎

We quote Lemma 3.6 of Levinson [8].

<u>Lemma 6.1.4</u> Let $k=(k_1,k_2)=\gcd(k_1,k_2)$. Then

$$\sum\sum_{k_1,k_2 \leqslant y} k/(k_1 k_2) = O(\log^3 y) = O(L^3).$$

We have the Euler-McClaurin summation formula.

Lemma 6.1.5 If f has a continuous derivative then

$$\sum_{x_1 \leqslant n \leqslant x_2} f(n)$$

$$= \int_{x_1}^{x_2} f(x)\,dx + \int_{x_1}^{x_2} f'(x)([x]-x)\,dx + f(x_2)([x_2]-x_2) - f(x_1)([x_1^-]-x_1).$$

Proof. By Stieltjes integration we have

$$\sum_{x_1 \leqslant n \leqslant x_2} f(n) = \int_{x_1^-}^{x_2} f(x)\,d([x]-x) + \int_{x_1}^{x_2} f(x)\,dx.$$

The Lemma follows by integrating the first integral by parts.∎

Corollary 6.1.6 If f has a continuous derivative, then

$$\sum_{x_1 \leqslant n \leqslant x_2} f(n)$$

$$= \int_{x_1}^{x_2} f(x)\,dx + O\left(\int_{x_1}^{x_2} |f'(x)|\,dx\right) + O(|f(x_1)|) + O(|f(x_2)|).$$

Corollary 6.1.7 Suppose that

$$\int_1^{\infty} |f'(x)|\,dx < \infty.$$

Then there is a constant K such that

$$\sum_{1\leqslant n\leqslant x_2} f(n)=\int_1^{x_2} f(x)\,dx+K+O(|f(x_2)|)+O\left(\int_{x_2}^{\infty} |f'(x)|\,dx\right).$$

Proof. Take $x_1=1$ in the Lemma. Then

$$K=\int_1^{\infty} f'(x)([x]-x)\,dx+f(1).\blacksquare$$

Finally we have estimates for the derivatives of ϕ and ϕ^*.

Lemma 6.1.8 For $0\leq x\leq 1$ and $k\geq 1$ we have (uniformly)

$$|\phi^{(k)}(x)|\ll d^k\sqrt{k}.$$

Proof. By Cauchy's formula and (4.1.3) we have

$$|\phi^{(k)}(x)|=|k!/(2\pi)|\left|\int_{|w-x|=k/d}\phi(w)(w-x)^{-k-1}dw\right|$$

$$\ll k!(d/k)^k\exp(d(k/d+1))\ll k!(e/k)^k d^k,$$

from which the Lemma follows by (1.1.10).■

Corollary 6.1.9 If $0\leqslant x\leqslant 1$, then uniformly for all $k\geqslant 1$

$$|\phi^{*(k)}(x)|\ll d^k k^{m+1/2}$$

uniformly for all $k\geq 1$.

Proof. By (5.2.36) we have

$$\phi^{*(k)}(x)$$

$$= \sum_{j=0}^{\min\{m,k\}} \binom{k}{j} \phi^{(k-j)}(x)\, m(m-1)\ldots(m-j+1)(-2)^j \left(1+\frac{\pi i}{2L}-2x\right)^{m-j}$$

$$<< \sum_{j=0}^{\min\{m,k\}} \binom{k}{j} |\phi^{(k-j)}(x)|$$

$$<< \sum_{j=0}^{\min\{m,k\}} \binom{k}{j} d^{k-j}\sqrt{k-j}$$

$$<< k^{m+1/2} d^k. \blacksquare$$

Lemma 6.1.10 If w is any complex number, then

$$|\phi'(w)| << \exp(d|w|).$$

Proof. By Cauchy's theorem and (4.1.3) we have

$$2\pi|\phi'(w_0)| = \left| \int_{|w-w_0|=1} \phi(w)(w-w_0)^{-2} dw \right|$$

$$<< \exp(d(|w_0|+1)) << \exp(d|w_0|). \blacksquare$$

Corollary 6.1.11 For any w we have

$$|\phi^{*\prime}(w)| << \exp(d|w|)(1+|w|)^m.$$

Proof. By (5.2.36) we have

$$\phi^{*\prime}(w)=\phi'(w)(1+\pi i/(2L)-2w)^{m}-2\phi(w)m(1+\pi i/(2L)-2w)^{m-1}$$

$$\ll \exp(d|w|)(1+|w|)^{m}. \blacksquare$$

6.2 A first look at J. We square out (5.2.53) to see that

(6.2.1) $$J=J_1+J_2+2\text{Re } J_3$$

where

(6.2.2) $$J_1=\int_T^{T+U} |\psi g_1^*(a+it)|^2 dt , \quad J_2=\int_T^{T+U} |\chi^*(t)\psi g_2^*(a+it)|^2 dt,$$

$$J_3=\int_T^{T+U} \psi g_1^*(a+it)\overline{\psi g_2^*(a+it)}\ \chi^*(t)\overline{\chi^*(t)}\ dt.$$

Since J_1 and J_2 are real, we may write (6.2.1) as

(6.2.3) $$J=\text{Re } (J_1+J_2+2J_3).$$

We treat J_1 first. We replace ψ and g_1^* by the sums which define them (see (4.5.1) and (5.2.44)), use the fact that

$$|\psi g_1^*(a+it)|^2=\psi g_1^*(a+it)\overline{\psi g_1^*(a+it)} ,$$

and interchange summation and integration to see that

(6.2.4)

$$J_1 = \sum_{k_1,k_2 \leq y}\sum \frac{b_{k_1} b_{k_2}}{k_1^a k_2^a} \sum_{j_1,j_2 \leq \tau}\sum \frac{c^*_{j_1} c^*_{j_2}}{j_1^a j_2^a} \int_T^{T+U} \exp(it \log\frac{j_2 k_2}{j_1 k_1})\,dt.$$

Let

(6.2.5) $$k=(k_1,k_2)$$

and

(6.2.6) $$k_1=kA_1 \ , \ k_2=kA_2$$

It follows that

(6.2.7) $$(A_1,A_2)=1.$$

We write

(6.2.8) $$J_1=J_{11}+J'_{11}$$

where J_{11} consists of those terms for which

(6.2.9) $$j_1=jA_2 \ , \ j_2=jA_1$$

for some j. In J'_{11} we have those terms for which

(6.2.10) $$j_1A_1 \neq j_2A_2.$$

In J_{11} the integrand is 1, by (6.2.9). With

(6.2.11) $$A_M=\max\{A_1,A_2\}, \ A_m=\min\{A_1,A_2\}$$

we have

(6.2.12)
$$J_{11}=U\sum_{k_1,k_2\leqslant y}\sum \frac{b_{k_1} b_{k_2}}{k_1^{2a}\, k_2^{2a}}\, k^{2a} \sum_{j\leqslant \tau/A_M} c^*_{jA_1} c^*_{jA_2}\, j^{-2a}.$$

We apply Lemma 5.1.1 to J'_{11}. Then, by (4.5.2), (5.2.2), (5.2.34), (6.2.7), and (6.2.10) we have

(6.2.13)
$$J'_{11} << \sum_{k_1,k_2\leqslant y}\sum k_1^{-a}\, k_2^{-a} \sum_{\substack{j_1,j_2\leqslant\tau\\ j_1A_1\neq j_2A_2}}\sum j_1^{-a}\, j_2^{-a}/|\log(j_2A_2j_1^{-1}A_1^{-1})|$$

$$<<T^{1/2}L\left[\sum_{k_1\leqslant y} k_1^{-1/2}\right]^2 << T^{1/2}Ly=UL^{-9}.$$

Thus, by (6.2.8) and (6.2.13) we have

(6.2.14)
$$J_1=J_{11}+O(UL^{-9}).$$

It is clear from (5.2.31) that

(6.2.15)
$$|\chi^*(t)|^2=(T/(2\pi))^{1-2a}=\tau^{2-4a}<<1 .$$

The treatment of J_2 is really no different from that of J_1. We use (4.5.1), (5.2.44), and (6.2.15) to see that

(6.2.16)
$$J_2=\tau^{2-4a}\sum_{k_1,k_2\leqslant y}\sum \frac{b_{k_1} b_{k_2}}{k_1^{a}\, k_2^{a}} \sum_{j_1,j_2\leqslant\tau}\sum \frac{d^*_{j_1} d^*_{j_2}}{j_1^{1-a} j_2^{1-a}} \int_T^{T+U} \exp\left(it \log\frac{j_2k_2}{j_1k_1}\right)dt.$$

We use (6.2.5), (6.2.6), and (6.2.7) and let

(6.2.17)
$$J_2=J_{21}+J_{21}$$

where, as for J_1, the terms with (6.2.9) are in J_{21} and the terms with (6.2.10) are in J'_{21}. Then we find that

(6.2.18)
$$J_{21}=U\tau^{2-4a}\sum_{k_1,k_2\leqslant y}\frac{b_{k_1}b_{k_2}}{k_1k_2}k^{2-2a}\sum_{j\leqslant\tau/A_M}d^*_{jA_1}d^*_{jA_2}j^{2a-2}.$$

Just as for J'_{11}, we find by Lemma 5.1.1 that

(6.2.19)
$$J'_{21}=O(UL^{-9}).$$

Hence by (6.2.17) and (6.2.19) we have

(6.2.20)
$$J_2=J_{21}+O(UL^{-9}).$$

6.3 The integral J_3 The treatment of J_3 is more difficult than that of J_1 or J_2 yet considerably easier than Levinson's treatment because of the simplifications (5.2.31) and (5.2.44). We use (4.5.1), (5.2.31), (5.2.44), (6.1.1), and (6.2.2) and we interchange summation and integration to see that

(6.3.1)
$$J_3=\tau^{1-2a}e^{-\pi i/4}\sum_{k_1,k_2\leqslant y}\frac{b_{k_1}b_{k_2}}{k_1^a k_2^a}\sum_{j_1,j_2\leqslant\tau}\frac{c^*_{j_1}d^*_{j_2}}{j_1^{-a}j_2^{1-a}}I(r)$$

where

(6.3.2)
$$r=2\pi j_1j_2k_1/k_2.$$

Then by (6.3.1) and Lemma 6.1.1 we have

(6.3.3)
$$J_3=J_{31}+J'_{31}$$

where by (6.3.2) we have

(6.3.4)
$$J_{31}=$$
$$=2\pi\tau^{1-2a}\sum_{k_1,k_2\leqslant y}\sum \frac{b_{k_1} b_{k_2}}{k_1^{a-1/2}\, k_2^{a+1/2}} \sum_{\substack{j_1,j_2\leqslant\tau\\ T\leqslant r\leqslant T+U}}\sum \frac{c^*_{j_1} d^*_{j_2}}{j_1^{a-1/2}\, j_2^{1/2-a}} \exp(-ir)$$

and by (4.5.2) and (5.2.39) we have

(6.3.5) $$J'_{31} \ll \sum_{k_1,k_2\leqslant y}\sum k_1^{-a}\, k_2^{-a} \sum_{j_1,j_2\leqslant\tau}\sum j_1^{-a}\, j_2^{a-1}\, E(r).$$

We estimate J'_{31}. Recall in Lemma 6.1.1 that $E(r)$ is a sum of three terms. We estimate the contribution of each of these terms to J'_{31}. By (6.3.5) and (5.2.2) the contribution of the $O(1)$ part of $E(r)$ to J'_{31} is

(6.3.6)
$$\ll \left[\sum_{k_1\leqslant y} k_1^{-1/2}\right]^2 \left[\sum_{j_1\leqslant\tau} j_1^{-1/2}\right]^2 \ll y\tau \ll TL^{-20} = UL^{-10}.$$

Also, if

(6.3.7) $$r<3T/4 \text{ or } r>5T/4$$

then by Lemma 6.1.1 we have

$$E(r)=O(1).$$

The case (6.3.7) is taken care of by the estimate (6.3.6).

If

(6.3.8) $$3T/4\leqslant r\leqslant 5T/4$$

then by (6.3.2) we have

(6.3.9) $k_1^{-a}\ k_2^{-a}\ j_1^{-a}\ j_2^{a-1} = (j_1 j_2 k_1/k_2)^{-a}\ k_2^{-2a}\ j_2^{2a-1}$

$$= (r/2\pi))^{-a}\ k_2^{-2a}\ j_2^{2a-1} << T^{-1/2}\ k_2^{-1}.$$

Thus by (6.3.5) and (6.3.9) the contribution to J'_{31} of the

$$O(T/(|T-r|+\sqrt{T}))$$

part of E(r) is

(6.3.10)

$$<<T^{-1/2} \sum_{k_1,k_2\leq y}\sum k_2^{-1} \sum_{j_1,j_2\leq T}\sum \frac{Tk_2/(2\pi j_2 k_1)}{\left|\frac{Tk_2}{2\pi j_2 k_1} - j\right| + \frac{T^{1/2}\ k_2}{2\pi j_2 k_1}}\ .$$

We estimate the inner sum on j_1 of (6.3.10). If the term in absolute values in the denominator of (6.3.10) is <1/2 then the quotient of the innermost summand of (6.3.10) is

(6.3.11) $$<<T^{1/2}.$$

There are O(1) such terms. For the rest of the values of j_1 we ignore the term $T^{1/2}\ k_2/(2\pi j_2 k_1)$ of the denominator of (6.3.10) so that the sum on j_1 in (6.3.10) is

(6.3.12) $$<<T^{1/2} + Tk_2 j_2^{-1}\ k_1^{-1} \sum_{j_1\leq T} j_1^{-1}$$

$$<<T^{1/2} + Tk_2 j_2^{-1}\ k_1^{-1}\ L.$$

Now we sum on j_2, k_1, and k_2 and use (6.3.12) to find that the expression in (6.3.10) is

(6.3.13)

$$<< \sum_{k_1,k_2\leqslant y}\sum k_2^{-1} \sum_{j_2\leqslant\tau} 1+T^{1/2} L \sum_{k_1,k_2\leqslant y}\sum k_1^{-1} \sum_{j_2\leqslant\tau} j_2^{-1}$$

$$<< y\tau L+T^{1/2} Ly(\log y)\log\tau =O(UL^{-7}).$$

In just the same way we find that the contribution to J'_{31} of the

$$O((T+U)/(|T+U-r|+\sqrt{T+U}))$$

part of E(r) is

(6.3.14) $$<<UL^{-7}.$$

By (6.3.6), (6.3.13), and (6.3.14) we have

(6.3.15) $$J'_{31}=O(UL^{-7}).$$

Hence by (6.3.3) and (6.3.15) we have

(6.3.16) $$J_3=J_{31}+O(UL^{-7}).$$

Now we use (6.2.5) and (6.2.6) as with J_1 and J_2 to obtain the main part of J_{31}. We write

(6.3.17) $$J_{31}=J_{32}+J'_{32}$$

where J_{32} is that part of (6.3.4) for which

(6.3.19) $A_1 j_2 \equiv j \bmod A_2 \quad , \quad 1 \leqslant j \leqslant A_2 - 1.$

We treat J'_{32} first. We consider the inner sum on j_1: let

(6.3.20) $$J''_{32} = \sum_{j_1} c^*_{j_1} j_1^{1/2-a} e(-j_1 j/A_2)$$

where we have used (6.1.2). By (6.3.2) and (6.3.4), the conditions of summation on j_1 are given by

(6.3.21) $Tk_2/(2\pi j_2 k_1) \leqslant j_1 \leqslant (T+U)/(2\pi j_2 k_1) \quad , \quad 1 \leqslant j_1 \leqslant \tau.$

We see that the sum on j_1 is for all integers in a certain interval. We apply Lemma 6.1.2 with

(6.3.22) $$\gamma_{j_1} = c^*_{j_1} j_1^{1/2-a} \quad , \quad \gamma'_{j_1} = e(-j_1 j/A_2).$$

Then by (6.3.19), (6.3.22), and Lemma 6.1.3 we have

(6.3.23) $$C_n = \sum_{j_1 \leqslant n} e(-j_1 j/A_2) << A_2/j + A_2/(A_2 - j).$$

To estimate $\gamma_{n-1} - \gamma_n$, first observe that for $n \leqslant \tau$, by the mean value theorem and Lemma 6.1.8 we have

(6.3.24) $$|D_{n,k}|$$

$$= \left| \phi\left(\frac{\log(n-1)}{L}\right) \log^k(n-1)(n-1)^{1/2-a} - \phi\left(\frac{\log n}{L}\right)(\log^k n) n^{1/2-a} \right|$$

$$\leqslant \max_{n-1 \leqslant \xi \leqslant n} \left| \frac{\log^k \xi}{\xi^{a+1/2}} \left(\phi(L^{-1}\log\phi)\left(\frac{k}{\log\xi} + 1/2 - a\right) + \frac{\phi'(L^{-1}\log\xi)}{L} \right) \right|$$

$$<< L^{k-1}/n.$$

Then by (4.1.4), (5.2.38), and (6.3.22) with the definition of $D_{n,k}$ implicit in (6.3.24) we have

(6.3.25) $$\gamma_{n-1} - \gamma_n =$$

$$=\phi(L^{-1}\log(n-1))(1+\pi i/(2L)-2L^{-1}\log(n-1))^m(n-1)^{1/2-a} -$$

$$\phi(L^{-1}\log n)(1+\pi i/(2L)-2L^{-1}\log n)^m n^{1/2-a}$$

$$= \sum_{k=0}^{m} \binom{m}{k}(1+\pi i/(2L))^{m-k}(-1)^k D_{n,k}$$

$$<<1/(nL).$$

Thus, by Lemma 6.1.2, (6.3.20), (6.3.22), (6.3.23), and (6.3.25) we have

(6.3.26) $$J''_{32} << L^{-1}\sum_{j_1} (A_2/j+A_2/(A_2-j))j_1^{-1}$$

$$<<A_2/j+A_2/(A_2-j).$$

We sum next on j_2 in J'_{32} where we have the restriction

$$A_2 \nmid j_2$$

as described in (6.3.19). We observe that j_2 runs through at most τ/A_2 sets of successive integers, each set containing A_2-1 successive integers, and for each set j runs from 1 to A_2-1. Thus, by (6.3.4), (6.3.20), and (6.3.26) we have

(6.3.27) $$\sum_{j_2} |d^*_{j_2}| j_2^{a-1/2} (A_2/j+A_2/(A_2-j))$$

$$<< (\tau/A_2) \sum_{j=1}^{A_2-1} (A_2/j+A_2/(A_2-j))$$

$$<<\tau \log A_2 << T^{1/2} L.$$

Finally, we sum on k_1 and k_2. Then by (6.3.4), (6.3.19), (6.3.20), (6.3.26), and (6.3.27) we have

(6.3.28)
$$J'_{32} << \sum\sum_{k_1,k_2\leqslant y} k_1^{1/2-a} k_2^{-1/2-a} T^{1/2} L << yT^{1/2} L^2 = UL^{-8}.$$

Hence by (6.3.16), (6.3.17), and (6.3.28) we have

(6.3.29) $$J_3 = J_{32} + O(UL^{-7}).$$

In J_{32} let

(6.3.30) $$j_2 = jA_2.$$

Then by (6.3.2) and (6.3.30), $\exp(-ir)=1$, so that

(6.3.31)
$$J_{32} = 2\pi\tau^{1-2a} \sum\sum_{k_1,k_2\leqslant y} b_{k_1} b_{k_2} k_1^{1/2-a} k_2^{-1/2-a} A_2^{a-1/2} \Sigma_1$$

where

(6.3.32) $$\Sigma_1 = \sum\sum_{j_1,j} c^*_{j_1} d^*_{jA_2} j_1^{1/2-a} j^{a-1/2}.$$

The conditions of summation on j_1 and j are

$$T\leqslant r \leqslant T+U \ , \ 1\leqslant j_1, \ jA_2 \leqslant \tau \ ,$$

which is, by (6.3.2), the same as

(6.3.33) $$\text{(i)}\ \frac{\tau^2}{A_1} \leqslant j_1 j \leqslant \frac{\tau_1^2}{A_1}\ ,\quad \text{(ii)}\ 1 \leqslant j \leqslant \frac{\tau}{A_2}\ ,\quad \text{(iii)}\ 1 \leqslant j_1 \leqslant \tau .$$

Conditions (i) and (iii) imply that

(6.3.34) $$j \geqslant \tau/A_1 .$$

Conditions (i) and (ii) imply that

(6.3.35) $$j_1 \geqslant \tau A_2/A_1 .$$

By (6.3.34) and (6.3.35) we may replace Conditions (6.3.33) by

(6.3.36) $$\text{(i)}\ \frac{\tau^2}{A_1} \leqslant j_1 j \leqslant \frac{\tau_1^2}{A_1}\ ,\quad \text{(iv)}\ \frac{\tau}{A_1} \leqslant j \leqslant \frac{\tau}{A_2}\ ,\quad \text{(v)}\ \frac{\tau A_2}{A_1} \leqslant j_1 \leqslant \tau .$$

Note that the region defined in (6.3.36) is empty if $k_2 > k_1$ (see (6.2.6)). Thus in (6.3.31) the double sum on k_1, k_2 may be written as

(6.3.37) $$1 \leqslant k_2 \leqslant k_1 \leqslant y .$$

We will show that Σ_2 is a good approximation for Σ_1, where

(6.3.38) $$\Sigma_2 = \sum_j d^*_{jA_2}\, j^{a-1/2} \sum_{j_1} c^*_{j_1}\, j_1^{1/2-a} .$$

The conditions of summation on j and j_1 are (iv) and (i) of (6.3.36). To show that $\Sigma_1 - \Sigma_2$ is small we consider two cases.

First, suppose

(6.3.39) $$A_1/A_2<1+U/T.$$

Then (iv) and (6.3.39) imply that

(6.3.40) $$\tau_1^2/(A_1 j)>\tau\ .$$

By (6.3.32), (6.3.36), (6.3.38), (6.3.39), (6.3.40), and since the lower bound for j_1 of (i) is never smaller than the lower bound of (v) we have

(6.3.41) $$\Sigma_1-\Sigma_2=-\sum_{\frac{\tau}{A_1}\leqslant j\leqslant\frac{\tau}{A_2}} d^*_{jA_2}\, j^{a-1/2}\sum_{\tau<j_1\leqslant\frac{\tau_1^2}{A_1 j}} c_{j_1}\, j_1^{1/2-a}$$

$$<<\left(\frac{\tau}{A_2}-\frac{\tau}{A_1}\right)(\tau_1^2/\tau-\tau)=(1/A_2-1/A_1)(\tau_1^2-\tau^2)$$

$$=\left((A_1/A_2-1)/A_1\right)U/(2\pi)<<UL^{-10}/A_1.$$

In the second case we have

$$A_1/A_2\geqslant 1+U/T.$$

Then, since

$$\tau\leqslant\tau_1^2/(A_1 j)\ \text{ for } j\leqslant\tau_1^2/(A_1\tau)$$

and

$$\tau_1^2/(A_1 j)\leqslant\tau\ \text{ for } j\geqslant\tau_1^2/(A_1\tau),$$

we have

(6.3.42)

$$\Sigma_1-\Sigma_2=\sum_{\frac{\tau}{A_1}\leqslant j\leqslant\frac{\tau_1^2}{A_1\tau}} d^*_{jA_2}\, j^{a-1/2} \sum_{\tau<j_1\leqslant\frac{\tau_1^2}{A_1 j}} c^*_{j_1} j_1^{1/2-a}$$

$$<< (\tau_1^2/(A_1\tau)-\tau/A_1)(\tau_1^2/\tau-\tau)=(\tau_1^2-\tau^2)^2/(\tau^2A_1)$$

$$<< U^2/(TA_1)=UL^{-10}/A_1.$$

Thus, in either case (6.3.41) or (6.3.42) we have

(6.3.43) $$\Sigma_1-\Sigma_2=O(UL^{-10}A_1^{-1}).$$

Let

(6.3.44) $$J_{33}=2\pi\tau^{1-2a}\sum\sum_{1\leqslant k_2\leqslant k_1\leqslant y} b_{k_1}b_{k_2}k_1^{1/2-a}\, k_2^{-1}\, k^{1/2-a}\,\Sigma_2.$$

Then by (6.3.31), (6.3.43), (6.3.44), and Lemma 6.1.4,

(6.3.45) $$J_{32}-J_{33}<<\sum\sum_{k_1,\ k_2\leqslant y} k_2^{-1}\, A_1^{-1}\, UL^{-10}$$

$$= UL^{-10}\sum\sum_{k_1,k_2\leqslant y} k/(k_1k_2)=O(UL^{-7}).$$

By (6.3.29) and (6.3.45) we have

(6.3.46) $$J_3=J_{33}+O(UL^{-7}).$$

Let

(6.3.47) $$\Sigma_3=\sum_{\frac{\tau^2}{A_1 j}\leqslant j_1\leqslant\frac{\tau_1^2}{A_1 j}} c^*_{j_1} j_1^{1/2-a}\ .$$

Then by (6.3.38) we have

(6.3.48)
$$\Sigma_2 = \sum_{\frac{\tau}{A_1} \leqslant j \leqslant \frac{\tau}{A_2}} d^*_{jA_2} \, j^{a-1/2} \, \Sigma_3 .$$

We would like to replace the sum Σ_3 by the length of the interval of summation times the value of the summand at the lower limit. For convenience, let

(6.3.49)
$$X_0 = \tau^2/(A_1 j) \ , \ X_1 = \tau_1^2/(A_1 j).$$

Observe that by the mean value theorem and Lemma 6.1.8 (just as in (6.3.24)), for $X_0 \leqslant j \leqslant X_1$ we have

(6.3.50)
$$|D'_{j_1,k}| =$$

$$= |\phi(L^{-1}\log j_1)(\log^k j_1) j_1^{1/2-a} - \phi(L^{-1}\log X_0)(\log^k X_0) X_0^{1/2-a}$$

$$<< (j_1 - X_0) L^{k-1}/X_0$$

where $D'_{j_1,k}$ has been defined implicitly. Therefore, for $X_0 \leqslant j_1 \leqslant X_1$, by (6.3.50) and (5.2.36) we have

(6.3.51)
$$|c^*_{j_1} j_1^{1/2-a} - \phi^*(L^{-1}\log X_0) X_0^{1/2-a}|$$

$$= |\sum_{k=0}^{m} \binom{m}{k} (1+\pi i/(2L))^{m-k} (-1)^k (2/L)^k D'_{j_1 k}|$$

$$<< (j_1 - X_0)/(LX_0).$$

Thus, by (6.3.47) we have

(6.3.52)

$$\Sigma_3=\phi^*(L^{-1}\log X_0)X_0^{1/2-a}((\tau_1^2-\tau^2)/(A_1 j)+O(1))+O(\Sigma_4)$$

where by (6.3.49) we have

(6.3.53)

$$\Sigma_4=\sum_{X_0\leqslant j_1\leqslant X_1}(j_1-X_0)/(LX_0)<<\frac{1}{LX_0}\sum_{j_1\leqslant X_1-X_0} j_1<<\frac{1}{LX_0}(X_1-X_0)^2$$

$$=(U/(2\pi A_1 j))^2/(L\tau^2/(A_1 j))<<UL^{-11}/(A_1 j).$$

By (6.3.49), (6.3.52), and (6.3.53) we have

(6.3.54)

$$\Sigma_3=\frac{U}{2\pi A_1 j}\phi^*\left(L^{-1}\log\frac{\tau^2}{A_1 j}\right)\left[\frac{\tau^2}{A_1 j}\right]^{1/2-a}+O(1)+O\left[\frac{UL^{-11}}{A_1 j}\right]$$

$$=\frac{\tau^{1-2a}U}{2\pi}(A_1 j)^{a-3/2}\phi^*(1-L^{-1}\log(A_1 j))+O(1)+O\left[\frac{UL^{-11}}{A_1 j}\right]$$

$$=\frac{\tau^{1-2a}U}{2\pi}(A_1 j)^{a-3/2}d^*_{jA_1}+O(1)+O\left[\frac{UL^{-11}}{A_1 j}\right].$$

By (6.3.48) the contribution of the O-terms of (6.3.54) to Σ_2 is

(6.3.55) $$<<\tau(A_2^{-1}-A_1^{-1})+UL^{-11}A_1^{-1}\sum_j j^{-1}$$

$$<<T^{1/2}+UL^{-10}/A_1.$$

By (6.3.48), (6.3.45), and (6.3.55) we have

(6.3.56)

$$\Sigma_2 = \tau^{1-2a} \frac{U}{2\pi} A_1^{a-3/2} \sum_{\frac{\tau}{A_1} \leqslant j \leqslant \frac{\tau}{A_2}} d^*_{jA_1} d^*_{jA_2} j^{2a-2} +$$

$$+ O(T^{1/2}) + O(UL^{-10}/A_1).$$

By Lemma 6.1.4 the contribution of the O-terms of (6.3.56) to (6.3.44) is

(6.3.57)

$$<< \sum_{k_1, k_2 \leqslant y} \sum k_2^{-1}(T^{1/2} + UL^{-10}/A_1) << T^{1/2} yL + UL^{-10} \sum_{k_1, k_2 \leqslant y} \sum k/(k_1 k_2)$$

$$= O(UL^{-7}).$$

Therefore, by (6.3.44), (6.3.46), (6.3.56), and (6.3.57),

(6.3.58)

$$J_3 = \tau^{2-4a} U \sum_{1 \leqslant k_2 \leqslant k_1 \leqslant y} \sum \frac{b_{k_1} b_{k_2}}{k_1 k_2} k^{2-2a} \sum_{\frac{\tau}{A_1} \leqslant j \leqslant \frac{\tau}{A_2}} d^*_{jA_1} d^*_{jA_2} j^{2a-2}$$

$$+ O(UL^{-7}).$$

6.4 Replacing sums by integrals. We will use the Euler-Maclaurin formula to approximate to the inner sums of J_1, J_2, and J_3. But first we have a simplification.

The summand of (6.3.58) is symmetric in k_1 and k_2

We can replace the conditions of summation on k_1 and k_2 by the conditions

$$1\leqslant k_1\leqslant k_2\leqslant y$$

and have another expression for J_3, provided we interchange A_1 and A_2. Thus, by (6.3.58) and these remarks we see that

(6.4.1)
$$2J_3=\tau^{2-4a}\, U\sum\sum_{k_1,k_2\leqslant y} \frac{b_{k_1}b_{k_2}}{k_1k_2}\, k^{2-2a} \sum_{\frac{\tau}{A_M}\leqslant j\leqslant\frac{\tau}{A_m}} d^*_{jA_1}\, d^*_{jA_2}\, j^{2a-2} + $$

$$+\ O(UL^{-7}).$$

Now J_2 and $2J_3$ combine nicely. We have by (6.2.1), (6.2.12), (6.2.14), (6.2.17), (6.2.18), and (6.4.1) that

(6.4.2)
$$J=\text{Re}\ (J_4+J_5)+O(UL^{-7})$$

where

(6.4.3)
$$J_4=U\sum\sum_{k_1,k_2\leqslant y} \frac{b_{k_1}b_{k_2}}{k_1^{2a}\ k_2^{2a}}\, k^{2a} \sum_{j\leqslant\tau/A_M} c^*_{jA_1}\, c^*_{jA_2}\, j^{-2a}$$

and

(6.4.4)
$$J_5=\tau^{2-4a}\ U\sum\sum_{k_1,k_2\leqslant y} \frac{b_{k_1}b_{k_2}}{k_1k_2}\, k^{2-2a} \sum_{j\leqslant\tau/A_m} d^*_{jA_1}\, d^*_{jA_2}\, j^{2a-2}.$$

We apply Corollary 6.1.7 with

(6.4.5) $$f(x)=\phi^*(L^{-1}\log(A_1 x))\phi^*(L^{-1}\log(A_2 x))x^{-2a}.$$

By (6.4.5) and Corollary 6.1.11 we have for large x the estimate

(6.4.6) $$|f'(x)| \ll x^{2/L-1-2a}(\log x)^{m_L - m}.$$

Also, by (6.4.5) we have

(6.4.7) $$f(\tau/A_M) \ll A_M/\tau.$$

Therefore, by Corollary 6.1.7, (6.4.5),(6.4.6), and (6.4.7),

(6.4.8) $$\sum_{j\leqslant\tau/A_M} c^*_{jA_1} c^*_{jA_2} j^{-2a}$$

$$= \int_1^{\tau/A_M} c^*(vA_1)c^*(vA_2)v^{-2a}dv+K_4+O(A_M/\tau)$$

where

(6.4.9) $$c^*(x)=\phi^*(L^{-1}\log x)$$

and K_4 is a constant.

In a similar way, we can use Corollary 6.1.7 to show that

(6.4.10) $$\sum_{j\leqslant\tau/A_m} d^*_{jA_1} d^*_{jA_2} j^{2a-2}$$

$$= \int_1^{\tau/A_m} d^*(vA_1)d^*(vA_2)v^{2a-2}\,dv+K_5+O(A_m/\tau)$$

where

$$d^*(x)=\phi^*(1-L^{-1}\log x). \tag{6.4.11}$$

When the $O(A_M/\tau)$ term of (6.4.8) is accounted for in (6.4.3) the resulting contribution to J_4 is

$$\ll U\tau^{-1}\sum_{k_1,k_2\leq y}\sum k_m^{-1}\ll U\tau^{-1}yL\ll UL^{-19}. \tag{6.4.12}$$

Here we use the notation

$$k_m=\min\{k_1,k_2\} \text{ and } k_M=\max\{k_1,k_2\}. \tag{6.4.13}$$

In a similar way, the $O(A_m/\tau)$ term of (6.4.10) contributes

$$\ll UL^{-19}$$

to J_5.

Therefore, by (6.4.3), (6.4.4), (6.4.8), (6.4.10), and (6.4.12) we have

(6.4.14)

$$J_4=U\sum_{k_1,k_2\leq y}\sum \frac{b_{k_1}b_{k_2}}{k_1^{2a}k_2^{2a}}k^{2a}\left[\int_1^{\tau/A_M} c^*(vA_1)c^*(vA_2)\frac{dv}{v^{2a}}+K_4\right]+O(UL^{-7}),$$

$$J_5=\tau^{2-4a}U\sum_{k_1,k_2\leq y}\sum \frac{b_{k_1}b_{k_2}}{k_1k_2}k^{2-2a}\left[\int_1^{\tau/A_m} d^*(vA_1)d^*(vA_2)\frac{dv}{v^{2-2a}}+K_5\right]$$

$$+\ O(UL^{-7}).$$

6.5 Replacing integrals by sums. We now integrate the integrals of (6.4.14) by parts. It will be convenient to let

$$\phi_*(x) = \phi^*(x) \tag{6.5.1}$$

so that the superscripts indicating derivatives will be more prominent.

Let

$$I_4 = \int_1^{\tau/A_M} \phi_*(L^{-1}\log(vA_1))\,\phi_*(L^{-1}\log(vA_2))\,v^{-2a}\,dv \tag{6.5.2}$$

and

$$I_5 = \int_1^{\tau/A_m} \phi_*(1-L^{-1}\log(vA_1))\,\phi_*(1-L^{-1}\log(vA_2))\,v^{2a-2}\,dv.$$

Note that $\phi_*(w)$ is an entire function (see (4.1.3) and (5.2.36)). We integrate (6.5.2) by parts N times to see that

$$I_4 = \frac{v^{1-2a}}{1-2a} \cdot \tag{6.5.4}$$

$$\cdot \sum_{n=0}^{N} \frac{(-1/L)^n}{(1-2a)^n} \sum_{j=0}^{n} \binom{n}{j} \phi_*^{(j)}\left(\frac{\log(vA_1)}{L}\right)\phi_*^{(n-j)}\left(\frac{\log(vA_2)}{L}\right)\Bigg|_{v=1}^{v=\frac{\tau}{A_M}} +$$

$$+ R_{N+1}$$

where

$$R_N = \frac{(-1)^N}{L^N(1-2a)^N} \int_1^{\tau/A_M} \sum_{j=0}^{N} \binom{N}{j} \phi_*^{(j)}\left(\frac{\log (vA_1)}{L}\right) \phi_*^{(N-j)}\left(\frac{\log(vA_2)}{L}\right) \frac{dv}{v^{2a}} .$$

By Lemma 6.1.9 we have

(6.5.6) $$\sum_{j=0}^{N} \binom{N}{j} \phi_*^{(j)} \left(L^{-1}\log(vA_1)\right) \phi_*^{(N-j)} \left(L^{-1}\log(vA_2)\right)$$

$$<<d^N N^{2m+1} \sum_{j=0}^{N} \binom{N}{j} = (2d)^N N^{2m+1}$$

for $1 \le v \le \tau$. Recall that

(6.5.7) $$(1-2a)L=2R .$$

Thus by (6.5.6) and (4.1.3) we have

(6.5.8) $$R_N \to 0 \text{ as } N\to\infty .$$

Moreover, if we let $N\to\infty$ then the infinite sum we get for I_4 is absolutely convergent by (4.1.3) and (6.5.6). Thus, since $\log\tau = L/2$, we have

(6.5.9) $$I_4 =$$

$$= \frac{\tau^{1-2a}}{1-2a} A_M^{2a-1} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \sum_{j=0}^{n} \binom{n}{j} \phi_*^{(j)}(1/2)\, \phi_*^{(n-j)} \left(\frac{1}{L} \log \frac{\tau A_m}{A_M}\right) -$$

$$\frac{1}{1-2a} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \sum_{j=0}^{n} \binom{n}{j} \phi_*^{(j)} \left(L^{-1}\log\frac{k_1}{k}\right) \phi_*^{(n-j)} \left(L^{-1}\log\frac{k_2}{k}\right).$$

The same treatment works for I_5. We integrate by parts to see that

(6.5.10)

$$I_5 = \frac{-\tau^{2a-1}}{(1-2a)} A_m^{1-2a} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \sum_{j=0}^{n} \binom{n}{j} \phi_*^{(j)} \left(\frac{1}{2}\right) \phi_*^{(n-j)} \left(1-\frac{1}{L} \log \frac{\tau A_M}{A_m}\right) +$$

$$+\frac{1}{1-2a} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \sum_{j=0}^{n} \binom{n}{j} \phi_*^{(j)} (1-L^{-1}\log\frac{k_1}{k}) \phi_*^{(n-j)} (1-L^{-1}\log\frac{k_2}{k}).$$

It is clear that

(6.5.11) $$k_1^{-2a} k_2^{-2a} k^{2a} \tau^{1-2a} A_M^{2a-1} = \tau^{1-2a} k_m^{-2a} k_M^{-1} k,$$

and

(6.5.12)

$$\tau^{2-4a} k_1^{-1} k_2^{-1} k^{2-2a} \tau^{2a-1} A_m^{1-2a} = \tau^{1-2a} k_m^{-2a} k_M^{-1} k.$$

Also we have

(6.5.13) $$1-L^{-1}\log(\tau A_m / A_M) = L^{-1}\log(\tau A_m / A_M).$$

Hence, by (6.4.14), (6.5.2), (6.5.3), and (6.5.9) through (6.5.13) the terms in J_4 and J_5 which involve k_m and k_M as introduced in I_4 and I_5 are negatives of each other, and so cancel when brought together in J as in (6.4.2).

Therefore, we have

(6.5.14) $$J = \mathrm{Re}\ (J_7 - J_6) + O(UL^{-7})$$

where

(6.5.15)

$$J_7 = \frac{Ue^{2R}}{1-2a} \sum_{k_1,k_2 \leqslant y}\sum \frac{b_{k_1} b_{k_2}}{k_1 k_2} k^{2-2a}\Big((1-2a)K_5 +$$

$$+ \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \sum_{j=0}^{n} \binom{n}{j} \phi_*^{(j)} \left(1-\frac{1}{L}\log\frac{k_1}{k}\right) \phi_*^{(n-j)} \left(1-\frac{1}{L}\log\frac{k_2}{k}\right)\Big)$$

and

$$J_6 = \frac{U}{1-2a} \sum_{k_1,k_2 \leqslant y}\sum \frac{b_{k_1} b_{k_2}}{k_1^{2a}\; k_2^{2a}} k^{2a} \, .$$

$$\cdot\Big((1-2a)K_4 + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \sum_{j=0}^{n} \binom{n}{j} \phi_*^{(j)} \left(\frac{1}{L}\log\frac{k_1}{k}\right) \phi_*^{(n-j)} \left(\frac{1}{L}\log\frac{k_2}{k}\right)\Big) .$$

In the above we have used the fact that

$$\tau^{2-4a} = e^{2R} .$$

In the next chapter, we specify the coefficients b_j of the mollifier and we evaluate J_6 and J_7.

CHAPTER VII

THE MOLLIFIER

7.1 Further lemmas. The following lemmas will be useful in evaluating the sums J_7 and J_6.

Lemma 7.1.1 If f is an arithmetical function, then

$$f(k) = \sum_{j|k} \Big(\sum_{d|j} \mu(d) f(j/d) \Big) .$$

Proof. Here μ is the Möbius function. It is multiplicative, -1 on primes, and 0 on prime powers higher than the first. Also (see Apostol [1, Theorem 2.1])

$$\sum_{d|n} \mu(d) = \begin{cases} 1 \text{ if } n=1 \\ 0 \text{ if } n>1 \end{cases} .$$

Thus, by (7.1.1) we have

$$f(k) = \sum_{n|k} f(n) \Big(\sum_{d|\frac{k}{n}} \mu(d) \Big) = \sum_{dn|k} \mu(d) f(n)$$

from which the Lemma follows when we let $dn=j$. ■

Lemma 7.1.2 Suppose that

$$A(w) = \sum_{n=1}^{\infty} a(n) n^{-w}$$

is absolutely convergent for $u>1$. Then if $c>1+\alpha$ we have for $\ell \geq 1$ that

$$(7.1.2) \qquad \sum_{n \leq x} \frac{a(n)}{n^{\alpha}} \left(\log \frac{x}{n}\right)^{\ell} = \frac{\ell!}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{A(w)\, x^{w-\alpha}}{(w-\alpha)^{\ell+1}}\, dw.$$

Proof. The right side of (7.1.2) is

$$(7.1.3) \qquad \ell! \sum_{n=1}^{\infty} \frac{a(n)}{n^{\alpha}} I(x/n)$$

where

$$I(y) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{y^{w-\alpha}}{(w-\alpha)^{\ell+1}}\, dw;$$

the interchange of summation and integration is justified by absolute convergence. By Cauchy's theorem

$$(7.1.4) \qquad I(y) = \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} y^{w}\, w^{-\ell-1}\, dw$$

for any $c_1 > 1$. The integrand of (7.1.4) has a pole of order $(\ell+1)$ at $w=0$ with residue

$$(\log y)^{\ell} / \ell! \; .$$

Suppose $y \geq 1$. Then by Cauchy's theorem and (7.1.4) we have

(7.1.5)

$$I(y)=\lim_{\substack{X\to\infty\\ Y\to\infty}}\frac{1}{2\pi i}\int_{c_1-iX}^{c_1+iY} y^w w^{-\ell-1}\,dw$$

$$=(\log y)^{\ell}/\ell!+\lim_{\substack{X\to\infty\\ Y\to\infty}}\frac{1}{2\pi i}\left(\int_{c_1-iX}^{-\infty-iX} y^w\, w^{-\ell-1}\,dw+\int_{-\infty+iY}^{c_1+iY} y^w\, w^{-\ell-1}\,dw\right)$$

$$=(\log y)^{\ell}/\ell!+\lim_{\substack{X\to\infty\\ Y\to\infty}} O\left((X^{-\ell-1}+Y^{-\ell-1})\int_{-\infty}^{c_1} y^u\,du\right)$$

$$=(\log y)^{\ell}/\ell!.$$

Suppose $0<y<1$. Then alter the path to $u-iX$, $c_1\leqslant u\leqslant\infty$ and $u+iY$, $\infty\geqslant u\geqslant c_1$ to see that in this case

(7.1.6) $$I(y)=0.$$

The Lemma follows from (7.1.3), (7.1.5), and (7.1.6).∎

<u>Lemma 7.1.3</u> For $j\leqslant y$ and $|u-1|\ll 1/\log L$ we have

$$\sum_{p|j} p^{-u}\log p\ll\log L.$$

<u>Proof</u>. Let $q_1, q_2, q_3, \ldots = 2, 3, 5, \ldots$ be the sequence of primes. Suppose that

$$q_1q_2\ldots q_r\leqslant j<q_1q_2\ldots q_{r+1}.$$

Then by (2.1.10) we have

(7.1.7) $$\sum_{p|j} p^{-u}\log p \ll \sum_{p\leqslant q_r} p^{-u}\log p = \int_1^{q_r} t^{-u}\, d\theta(t)$$

$$=\theta(q_r)q_r^{-u} - u\int_1^{q_r} t^{-u-1}\,\theta(t)\, dt.$$

By (2.1.11) we have

(7.1.8) $$q_r \ll \theta(q_r) = \log(q_1 q_2 \cdots q_r) \leqslant \log j < L.$$

By (7.1.8) and the hypothesis on u, we have for $t \le q_r$ that

(7.1.9) $$t^{-u} = t^{-1}\, t^{1-u} \ll t^{-1}\, L^{c/\log L} \ll t^{-1}.$$

The Lemma follows from (7.1.7), (7.1.8), and (7.1.9).■

<u>Lemma 7.1.4</u> Let

$$f(r) = \prod_{p|r} f(p) \quad , \; f(p) = 1 + O(p^{-c}),$$

where $c>0$. For d a fixed non-negative integer, let

$$J_d(x) = \sum_{r\leqslant x} \frac{\mu^2(r)}{r} f(r)\left(\log\frac{x}{r}\right)^d .$$

Then

(7.1.10) $$J_d(x) = \prod_p\left(1 + \frac{f(p)-1}{p+1}\right)(1-p^{-2})\frac{\log^{d+1}x}{d+1} + O(\log^d x).$$

Proof. The case d=0 is Lemma 3.11 of Levinson [8]. For $d \geq 1$ we apply Lemma 7.1.2 with

$$a(n)=\mu^2(n)f(n)$$

and

$$A(w)=\sum_{n=1}^{\infty}\frac{\mu^2(n)f(n)}{n^w}=\prod_p\left(1+\frac{f(p)}{p^w}\right) \tag{7.1.11}$$

$$=\zeta(w)\prod_p\left(1+\frac{(f(p)-1)}{p^w}-\frac{f(p)}{p^{2w}}\right).$$

The product on the right side of (7.1.11) is absolutely convergent for $u \geq 1-c/2$ and $A(w)$ is analytic in this region except for a simple pole at $w=1$ with residue

$$\prod_p\left(1+\frac{f(p)-1}{p}-\frac{f(p)}{p^2}\right)=\prod_p\left(1+\frac{f(p)-1}{p+1}\right)(1-p^{-2}). \tag{7.1.12}$$

Also $A(w)$ is majorized by $\zeta(w)$ for $u \geq 1-c/2$. By Lemma 7.1.2 we have

$$J_d(x)=\frac{d!}{2\pi i}\int_{3-i\infty}^{3+i\infty}\frac{A(w)x^{w-1}}{(w-1)^{d+1}}\,dw. \tag{7.1.13}$$

The integrand of (7.1.13) has a pole of order d+2 at w=1. By (7.1.12) the residue is the expression on the right side of (7.1.10). The Lemma follows when we shift the path of integration to the line $w=1-c/2+iv$, $-\infty<v<\infty$ and use the estimate

$$A(w) \ll \zeta(w) \ll (1+|v|)^{1/2},$$

which is valid on the new path by the equation before (2.4.5).

Lemma 7.1.5 Let $\alpha=1$ or $\alpha=2a$. Let

(7.1.14) $$f(p)=(1-p^{2\alpha-2a})/(1-p^{a-1/2-\alpha})^2.$$

Let

(7.1.15) $$Y(a)=\prod_p (1+(f(p)-1)/(p+1)).$$

Then we have

(7.1.16) $$(\prod_p (1-p^{-2}))Y(a)=1+O(|1/2-a|).$$

Proof. By (7.1.14) we have

(7.1.17) $$f(p)=1+O(p^{-c})$$

where

$$c=\min\{2\alpha-2a,\ \alpha+1/2-a\}.$$

It is easy to see that $Y(a)$ is an analytic function of a for $|a-1/2|<1/4$. Therefore, for $|a-1/2|$ small we have

(7.1.18) $$Y(a)=Y(1/2)+O(|1/2-a|).$$

By (7.1.14) with $a=1/2$ and $\alpha=2a=1$ we have

(7.1.19) $$f(p)=(1-p^{-1})/(1-p^{-1})^2=1+1/(p-1).$$

By (7.1.19) we have

(7.1.20) $$1+(f(p)-1)/(p+1)=p^2/(p^2-1)=(1-p^{-2})^{-1}.$$

By (7.1.15) and (7.1.20) we have

(7.1.21) $$Y(1/2)=\prod_p (1-p^{-2})^{-1}.$$

The Lemma follows from (7.1.9), (7.1.13), and (7.1.16).∎

7.2 The choice of the mollifier. Because of

$$\log\tau = L/2$$

we have

(7.2.1) $$1-L^{-1}\log\left(\frac{k_i}{k}\right)=1/2-L^{-1}\log(k_i/(k\tau))$$

and

(7.2.2) $$L^{-1}\log(k_i/k)=1/2+L^{-1}\log(k_i/(k\tau)).$$

Now we consider ϕ_* as a Taylor series expanded around 1/2, and we obtain the derivative $\phi_*^{(j)}$ by term differentiation of the Taylor series. Then by the above, the inner sums of (6.5.15) will involve sums of

$$(L^{-1}\log(k_1/(k\tau)))^{r_1}(L^{-1}\log(k_2/(k\tau)))^{r_2}$$

for various r_1, r_2. Thus we are led to consider the sums

(7.2.3)
$$S_{r_1,r_2}(\alpha)=\sum\sum_{k_1,k_2\leq y}\frac{b_{k_1}b_{k_2}}{k_1^{\alpha}\,k_2^{\alpha}}k^{2\alpha-2a}\left(L^{-1}\log\frac{k_1}{k\tau}\right)^{r_1}\left(L^{-1}\log\frac{k_2}{k\tau}\right)^{r_2}.$$

Then J_7 involves the sums $S_{r_1,r_2}(1)$ and J_6 the sums $S_{r_1,r_2}(2a)$.

For convenience, let

(7.2.4) $$\gamma=2\alpha-2a.$$

We apply Lemma 7.1.1 with

(7.2.5) $$f(k)=k^{\gamma}\left(\log\frac{k_1}{k\tau}\right)^{r_1}\left(\log\frac{k_2}{k\tau}\right)^{r_2}.$$

Then we see that

(7.2.6) $$f(k)=\sum_{j|k} j^{\gamma}\sum_{d|j}\mu(d)d^{-\gamma}\left(\log\frac{k_1 d}{\tau j}\right)^{r_1}\left(\log\frac{k_2 d}{\tau j}\right)^{r_2}$$

$$=\sum_{j|k} j^{\gamma}\sum_{i_1=0}^{r_1}\sum_{i_2=0}^{r_2}\binom{r_1}{i_1}\binom{r_2}{i_2}\left(\log\frac{k_1}{\tau j}\right)^{i_1}\left(\log\frac{k_2}{\tau j}\right)^{i_2} g_j(r_1-i_1+r_2-i_2,\gamma)$$

where

(7.2.7) $$g_j(e,\gamma)=\sum_{d|j}\mu(d)d^{-\gamma}(\log d)^{e}.$$

Let

(7.2.8) $$M_j(i,\alpha)=\sum_{\substack{k_1\leq y\\ k_1\equiv 0 \bmod j}} b_{k_1}k_1^{-\alpha}\left(\log\frac{k_1}{\tau j}\right)^{i}.$$

We use (7.2.3), (7.2.4), (7.2.5), (7.2.6), (7.2.7), and (7.2.8) and recall that $k=(k_1,k_2)$. We interchange the sums on k_1,k_2 with that on j and have that

(7.2.9) $$L^{r_1+r_2}S_{r_1,r_2}(\alpha)=$$

$$=\sum_{i_1=0}^{r_1}\sum_{i_2=0}^{r_2}\binom{r_1}{i_1}\binom{r_2}{i_2}\sum_{j\leq y} j^{\gamma}g_j(r_1-i_1+r_2-i_2,\gamma)M_j(i_1,\alpha)M_j(i_2,\alpha).$$

In (7.2.8) make the change of variable $k_1=nj$ and let

(7.2.10) $$x=y/j.$$

Then

$$(7.2.11)\quad M_j(i,\alpha)=j^{-\alpha}\sum_{n\leq x} b_{nj}n^{-\alpha}\log^i(n/\tau)$$

$$=j^{-\alpha}\sum_{e=0}^{i}\binom{i}{e}(-\log\tau)^{i-e}\sum_{n\leq x} b_{nj}n^{-\alpha}\log^e n$$

$$=j^{-\alpha}(-1)^i\sum_{e=0}^{i}\binom{i}{e}(L/2)^{i-e}\left(\frac{\partial}{\partial\alpha}\right)^e\left(\sum_{n\leq x} b_{nj}n^{-\alpha}\right).$$

Now we specify b_j. More general than Levinson's choice [4, equation 2.4], we take the mollifier coefficients to be

$$(7.2.12)\qquad b_j=\frac{\mu(j)}{j^{1/2-a}}P\left(\frac{\log(y/j)}{\log y}\right).$$

where P is a polynomial which satisfies

$$P(0)=0\ ,\ P(1)=1.$$

Let

$$(7.2.13)\qquad \beta=\alpha-a+1/2.$$

Then for $\alpha=1$ or $\alpha=2a$ we have

$$(7.2.14)\qquad |1-\beta|\ll L^{-1}.$$

Let

$$(7.2.15)\qquad G(\alpha)=G(\alpha,j)=\sum_{\substack{n\leq x\\(n,j)=1}}\mu(n)n^{-\beta}P\left(\frac{\log(x/n)}{\log y}\right).$$

Then by (7.2.12), (7.2.13), and (7.2.15) we have

$$(7.2.16) \qquad \sum_{n\leqslant x} b_{nj}\, n^{-\alpha}=\mu(j)j^{a-1/2}\, G(\alpha).$$

We will use Lemma 7.1.2 to evaluate $G(\alpha)$ which can then be used to evaluate $M_j(i,\alpha)$ which is essential to $S_{r_1,r_2}(\alpha)$.

Let

$$(7.2.17) \qquad F(j,w)=\prod_{p|j}(1-p^{-w}).$$

Then, for $u>1$, by (2.1.2) and (7.2.17) we have

$$(7.2.18) \qquad \sum_{\substack{n=1\\(n,j)=1}}^{\infty} \mu(n)n^{-w}=\prod_{\substack{p\\p\nmid j}}(1-p^{-w})=1/(\zeta(w)F(j,w)).$$

Hence, by (7.2.15), (7.2.18), and Lemma 7.1.2 we have

$$(7.2.19) \quad G(\alpha)=\sum_{\ell\geqslant 1}\frac{P^{(\ell)}(0)}{\log^{\ell} y}\,\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{x^{w-\beta}}{\zeta(w)F(j,w)}\,\frac{dw}{(w-\beta)^{\ell+1}}.$$

We want to evaluate $G(\alpha)$ for the values

$$(7.2.20) \qquad \alpha=2a\ ,\ \alpha=1.$$

<u>7.3 The evaluation of $G^{(e)}(\alpha)$.</u> The integrand of (7.2.19) has a pole of order $\ell+1$ at $w=\beta$. We move the path of integration to the other side of the pole and use Cauchy's theorem. Let $P_0(\alpha)$ be the residue of the pole, let $P_1(\alpha)$ be the integral on $w=1+iv$, $-\infty<v\leqslant -L^{10}$, let $P_2(\alpha)$ be the integral on $w=u-iL^{10}$, $1-b\leqslant u\leqslant 1$, let $P_3(\alpha)$ be the integral on $w=(1-b)+iv$, $-L^{10}\leqslant v\leqslant L^{10}$, and let $P_4(\alpha)$ and

$P_5(\alpha)$ be the integrals on paths conjugate to the paths of $P_2(\alpha)$ and $P_1(\alpha)$ respectively. Here b is given by

$$b=1/(M \log L) \tag{7.3.1}$$

where M is a large constant. By (2.1.9) and since

$$\zeta(\bar{w})=\overline{\zeta(w)},$$

we may use the estimate

$$1/\zeta(1+iv)=O(\log|v|) \tag{7.3.2}$$

in P_1 and P_5. As long as M is sufficiently large, by (7.3.1), (2.1.7), and (2.1.8) we have that

$$1/\zeta(w)=O(\log L) \tag{7.3.3}$$

is valid for the paths of P_2, P_3, and P_4. For $u>0$ we have

$$|F(j,w)|=\prod_{p|j}|1-p^{-u-iv}|\geqslant \prod_{p|j}|1-p^{-u}|=F(j,u). \tag{7.3.4}$$

Let

$$F_1(j,u)=\prod_{p|j}(1+p^{-u}). \tag{7.3.5}$$

Then if $u>3/4$ we have

(7.3.6)

$$F_1(j,u)F(j,u)=\prod_{p|j}(1-p^{-2u})>\prod_{p}(1-p^{-3/2})=1/\zeta(3/2)>>1.$$

Hence (7.3.4) and (7.3.6) imply that for $u>3/4$ we have

$$1/F(j,w)>>F_1(j,u). \tag{7.3.7}$$

It follows from (7.2.19), (7.3.2), (7.3.3), (7.3.5), and (7.3.7) that the integrals $P_i(\alpha)$ are uniformly convergent for

(7.3.8) $$|\alpha-1|\leqslant 1/5 .$$

Therefore,

(7.3.9) $$G^{(e)}(\alpha)=\sum_{i=0}^{5} P_i^{(e)}(\alpha)$$

where the derivatives $P_i^{(e)}(\alpha)$ may be computed by differentiating under the integral sign.

By Cauchy's theorem and (7.2.13) we have

(7.3.10)
$$\left(\frac{\partial}{\partial\alpha}\right)^e\left(\frac{x^{w-\beta}}{(w-\beta)^{\ell+1}}\right)=\frac{e!}{2\pi i}\int_{|z-\alpha|=L^{-1}}\frac{x^{w-z+a-1/2}}{(w-z+a-1/2)^{\ell+1}}(z-\alpha)^{-e-1}dz.$$

For $|z-\alpha|=L^{-1}$ it follows from (7.2.20) and (7.2.10) that

(7.3.11) $$|x^{w-z+a-1/2}|<<x^{u-1} .$$

For $|z-\alpha|=L^{-1}$ and w on the new path of integration described above (7.3.1), it follows from (7.2.20) that

(7.3.12) $$|w-z+a-1/2|^{\ell+1}>>[(u-1)^2+v^2]^{(\ell+1)/2}.$$

By (7.3.10), (7.3.11), and (7.3.12), for any e we have

(7.3.13) $$\left(\frac{\partial}{\partial\alpha}\right)^e\left(\frac{x^{w-\beta}}{(w-\beta)^{\ell+1}}\right)<<e!L^e x^{u-1}/[(u-1)^2+v^2]^{(\ell+1)/2}.$$

Let

(7.3.14) $$\delta = 1/\log L.$$

By (7.3.2), (7.3.7), (7.3.13), and (7.3.14) for $\ell \geqslant 1$ we have

(7.3.15) $$P_5^{(e)}(\alpha) \ll e!L^e F_1(j,1) \int_{L^{10}}^{\infty} v^{-2}\log v\,dv$$

$$\ll e!F_1(j,1)L^{e-10}\log L \ll e!F_1(j,1-2\delta)L^{e-9}.$$

Here we have used the fact that $u_1 < u_2$ implies that

(7.3.16) $$F_1(j,u_1) > F_1(j,u_2).$$

The estimate obtained in (7.3.15) is valid for $P_1^{(e)}(\alpha)$ as well. By (7.3.1), (7.3.3), (7.3.7), (7.3.13), (7.3.14), and (7.3.16) we have

(7.3.17)

$$P_4^{(e)}(\alpha) \ll e!L^e b(\log L)L^{-20}F_1(j,1-b) \ll e!L^{e-20}F_1(j,1-2\delta),$$

and the same estimate holds for $P_2^{(e)}(\alpha)$. Observe that by (1.2.14) we have

(7.3.18) $$\int_{-L^{10}}^{L^{10}} \frac{dv}{(v^2+\delta^2)^{(\ell+1)/2}} \ll \delta^{-\ell}.$$

Therefore, by (7.2.10), (7.3.1), (7.3.3), (7.3.7), (7.3.13), (7.3.14), (7.3.16), and (7.3.18) we have

(7.3.19) $$P_3^{(e)}(\alpha) \ll e!L^e\, j^b\, y^{-b}\, F_1(j,1-2\delta)\log^{\ell+1} L.$$

Let

$$(7.3.20)\qquad Z(w)=1/(F(j,w)\zeta(w)).$$

The residue $P_0(\alpha)$ is given by the coefficient of $(w-\beta)^{-1}$ in

$$(w-\beta)^{-\ell-1}\left[\sum_{q=0}^{\infty}\frac{Z^{(q)}(\beta)}{q!}(w-\beta)^q\right]\left[\sum_{n=0}^{\infty}\frac{(\log x)^n(w-\beta)^n}{n!}\right].$$

From (7.2.13) and the above it follows easily that

$$(7.3.21)\qquad P_0^{(e)}(\alpha)=\frac{1}{\ell!}\sum_{q=0}^{\ell}\binom{\ell}{q}(\log x)^{\ell-q}Z^{(q+e)}(\beta).$$

By (2.1.4) and (7.3.20) we have

$$(7.3.22)\qquad Z(\beta)=1/(F(j,\beta)\zeta(\beta))=\frac{(\beta-1)+O(|\beta-1|^2)}{F(j,\beta)}.$$

By (7.3.20), (2.1.4), and (2.1.5) we have

$$(7.3.23)\qquad Z'(\beta)=\frac{1}{F(j,\beta)\zeta(\beta)}\left(-\frac{F'}{F}(j,\beta)-\frac{\zeta'}{\zeta}(\beta)\right)$$

$$=\frac{1}{F(j,\beta)}+O(F_1(j,1-2\delta)|\beta-1|\log L),$$

since by (7.2.17), (7.3.14), and Lemma 7.1.3,

(7.3.24)

$$\frac{F'}{F}(j,w)=\frac{d}{dw}\log F(j,w)=\sum_{p|j}\frac{\log p}{p^w-1}\ll\sum_{p|j}\frac{\log p}{p^u}\ll\log L$$

for $|w-1|\ll\delta$ and $j\leq y$.

It is clear that Z is regular for $|w-1|\leq 1/4$. Therefore, by Cauchy's theorem, (7.3.14), and (2.1.4) we have

(7.3.25) $$Z^{(k)}(\beta)=\frac{k!}{2\pi i}\int_{|w-\beta|=\delta} Z(w)(w-\beta)^{-k-1}\,dw$$

$$\ll k!F_1(j,1-2\delta)(\log L)^{k-1}.$$

By (7.3.21), (7.3.22), (7.3.23), (7.3.25), and (7.2.14),

(7.3.26) $$P_0(\alpha)=$$

$$=\frac{1}{F(j,\beta)}\left(\frac{(\log^{\ell}x)(\beta-1)}{\ell!}+\frac{(\log^{\ell-1}x)}{(\ell-1)!}\right)+O(F_1(j,1-2\delta)L^{\ell-2}\log L),$$

(7.3.27) $$P_0'(\alpha)=\frac{\log^{\ell}x}{\ell!F(j,\beta)}+O(F_1(j,1-2\delta)L^{\ell-1}\log L),$$

and

(7.3.28) $$P_0^{(e)}(\alpha)=O(e!L^{\ell}(\log L)^{e-1}F_1(j,1-2\delta)).$$

In summary, by (7.2.19), (7.3.15), (7.3.17), (7.3.19), (7.3.26), (7.3.27), and (7.3.28) we have

(7.3.29)
$$G^e(\alpha)=D_e(\alpha)+O(e!F_1(j,1-2\delta)L^{e-2}\log^2 L(1+L(j/y)^b)$$

where

$$(7.3.30)\quad \begin{cases} D_0(\alpha) = \dfrac{1}{F(j,\beta)}\left[(\beta-1)P\left(\dfrac{\log x}{\log y}\right) + \dfrac{1}{\log y}P'\left(\dfrac{\log x}{\log y}\right)\right] \\ D_1(\alpha) = \dfrac{1}{F(j,\beta)}P\left(\dfrac{\log x}{\log y}\right) \\ D_e(\alpha) = 0\ ,\ e\geqslant 2\ . \end{cases}$$

A convenient estimate for $G^{(e)}(\alpha)$ valid for all e follows from (7.3.29) and (7.3.30) and is given by

$$(7.3.31)\qquad G^{(e)}(\alpha) \ll e!F_1(j,1-2\delta)L^{e-1}\log^2 L.$$

7.4 <u>The evaluation of $S_{r_1,r_2}(\alpha)$.</u> By (7.2.9) we must consider $M_j(i_1,\alpha)M_j(i_2,\alpha)$. By (7.2.11), (7.2.16), and (7.2.13) we have

$$(7.4.1)\qquad M_j(i_1,\alpha)M_j(i_2,\alpha) = \frac{\mu^2(j)(-1)^{i_1+i_2}}{j^{2\beta}}\ .$$

$$\cdot \sum_{e_1=0}^{i_1}\ \sum_{e_2=0}^{i_2}\binom{i_1}{e_1}\binom{i_2}{e_2}(L/2)^{i_1-e_1+i_2-e_2}G^{(e_1)}(\alpha)G^{(e_2)}(\alpha).$$

We would like to sum on j in (7.2.9); rearrange the order so that the sum on j is innermost in (7.2.9). By (7.2.4) and (7.2.13) we have

$$(7.4.2)\qquad \gamma-2\beta=2\alpha-2a-2(\alpha-a+1/2)=-1$$

regardless of whether $\alpha=1$ or $\alpha=2a$. Therefore, by (7.2.9), (7.4.1), and (7.4.2) we have

(7.4.3)

$$S_{r_1,r_2}(\alpha)=L^{-r_1-r_2}\sum_{i_1=0}^{r_1}\sum_{i_2=0}^{r_2}\binom{r_1}{i_1}\binom{r_2}{i_2}(-1)^{i_1+i_2}\;.$$

$$\cdot\sum_{e_1=0}^{i_1}\sum_{e_2=0}^{i_2}\binom{i_1}{e_1}\binom{i_2}{e_2}(L/2)^{i_1-e_1+i_2-e_2}V_{r_1-i_1+r_2-i_2,e_1,e_2}(\alpha)$$

where

(7.4.4)

$$V_{e,e_1,e_2}(\alpha)=\sum_{j\leq y}\frac{\mu^2(j)}{j}g_j(e,2\alpha-2a)G^{(e_1)}(\alpha)G^{(e_2)}(\alpha).$$

It is $V_{e,e_1,e_2}(\alpha)$ that we will now evaluate.

By (7.2.7) and (7.2.17) we have

(7.4.5)
$$g_j(0,\gamma)=F(j,\gamma).$$

By (7.2.7) and (7.4.5) we have for $e\geq 0$ that

(7.4.6)
$$g_j(e,\gamma)=(-1)^e\left(\frac{d}{d\gamma}\right)^e g_j(0,\gamma)$$

$$=(-1)^e\left(\frac{d}{d\gamma}\right)^e F(j,\gamma)\;.$$

By (7.4.6), Cauchy's formula, (7.3.7), (7.3.14), and (7.3.16),

(7.4.7)
$$g_j(e,\gamma)=\frac{(-1)^e e!}{2\pi i}\int_{|w-\gamma|=\delta}\frac{F(j,w)}{(w-\gamma)^{e+1}}\,dw\ll e!F_1(j,1-2\delta)\log^e L.$$

To deal with the sums $V_{e,e_1,e_2}(\alpha)$, we make use of Lemma 7.1.4. Let

(7.4.8) $$f(p)=(F_1(p,1-2\delta))^3$$

for primes p. By (7.2.17) and (7.4.8) we have

(7.4.9) $$f(p)=1+O(p^{-(1-2\delta)})$$

and for squarefree r, by (7.2.17) we have

$$f(r)=\prod_{p|r} f(p)$$

so that (7.1.6) is valid. Then by Lemma 7.1.4 we have

(7.4.10) $$J(y)=\sum_{j\leqslant y}\frac{\mu^2(j)}{j}F_1^3(j,1-2\delta)=K\log y+O(1)$$

for some constant K. For $\varepsilon>0$ let

(7.4.11) $$J_\varepsilon(y)=\sum_{j\leqslant y}\frac{\mu^2(j)}{j}F_1^3(j,1-2\delta)j^\varepsilon.$$

Then by (7.4.10) we have

(7.4.12) $$J_\varepsilon(g)=\int_{1-}^{y} v^\varepsilon dJ(v)=y^\varepsilon J(y)-\varepsilon\int_1^y v^{\varepsilon-1}J(v)\,dv$$

$$=Ky^\varepsilon\log y+O(y^\varepsilon)-\varepsilon\int_1^y Kv^{\varepsilon-1}\log v\,dv+O(\varepsilon\int_1^y v^{\varepsilon-1}dv)$$

$$=O(y^\varepsilon/\varepsilon).$$

Suppose that $e>0$. Then by (7.3.35), (7.4.4), (7.4.7), and (7.4.10) we have

(7.4.13) $$V_{e,e_1,e_2}(\alpha) \ll e!e_1!e_2!L^{e_1+e_2-1}(\log L)^{e+4}.$$

Let $i=i_1-e_1+i_2-e_2$ and let $r=r_1-i_1+r_2-i_2$. Let

(7.4.14) $$E_{r_1,r_2}(\alpha)=\sum_{\substack{i_1=0 \\ (i_1,i_2)\neq(r_1,r_2)}}^{r_1}\sum_{i_2=0}^{r_2}\binom{r_1}{i_1}\binom{r_2}{i_2}(-1)^{i_1+i_2}L^{-r_1-r_2} \cdot \sum_{e_1=0}^{i_1}\sum_{e_2=0}^{i_2}\binom{i_1}{e_1}\binom{i_2}{e_2}\left(\frac{L}{2}\right)^i V_{r,e_1,e_2}(\alpha).$$

Then by (7.4.3) and (7.4.14) we have

(7.4.15) $$S_{r_1,r_2}(\alpha)=(-1/2)^{r_1+r_2} \cdot \sum_{e_1=0}^{r_1}\sum_{e_2=0}^{r_2}\binom{r_1}{e_1}\binom{r_2}{e_2}(2/L)^{e_1+e_2} V_{0,e_1,e_2}(\alpha)+E_{r_1,r_2}(\alpha).$$

To estimate $E_{r_1,r_2}(\alpha)$ we use the easy inequalities

(7.4.16) $$\sum_{e_1=0}^{i_1}\binom{i_1}{e_1}e_1!2^{e_1}= i_1!\sum_{e_1=0}^{i_1}2^{e_1}/(i_1-e_1)!=i_1!\sum_{e_1=0}^{i_1}2^{i_1-e_1}/e_1! \ll 2^{i_1}i_1!$$

and

(7.4.17) $$r!=\binom{r}{r_1-i_1}(r_1-i_1)!(r_2-i_2)! \leqslant 2^r(r_1-i_1)!(r_2-i_2)!$$

where $r=r_1-i_1+r_2-i_2$. Thus by (7.4.13), (7.4.14), (7.4.16), and (7.4.17) we have

(7.4.18) $$E_{r_1,r_2}(\alpha) \ll L^{-1}\log^4 L \ \cdot$$

$$\cdot \sum_{\substack{i_1=0 \\ (i_1,i_2)\neq(r_1,r_2)}}^{r_1} \sum_{i_2=0}^{r_2} \binom{r_1}{i_1}\binom{r_2}{i_2}(r_1-i_1)!(r_2-i_2)!i_1!i_2!\left(\frac{2\log L}{L}\right)^r$$

$$= \frac{(\log^4 L)}{L} r_1!r_2!\left[\sum_{i_1=0}^{r_1}\left(\frac{L}{2\log L}\right)^{i_1-r_1}\sum_{i_2=0}^{r_2}\left(\frac{L}{2\log L}\right)^{i_2-r_2}-1\right]$$

$$\ll L^{-1}(\log^4 L)r_1!r_2!((1-(2\log L)/L)^{-2}-1)$$

$$\ll r_1!r_2!L^{-2}\log^5 L.$$

We now need to estimate $V_{0,e_1,e_2}(\alpha)$. When $e_1 \geqslant 2$ or $e_2 \geqslant 2$ we will show that $V_{0,e_1,e_2}(\alpha)$ is small. Suppose that $e_1 \geqslant 2$. Then for $G^{(e_1)}(\alpha)$ we use the estimate (7.3.29) and for $G^{(e_2)}(\alpha)$ we use (7.3.31). Then by (7.4.4), (7.4.5), (7.4.10), (7.4.11), and (7.3.1) we have

(7.4.19) $$V_{0,e_1,e_2}(\alpha) \ll e_1!e_2!L^{e_1+e_2-3}\log^3 L(J(y)+Ly^{-b}J_b(y))$$

$$\ll e_1!e_2!L^{e_1+e_2-2}\log^4 L\ , \ e_1 \geqslant 2.$$

Of course, the same estimate is valid if $e_2 \geqslant 2$. Let

(7.4.20) $$r_1' = \min\{1,r_1\},\ r_2' = \min\{1,r_2\}.$$

Let

(7.4.21) $$E^*_{r_1,r_2}(\alpha) =$$

$$(-1/2)^{r_1+r_2} \sum_{\substack{e_1=0 \\ \max\{e_1,e_2\}\geq 2}}^{r_1} \sum_{e_2=0}^{r_2} \binom{r_1}{e_1}\binom{r_2}{e_2}(2/L)^{e_1+e_2} V_{0,e_1,e_2}(\alpha)$$

so that by (7.4.15) and (7.4.21) we have

(7.4.22) $$S_{r_1,r_2}(\alpha) =$$

$$(-1/2)^{r_1+r_2} \sum_{e_1=0}^{r_1} \sum_{e_2=0}^{r_2} \binom{r_1}{e_1}\binom{r_2}{e_2}(2/L)^{e_1+e_2} V_{0,e_1,e_2}(\alpha) +$$

$$+ E_{r_1,r_2}(\alpha)+E^*_{r_1,r_2}(\alpha).$$

By (7.4.16), (7.4.19), and (7.4.21) we have

(7.4.23) $$E^*_{r_1,r_2}(\alpha)$$

$$<<2^{-r_1-r_2}L^{-2}\log^4 L \sum_{e_1=0}^{r_1} \sum_{e_2=0}^{r_2} 2^{r_1+r_2} r_1!r_2!$$

$$<<r_1!r_2!L^{-2}\log^4 L.$$

We now evaluate $V_{0,e_1,e_2}(\alpha)$ when $\max\{e_1,e_2\}\leq 1$. By (7.3.7), (7.3.16), and (7.3.30) we have

(7.4.24) $$D_e(\alpha)<<L^{e-1} F_1(j,1-2\delta).$$

Since $b>0$ and $x\geq 1$ it follows from (7.3.29) and (7.4.24) that for $e_1, e_2 \leq 1$ we have

$$G^{(e_1)}(\alpha)G^{(e_2)}(\alpha) - D_{e_1}(\alpha)D_{e_2}(\alpha) \tag{7.4.25}$$

$$\ll F_1^2(j,1-2\delta)L^{e_1+e_2-3}(\log^4 L)(1+Lj^b y^{-b}).$$

Hence, by (7.2.10), (7.3.1), (7.4.5), (7.4.10), and (7.4.11) the error in replacing $G^{(e_1)}(\alpha)G^{(e_2)}(\alpha)$ by $D_{e_1}(\alpha)D_{e_2}(\alpha)$ in (7.4.4) is

$$\ll L^{e_1+e_2-3}(\log^4 L)(J(y)+Ly^{-b}\ J_b(y)) \tag{7.4.26}$$

$$\ll L^{e_1+e_2-2}\log^5 L.$$

Now by (7.4.4), (7.4.5), and (7.4.27) for $e_1, e_2 \leqslant 1$ we have

$$V_{0,e_1,e_2}(\alpha) = \tag{7.4.27}$$

$$\sum_{j\leqslant y} \frac{\mu^2(j)}{j} F(j,2\alpha-2a)D_{e_1}(\alpha)D_{e_2}(\alpha)+O(L^{e_1+e_2-2}\ \log^5 L).$$

We now apply Lemmas 7.1.4 and 7.1.5 to evaluate the main term of (7.4.27), keeping in mind (7.2.10), (7.2.13), and (7.2.17). Since

$$\frac{(\log\ y)^{n+1}}{(n+1)(\log\ y)^n} = (\log\ y)\int_0^1 t^n\,dt,$$

we have

$$(7.4.28)\quad \begin{cases} V_{000}(\alpha) = \Gamma_1(\beta-1)^2\log y+\Gamma_2 2(\beta-1) + \\ \qquad + \dfrac{\Gamma_3}{\log y} + O(L^{-2}\log^5 L) \\ V_{001}(\alpha) = V_{010}(\alpha) = \Gamma_1(\beta-1)\log y+\Gamma_2+O(L^{-1}\log^5 L) \\ V_{011}(\alpha) = \Gamma_1 \log y+O(\log^5 L) \end{cases}$$

where

$$(7.4.29)\quad \Gamma_1 = \int_0^1 [P(t)]^2 dt,\ \Gamma_2 = \int_0^1 P(t)P'(t)\,dt,\ \Gamma_3 = \int_0^1 [P'(t)]^2\ dt.$$

Equations (7.4.28) can be substituted into (7.4.22) and $S_{r_1,r_2}(\alpha)$ is then evaluated. Observe that

$$(7.4.30)\quad \log y=\log(T^{1/2}/L^{20})=L/2+O(\log L).$$

To simplify the expression we get for $S_{r_1,r_2}(\alpha)$, let

$$(7.4.31)\quad \begin{cases} \Lambda_1(\alpha) = \dfrac{\Gamma_1}{2}(\beta-1)^2\ L^2+2\Gamma_2(\beta-1)L+2\Gamma_3 \\ \Lambda_2(\alpha) = \Gamma_1(\beta-1)L+2\Gamma_2 \\ \Lambda_3(\alpha) = 2\Gamma_1\ . \end{cases}$$

Then by (7.4.18), (7.4.20), (7.4.22), (7.4.23), (7.4.28), (7.4.30), and (7.4.31) we have

(7.4.32)

$$S_{r_1,r_2}(\alpha)= \frac{(-1)^{r_1+r_2}}{2^{r_1+r_2}\,L}(\Lambda_1(\alpha)+(r_1+r_2)\Lambda_2(\alpha)+r_1r_3\Lambda_3(\alpha)) +$$

$$+ O(r_1!r_2!L^{-2}\log^5 L).$$

7.5 The evaluation of J. We want to include the result (7.4.32) in the equations (6.5.15). By (7.2.3) and (7.4.32) with

$$r_1=r_2=0\ ,$$

the portions of J_6 and J_7 which involve K_4 and K_5 are

(7.5.1) $$<<UL^{-1}.$$

We have

(7.5.2) $$\phi_*(w)=\sum_{r=0}^{\infty}\phi_*^{(r)}(1/2)\ (w-1/2)^r/r!.$$

We differentiate j times in (7.5.2) to see that

(7.5.3)

$$\phi_*^{(j)}(w)=\sum_{r=j}^{\infty}\phi_*^{(r)}(1/2)\ r(r-1)\ \ldots\ (r-j+1)(w-1/2)^{r-j}\ /r!$$

$$=\sum_{r=0}^{\infty}\phi_*^{(r+j)}(1/2)\ (w-1/2)^r\ /r!.$$

Therefore, by (6.5.15), (7.2.1), (7.2.2), (7.2.3), (7.5.1), and (7.5.3) we have

(7.5.4) $$J_7=\frac{Ue^{2r}}{(1-2a)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2R)^n}\ .$$

$$\cdot\sum_{j=0}^{n}\binom{n}{j}\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{\phi_*^{(r_1+j)}(1/2)}{r_1!}\ \frac{\phi_*^{(r_2+n-j)}(1/2)}{r_2!}\ \frac{S_{r_1,r_2}(1)}{(-1)^{r_1+r_2}}+$$

$$+ O(UL^{-1})$$

and

(7.5.5)
$$J_6 = \frac{U}{(1-2a)} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \cdot$$

$$\cdot \sum_{j=0}^{n} \binom{n}{j} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \frac{\phi_*^{(r_1+j)}(1/2)}{r_1!} \frac{\phi_*^{(r_2+n-j)}(1/2)}{r_2!} S_{r_1,r_2}(2a)$$

$$+ O(UL^{-1}).$$

The contribution of the $O(r_1!r_2!L^{-2}\log^5 L)$ term of $S_{r_1,r_2}(\alpha)$ to J_6 and J_7 is

(7.5.6)
$$\ll UL^{-1}\log^5 L \sum_{n=0}^{\infty} (2R)^{-n} \cdot$$

$$\cdot \sum_{j=0}^{n} \binom{n}{j} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} d^{r_1+r_2+n} [(r_1+n)(r_2+n)]^{m+1/2}$$

$$\ll UL^{-1}\log^5 L \sum_{n=0}^{\infty} (d/R)^n (n+1)^{2m+1} \cdot$$

$$\cdot \sum_{r_1=0}^{\infty} (r_1+1)^{m+1/2} d^{r_1} \sum_{r_2=0}^{\infty} (r_2+1)^{m+1/2} d^{r_2}$$

$$\ll UL^{-1}\log^5 L$$

by Corollary 6.1.9 and Equation (4.1.3). Let

(7.5.7)
$$J_{r_1,r_2}(\alpha) = \Lambda_1(\alpha) + (r_1+r_2)\Lambda_2(\alpha) + r_1 r_2 \Lambda_3(\alpha).$$

Then by (7.4.32), (7.5.4), (7.5.5), (7.5.6), and (7.5.7),

(7.5.8)

$$J_7 = \frac{Ue^{2R}}{2R} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \cdot$$

$$\cdot \sum_{j=0}^{n} \binom{n}{j} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \frac{\phi_*^{(r_1+j)}(1/2)}{2^{r_1} r_1!} \frac{\phi_*^{(r_2+n-j)}(1/2)}{2^{r_2} r_2!} J_{r_1,r_2}(1) +$$

$$+ O(UL^{-1}\log^5 L),$$

$$J_6 = \frac{U}{2R} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \cdot$$

$$\cdot \sum_{j=0}^{n} \binom{n}{j} \sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} \frac{\phi_*^{(r_1+j)}(1/2)}{(-2)^{r_1} r_1!} \frac{\phi_*^{(r_2+n-j)}(1/2)}{(-2)^{r_2} r_2!} \cdot$$

$$\cdot J_{r_1,r_2}(2a) + O(UL^{-1}\log^5 L).$$

Now it is clear that

(7.5.9)

$$\sum_{r_1=0}^{\infty} \frac{\phi_*^{(r_1+j)}(1/2)}{2^{r_1} r_1!} = \phi_*^{(j)}(1), \quad \sum_{r_1=0}^{\infty} \frac{\phi_*^{(r_1+j)}(1/2)}{(-2)^{r_1} r_1!} = \phi_*^{(j)}(0)$$

and

(7.5.10)

$$\sum_{r_1=0}^{\infty} \frac{r_1 \phi_*^{(r_1+j)}(1/2)}{2^{r_1}\, r_1!} = \frac{1}{2} \sum_{r_1=1}^{\infty} \frac{\phi_*^{(r_1-1+j+1)}(1/2)}{2^{r_1-1}\,(r_1-1)!} = \frac{1}{2}\, \phi_*^{(j+1)}(1),$$

$$\sum_{r_1=0}^{\infty} \frac{r_1 \phi_*^{(r_1+j)}(1/2)}{(-2)^{r_1}\, r_1!} = -\frac{1}{2}\, \phi_*^{(j+1)}(0).$$

Thus we use Equations (7.5.7), (7.5.9), and (7.5.10) in (7.5.8) and have

(7.5.11)
$$J_7 = \frac{Ue^{2R}}{2R} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \sum_{j=0}^{n} \binom{n}{j} \cdot$$

$$\cdot \Big[\phi_*^{(j)}(1)\ \phi_*^{(n-j)}(1)\Lambda_1(1) + \phi_*^{(j+1)}(1)\phi_*^{(n-j)}(1)\ \frac{\Lambda_2(1)}{2} +$$

$$+\ \phi_*^{(j)}(1)\phi_*^{(n-j+1)}(1)\ \frac{\Lambda_2(1)}{2} + (\phi_*^{(j+1)}(1)\phi_*^{(n-j+1)}(1))\frac{\Lambda_3(1)}{4} \Big] +$$

$$+\ O(UL^{-1}\log^5 L)$$

and

$$J_6 = \frac{U}{2R} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2R)^n} \sum_{j=0}^{n} \binom{n}{j} \Big[\phi_*^{(j)}(0)\phi_*^{(n-j)}(0)\Lambda_1(2a) -$$

$$\phi_*^{(j+1)}(0)\phi_*^{(n-j)}(0)\frac{\Lambda_2(2a)}{2a} - \phi_*^{(j)}(0)\phi_*^{(n-j+1)}(0)\ \frac{\Lambda_2(2a)}{2a} +$$

$$+\ \phi_*^{(j+1)}(0)\phi_*^{(n-j+1)}(0)\ \frac{\Lambda_3(2a)}{4} \Big] + O(UL^{-1}\log^5 L).$$

We can eliminate the $\frac{\pi i}{2L}$ term from ϕ_* as follows. Let

$$\phi_1(x)=\phi(x)(1-2x)^m . \tag{7.5.12}$$

By (5.2.36) we have

$$\phi_*(x)-\phi_1(x)=\phi(x)\sum_{k=1}^{m}\binom{m}{k}\left(\frac{\pi i}{2L}\right)^k(1-2x)^{m-k} . \tag{7.5.13}$$

Then by Lemma 6.1.8 we have for $0\le x\le 1$ and any j that

$$\phi_*^{(j)}(x) - \phi_1^{(j)}(x) = \tag{7.5.14}$$

$$\sum_{e=0}^{\min\{j,m\}}\binom{j}{e}\phi^{(j-e)}(x)\sum_{k=1}^{m}\binom{m}{k}\left(\frac{\pi i}{2L}\right)^k\left(\frac{d}{dx}\right)^e\left((1-2x)^{m-k}\right)$$

$$<<j^m d^{j-m}\sqrt{j}\; L^{-1}<<L^{-1}d^j j^{m+1/2}.$$

By Corollary 6.1.9, (7.5.11), and (7.5.14) the error in replacing $\phi_*^{(j)}$ by $\phi_1^{(j)}$ in J_6 and J_7 is

$$<<U\sum_{n=0}^{\infty}\frac{1}{(2R)^n}\,d^n n^{2m+1}\,L^{-1}\sum_{j=0}^{n}\binom{n}{j} \tag{7.5.15}$$

$$=UL^{-1}\sum_{n=0}^{\infty}(d/R)^n n^{2m+1}<<UL^{-1}.$$

By (7.2.13) and (7.4.31) we have

$$\begin{cases} \Lambda_1(1)=(\Gamma_1/2)R^2+2\Gamma_2R+2\Gamma_3 \\ \Lambda_2(1)=\Gamma_1R+2\Gamma_2,\quad \Lambda_3(1)=2\Gamma_1 \\ \Lambda_1(2a)=(\Gamma_1/2)R^2-2\Gamma_2R+2\Gamma_3 \\ \Lambda_2(2a)=-\Gamma_1R+2\Gamma_2,\quad \Lambda_3(2a)=2\Gamma_1 . \end{cases} \tag{7.5.16}$$

Consider J_6 and J_7 e^{-2R} (with ϕ_* replaced by ϕ_1) as series in powers of R. Now we use Equation (7.5.16). The coefficient of R^1 in $\frac{J_6}{U}$ is

(7.5.17) $$(\phi_1(0))^2\Gamma_1/4.$$

The coefficient of R^0 in J_6/U is

(7.5.18)
$$-\Gamma_2(\phi_1(0)^2+(\Gamma_1/2)\phi_1(0)\phi_1'(0)-(\Gamma_1/4)\phi_1(0)\phi_1'(0)$$

$$= -\Gamma_2[\phi_1(0)]^2+(\Gamma_1/4)\phi_1(0)\phi_1'(0).$$

For $n \geqslant 1$, the coefficient of R^{-n} in J_6/U is

(7.5.19)
$$\frac{(-1)^{n+1}}{2^{n+1}}\sum_{j=0}^{n+1}\binom{n+1}{j}\phi_1^{(j)}(0)\phi_1^{(n+1-j)}(0)\Gamma_1/2$$

$$+\frac{(-1)^n}{2^{n+1}}\sum_{j=0}^{n}\binom{n}{j}\Big(\phi_1^{(j)}(0)\phi_1^{(n-j)}(0)(-2\Gamma_2)-$$

$$(\phi_1^{(j+1)}(0)\phi_1^{(n-j)}(0)+\phi_1^{(j)}(0)\phi_1^{(n+1-j)}(0))(-\Gamma_1/2)\Big)$$

$$+\frac{(-1)^{n-1}}{2^n}\sum_{j=0}^{n-1}\binom{n-1}{j}\Big(\phi_1^{(j)}(0)\ \phi_1^{(n-1-j)}(0)\ 2\Gamma_3-$$

$$(\phi_1^{(j+1)}(0)\phi_1^{(n-1-j)}(0)+\phi_1^{(j)}(0)\phi_1^{(n-j)}(0))\Gamma_2$$

$$+\ \phi_1^{(j+1)}(0)\ \phi_1^{(n-j)}(0)\Gamma_1/2\Big).$$

It is clear that

$$\sum_{j=0}^{n}\binom{n-1}{j}(a_{j+1}\, a_{n-1-j}+a_j a_{n-j})$$

$$=\sum_{j=0}^{n} a_j a_{n-j}\left(\binom{n-1}{j}+\binom{n-1}{j-1}\right)=\sum_{j=0}^{n}\binom{n}{j} a_j a_{n-j} .$$

The terms involving Γ_2 in (7.5.19) cancel out. The terms involving Γ_1 in (7.5.19) simplify to

(7.5.20)

$$\Gamma_1\left(\frac{(-1)^{n+1}}{2^{n+2}}\sum_{j=0}^{n+1}\binom{n+1}{j}\phi_1^{(j)}(0)\ \phi_1^{(n+1-j)}(0)\ (-1/2)\ +\right.$$

$$\left.+\ 2\sum_{j=0}^{n-1}\binom{n-1}{j}\phi_1^{(j+1)}(0)\ \phi_1^{(n-j+1)}(0)\right).$$

Let

(7.5.21) $\qquad \Phi_1(x)=[\phi_1(x)]^2\ ,\ \Phi_3(x)=[\phi_1'(x)]^2.$

Then by (7.5.19), (7.5.20), and (7.5.21) the coefficient of R^{-n} $(n\geq 1)$ in J_6/U is

(7.5.22)

$$(-1/2)^n[\ \Phi_1^{(n+1)}(0)\ \Gamma_1/8\ -\ \Phi_3^{(n-1)}(0)\ \Gamma_1/2\ -\ \Phi_1^{(n-1)}(0)\ 2\Gamma_3].$$

In a similar way, the coefficient of R^1 in J_7e^{-2R}/U is

(7.5.23) $\qquad (\phi_1(0))^2\Gamma_1/4;$

the coefficient of R^0 is

(7.5.24) $$\Gamma_2[\phi_1(1)]^2+(\Gamma_1/4)\phi_1(1)\phi_1'(1)$$

and the coefficient of R^{-n} $(n\geqslant 1)$ in J_7e^{-2R}/U is

(7.5.25)

$$(-1/2)^n\ [\phi_1^{(n+1)}(1)\Gamma_1/8 - \phi_3^{(n-1)}(1)\Gamma_1/2 - \phi_1^{(n-1)}(1)2\Gamma_3].$$

Let

(7.5.26) $$\psi(x)=\phi_1''(x)\Gamma_1/8 - \phi_3(x)\Gamma_1/2 - \phi_1(x)\ 2\Gamma_3$$

$$= \frac{\Gamma_1}{4}[\phi_1(x)\phi_1''(x)-\phi_1'(x)\phi_1'(x)]-2\Gamma_3\phi(x)\phi(x).$$

Then by (7.5.25) we have

(7.5.27)

$$\psi^{(n-1)}(x)=\phi_1^{(n+1)}(x)\Gamma_1/8 - \phi_3^{(n-1)}(x)\Gamma_1/2 - \phi_1^{(n-1)}(x)2\Gamma_3.$$

Let

(7.5.28) $$F(R)=(R\Gamma_1/4)[e^{2R}\phi_1^2(1)-\phi_1^2(0)] +$$

$$+\Gamma_2(e^{2R}\phi_1^2(1)+\phi_1^2(0))+\frac{\Gamma_1}{4}[e^{2R}\phi_1(1)\phi_1'(1)-\phi_1(0)\phi_1'(0)]$$

$$-\frac{1}{2R}\sum_{n=0}^{\infty}\left(\frac{-1}{2R}\right)^n [e^{2R}\psi^{(n)}(1)-\psi^{(n)}(0)].$$

Then by (7.5.11), (7.5.15), (7.5.17), (7.5.18), (7.5.22), (7.5.23), (7.5.24), (7.5.25), (7.5.27), and (7.5.28) we have

(7.5.29) $$J_7-J_6=UF(R)+O(UL^{-1}\log^5 L).$$

Since $UF(R)$ is real, we have by (6.5.14) and (7.5.29) that

(7.5.30) $$J=UF(R)+O(UL^{-1}\log^5 L).$$

7.6 A theorem. We gather the results together. By (3.3.26), (3.3.27), (4.1.3), (5.2.30), (5.2.54), (5.2.55), (7.5.28), and (7.5.30) we have proved the following

Theorem 7.6.1. Let $C\neq 0$ and $R>0$, and let ϕ be an entire function of w which satisfies

$$\phi(\bar{w})=\overline{\phi(w)},\quad \phi(w)+\phi(1-w)=C,\quad \phi(0)=1,\quad |\phi(w)|\ll\exp(d|w|)$$

where

$$d<\min\{R,1\}.$$

Let

$$\phi_1(w)=\phi(w)(1-2w)^m$$

and let $P(x)$ be a polynomial which satisfies

$$P(0)=0\ ,\ P(1)=1\ .$$

Let

$$\Gamma_1=\int_0^1[P(x)]^2dx,\quad \Gamma_2=\int_0^1 P(x)P'(x)\,dx,\quad \Gamma_3=\int_0^1[P'(x)]^2dx.$$

Let

$$\Psi(x)=\frac{\Gamma_1}{4}[\phi_1(x)\phi_1''(x)-\phi_1'(x)\phi_1'(x)]-2\Gamma_3\phi_1(x)\phi_1(x)$$

and let

$$F_m(R)=(R\Gamma_1/4)[e^{2R}\phi^2(1)-1]$$

$$+\Gamma_2(e^{2R}\phi^2(1)+1)+\frac{\Gamma_1}{4}[e^{2R}\phi_1(1)\phi_1'(1)-\phi_1'(0)]$$

$$- \frac{1}{2R} \sum_{n=0}^{\infty} \left(\frac{-1}{2R}\right)^n [e^{2R}\psi^{(n)}(1) - \psi^{(n)}(0)].$$

Then the proportion of zeroes of $\xi^{(m)}(s)$ with real part 1/2 is at least

$$1-[\log F_m(R)]/R.$$

It should be noted that the infinite sum in $F_m(R)$ can be written as an integral. We integrate by parts to see that

(7.6.1)
$$\int_0^1 e^{2Rx}\psi(x)\,dx = \frac{1}{2R}\sum_{n=0}^{N}\left(\frac{-1}{2R}\right)^n [e^{2R}\psi^{(n)}(1) - \psi^{(n)}(0)]+R_{N+1}$$

where

(7.6.2)
$$R_{N+1}=\frac{(-1)^{N+1}}{(2R)^{N+1}}\int_0^1 e^{2Rx}\psi^{(N+1)}(x)\,dx.$$

By Corollary 6.1.9 and Equations (7.5.14) and (7.5.26),

$$R_{N+1} \to 0$$

as $N\to\infty$. Thus we can express F_m as

$$F_m(R)=\frac{R\Gamma_1}{4}(e^{2R}[\phi_1(1)]^2-1)+\Gamma_2(e^{2R}[\phi_1(1)]^2+1)$$

$$+\frac{\Gamma_1}{4}[e^{2R}\phi_1(1)\phi_1'(1)-\phi_1'(0)]-\int_0^1 e^{2Rx}\Psi(x)\,dx$$

in the Theorem. If C=1, then $\phi_1(1)=0$ and $F_m(R)$ further simplifies to

$$(7.6.4) \qquad F_m(R)=\Gamma_2-\frac{\Gamma_1}{4}(R+\phi_1'(0))-\int_0^1 e^{2Rx}\,\Psi(x)\,dx.$$

CHAPTER VIII

COMPUTATIONS

8.1 Generalities. To obtain Levinson's result [8] that $>34.20\%$ of the zeroes of the zeta-function have real part 1/2, we use

$$P(x)=x,\ \phi_1(x)=1-x,\ R=1.3$$

and the computation is a simple matter. To obtain Levinson's result [9] that $>71.72\%$ of the zeroes of the ξ'-function have real part 1/2, we use

$$P(x)=x,\ \phi_1(x)=\phi(x)(1-2x)=(1-x)(1-2x),\ R=1.1\ .$$

We will carry out computations in the cases m=0,1,2. In all three cases we use

$$P(x)=x$$

so that by (7.4.29) we have

$$\Gamma_1=1/3,\ \Gamma_2=1/2,\ \Gamma_3=1. \tag{8.1.1}$$

In our calculations we will use the following notation.

$$\begin{cases} \phi_1(x)=\Sigma\beta_i x^i \\ \psi(x)=\Sigma\gamma_i x^i \\ \psi^{(n)}(1)=\delta_n,\ \psi^{(n)}(0)=\varepsilon_n. \end{cases} \tag{8.1.2}$$

Thus, given the β_i, we have by (7.5.26) and (8.1.2) that

(8.1.3)

$$\gamma_i = \frac{\Gamma_1}{4}\left[\sum_j (j+1)(j+2)\beta_{j+2}\beta_{i-j} - (j+1)\beta_{j+1}(i-j+1)\beta_{i-j+1}\right]$$

$$- 2\Gamma_3 \sum_j \beta_j \beta_{i-j}$$

$$= \frac{\Gamma_1}{4}\left[\sum_j (2j-i-3)j\beta_j\beta_{i+2-j}\right] - 2\Gamma_3\sum_j \beta_j\beta_{i-j}.$$

Then by (8.1.2) we have

(8.1.4) $$\delta_n = n!\sum_i \binom{i+n}{i}\gamma_{i+n}$$

and

(8.1.5) $$\varepsilon_n = n!\gamma_n.$$

Then, as in Theorem 7.6.1, we have

(8.1.6) $$F_m(R) = \frac{R}{12}\left[e^{2R}\phi_1^2(1) - 1\right] + 1/2\left[e^{2R}\phi_1^2(1) + 1\right] +$$

$$+ \frac{1}{12}\left[e^{2R}\phi_1(1)\phi_1'(1) - \phi_1'(0)\right] - \frac{1}{2R}\sum_n \left(\frac{-1}{2R}\right)^n \left[e^{2R}\delta_n - \varepsilon_n\right].$$

Clearly we have

(8.1.7) $$\phi_1(1) = \sum\beta_i,\quad \phi_1'(1) = \sum i\beta_i,\ \phi_1'(0) = \beta_1.$$

In the computations, we use a calculator which rounds in the 14th digit. The values of γ_n, δ_n, and ε_n in the Tables 1, 2, and 3 and the values we obtain for $F_m(R)$ and $[\log F_m(R)]/R$ are truncates of the actual values.

8.2 The case m=0. We take

(8.2.1) $\phi_1(x)=\phi(x)=1-(\alpha_1+3\alpha_3)x+6\alpha_3x^2-4\alpha_3x^3.$

Then it is easily checked that the conditions of Theorem 7.6.1 for ϕ are satisfied, provided that

$$\alpha_1+\alpha_3\neq 2.$$

We take

(8.2.2) $\alpha_1=1.279,\ \alpha_3=-.265,\ R=1.49.$

We give a table of values obtained from (8.1.2), (8.1.3), (8.1.4), (8.1.5), and (8.2.2).

n	β_n	γ_n	δ_n	ε_n
0	1	-2.2845	-.0236	-2.2845
1	-.484	2.3377	.09373	2.3377
2	-1.159	5.4701	-1.6016	10.9402
3	1.06	-6.7564	15.4548	-40.538
4		-3.2849	-78.838	-78.838
5		6.7416	-808.992	808.992
6		-2.2472	-1617.984	-1617.984

Table 1-- Intermediate computations for $F_0(R)$.

Also by (8.1.7) and Table 1 we have

(8.2.3) $\phi_1(1)=-.014,\ \phi_1'(1)=-.484,\ \phi_1'(0)=-.484.$

By Table 1 and Equations (8.2.3), (8.2.2), and (8.1.6) we have

$$F_0(R)=2.6006429 \ldots$$

so that

(8.2.4) $$[\log F_0(1.49)]/(1.49)=.6414488 \ldots .$$

By Theorem 7.6.1 and Equation (8.2.4) we have

Theorem 8.2.1 The proportion of zeroes of $\zeta(s)$ with real part 1/2 exceeds .3585.

8.3 The case m=1. To calculate a lower bound for the proportion of zeroes of $\xi'(s)$ with real part 1/2 we take

(8.3.1) $$\phi(x)=1-\alpha x$$

so that

(8.3.2) $$\phi_1(x)=(1-\alpha x)(1-2x).$$

Then the conditions of Theorem 7.6.1 for ϕ are satisfied provided that

$$\alpha \neq 2.$$

We take

(8.3.3) $$\alpha=.991 \ , \ R=1.09.$$

By (8.1.2), (8.1.3), (8.1.4), (8.1.5), (8.3.2), and (8.3.3) we construct the following table.

n	β_n	γ_n	δ_n	ε_n
0	1	-2.4151	-.0820	-2.4151
1	-2.991	12.9520	-.2863	12.9520
2	1.982	-26.474	-4.953	-52.949
3		23.712	-46.283	142.275
4		-7.8566	-188.559	-188.559

Table 2 -- Intermediate Computations for $F_1(R)$

Also, by (8.1.7) and Table 2 we have

(8.3.4) $\phi_1(1)=-.009$, $\phi_1'(1)=.973$, $\phi_1'(0)=-2.991$.

By Table 2, (8.3.3), (8.3.4), and (8.1.6) we have

(8.3.5) $F_1(R)=1.358859\ ...$

and $[\log F_1(1.09)]/1.09=.281326\ ....$

By Theorem 7.1.6 and (8.3.7) we have

<u>Theorem 8.3.1</u> The proportion of zeroes of $\xi'(s)$ with real part 1/2 exceeds .7186.

<u>8.4 The case m=2.</u> As with the case m=1 we use

(8.4.1) $\phi(x)=1-\alpha x$, $\alpha\neq 2$.

Then we have

(8.4.2) $$\phi_1(x)=(1-\alpha x)(1-2x)^2.$$

We take

(8.4.3) $$\alpha=.974 \ , \ R=1.38 \ .$$

By (8.1.2), (8.1.3), (8.1.4), (8.1.5), (8.4.2), and (8.4.3) we construct the following table.

n	β_n	γ_n	δ_n	ε_n
0	1	-2.745	-.0808	-2.745
1	-4.974	24.493	-.5100	24.493
2	7.896	-91.456	-7.031	-182.91
3	-3.896	182.937	-106.293	1097.62
4		-206.003	-1106.67	-4944.07
5		123.051	-7091.343	14766.1
6		-30.357	-21857.4	-21857.4

Table 3 -- Intermediate Computations for $F_2(R)$

Also, by (8.1.7) and Table 3 we have

(8.4.4) $$\phi_1(1)=.026 \ , \ \phi_1'(1)=-.87 \ , \ \phi_1'(0)=-4.974 \ .$$

By (8.4.3), (8.4.4), Table 3, and (8.1.6) we have

$$F_2(R)=1.28029 \ldots$$

so that

(8.4.5) $$[\log F_2(1.38)]/(1.38) = .179049 \ldots.$$

By Theorem 6.7.1 and (8.4.5) we have

Theorem 8.4.1. The proportion of zeroes of $\xi''(s)$ with real part 1/2 exceeds .8209.

BIBLIOGRAPHY

[1] T. Apostol, Introduction to Analytic Number Theory, Springer-Verlag New York, Inc., 1976.

[2] E. Bombieri, "A Lower Bound for the Zeros of Riemann's Zeta-Function on the Critical Line," Seminaire Bourbaki 469 (1975), 176-181.

[3] E. T. Copson, Asymptotic Expansions, Cambridge University Press, 1965.

[4] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford University Press, 1938.

[5] A. E. Ingham, The Distribution of Prime Numbers, Cambridge University Press, 1932.

[6] N. Levinson, "Remarks on a Formula of Riemann for his Zeta-Function," Journal of Mathematical Analysis and Applications 41 (1973), 345-351.

[7] N. Levinson, "At Least One-Third of Zeros of Riemann's Zeta-Function are on $\sigma=1/2$," Proc. Nat. Acad. Sci. USA 71 (1974), 1013-1015.

[8] N. Levinson, "More Than One Third of Zeros of Riemann's Zeta-Function are on $\sigma=1/2$," Advances in Mathematics 13 (1974), 383-436.

[9] N. Levinson, "Zeros of Derivative of Riemann's ξ-Function," Bulletin of the American Mathematical Society 80, No. 5, (1974), 951-954.

[10] N. Levinson, "Generalization of Recent Method Giving Lower Bound for $N_0(T)$ of Riemann's Zeta-Function," Proc. Nat. Acad. Sci. USA 71, No. 10, (1974), 3984-3987.

[11] N. Levinson, "Deduction of Semi-Optimal Mollifier for Obtaining Lower Bound for $N_0(T)$ for Riemann's Zeta-Function," Proc. Nat. Acad. Sci. USA 72, No. 1, (1975), 294-297.

[12] N. Levinson, "A Simplification in the Proof that $N_0(T) > (1/3)N(T)$ for Riemann's Zeta-Function," Advances in Mathematics 18 (1975), 239-242.

[13] Pan Cheng-biao, "A Simplification of the Proof of Levinson's Theorem," Acta Mathematica Sinica 22 (1979) 344-353.

[14] H. Rademacher, Topics in Analytic Number Theory, Springer-Verlag, Berlin · Heidelberg, 1973.

[15] A. Selberg, "On the Zeroes of Riemann's Zeta-Function," Skr. Norske Vid. Akad. Oslo 10 (1942), 1-59.

[16] C. L. Siegel, "Über Riemann's Nachlass zur analytischen Zahlentheorie," Quellen und Studien zur Geschichte der Math. Astr. und Physik, Abt. B: Studien 2 (1932), 45-80.

[17] E. C. Titchmarsh, The Theory of Functions, Oxford University Press, 1932.

[18] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, Oxford University Press, 1951.

[19] E. T. Whitaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, 1902.

Jim

Further results based on this paper:

Let α_m denote the proportion of zeros of $\xi^{(m)}(s)$ on the critical line. Then

$$\alpha_0 > .3658$$

$$\alpha_1 > .8137$$

$$\alpha_2 > .9584$$

$$\alpha_3 > .9873$$

$$\alpha_4 > .9947$$

also,

$$\lim_{m \to \infty} \alpha_m = 1$$

I will send you a copy when I write up these results.

Brian