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On the Distribution of the Zeros of the Riemann Zeta-Function

J. B. Conrey

In this article we shall describe recent results concerning the zeros of the zeta-function. In particular we are interested in the proportion of zeros of $\zeta(s)$ on the $\frac{1}{2}$-line, the proportion of simple zeros of $\zeta(s)$ on the $\frac{1}{2}$-line, and extreme gaps between the ordinates of consecutive zeros of $\zeta(s)$. Some of the results we quote are conditional; this will be appropriately indicated.

Selberg [19] was the first to show that a positive proportion of the zeros of $\zeta(s)$ are on $\sigma = \frac{1}{2}$. Levinson [11], by a different method, showed that this proportion, which we shall denote α, exceeds 0.3420. We briefly describe Levinson's method and then indicate modifications of it which have led to small improvements.

The starting point of Levinson's method is the identity

$$H(s)\zeta(s) = H(s)G(s) + H(1-s)G(1-s) \qquad (1)$$

where

$$H(s) = \tfrac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$$

and

$$G(s) = \zeta(s) - \frac{\zeta'(s)}{\frac{\chi'}{\chi}(s)} . \qquad (2)$$

Here

$$\frac{\chi'}{\chi}(s) = -\log\frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right) \qquad (3)$$

uniformly for $|\sigma| \leqslant 10$, $|t| \geqslant 1$. It follows that $\zeta(\frac{1}{2} + it) = 0$ when

$$\arg HG(\tfrac{1}{2} + it) \equiv \frac{\pi}{2} \bmod \pi \ . \tag{4}$$

Now

$$\arg H(\tfrac{1}{2} + it) = \frac{t}{2} \log \frac{|t|}{2\pi e} + O(1) \tag{5}$$

so that a bound of the shape

$$\left| \arg G(\tfrac{1}{2} + it) \Big|_{t=T}^{T+U} \right| \leqslant \beta \frac{UL}{2} \tag{6}$$

where

$$L = \log \frac{T}{2\pi} \ , \quad U = TL^{-10} \tag{7}$$

implies that

$$\alpha \geqslant 1 - \beta \ . \tag{8}$$

(It is easier to work on the interval $[T, T+U]$ than on $[0, T]$.) To obtain the bound (6) we use the argument principal on the rectangle with vertices $2 + iT$, $2 + i(T+U)$, $\tfrac{1}{2} + i(T+U)$, $\tfrac{1}{2} + iT$. This leads to (6) with

$$\beta UL = 4\pi N_G \tag{9}$$

where N_G is the number of zeros of $G(s)$ in $\sigma \geqslant \tfrac{1}{2}$, $T \leqslant t \leqslant T + U$. To bound N_G we apply Littlewood's lemma to ψG on the rectangle with vertices $2 + iT$, $2 + i(T+U)$, $a + iT$ where

$$\psi(s) = \sum_{n \leqslant y} b(n) n^{-s} \tag{10}$$

is a Dirichlet polynomial with $b(1) = 1$, $b(n) \ll 1$, and

$$a = \frac{1}{2} - \frac{R}{L} \tag{11}$$

for some fixed $R > 0$. This leads to

$$2\pi\left(\tfrac{1}{2} - a\right)N_G \leqslant \int_T^{T+U} \log|\psi G(a+it)|\,dt$$
$$\leqslant \frac{U}{2}\log\left(\frac{1}{U}\int_T^{T+U} |\psi G(a+it)|^2 dt\right) \tag{12}$$

so that

$$\alpha \geqslant 1 - \frac{\log\left(\frac{1}{U}\int_T^{T+U} |\psi G(a+it)|^2 dt\right)}{R} . \tag{13}$$

Levinson evaluated the integral in (13) in the case that

$$y = T^{\frac{1}{2}} L^{-20} \tag{14}$$

and

$$b(n) = \frac{\mu(n)}{n^{\frac{1}{2}-a}} \frac{\log y/n}{\log y} \tag{15}$$

and obtained

$$\frac{1}{U}\int_T^{T+U} |\psi G(a+it)|^2 dt \sim F(R) \tag{16}$$

where

$$F(R) = e^{2R}\left(\frac{1}{2R^3} + \frac{1}{24R}\right) - \frac{1}{2R^3} - \frac{1}{R^2} - \frac{25}{24R} + \frac{7}{12} - \frac{R}{12} . \tag{17}$$

With $R = 1.3$ this led to

$$\alpha \geqslant 0.3420 . \tag{18}$$

Subsequently, in an attempt to optimize the coefficients of the mollifier $\psi(s)$, Levinson [13] was led to the choice

$$b(n) = \frac{\mu(n)}{n^{1-2a}} \frac{y^{1-2a} - j^{1-2a}}{y^{1-2a} - 1} . \tag{19}$$

This choice gives the result

$$\alpha \geqslant 0.3474 . \tag{20}$$

Heath-Brown [9] and Selberg independently noticed that the zeros located by Levinson's method are simple zeros of $\zeta(s)$. Thus

$$\alpha_s \geqslant 0.3474 \tag{21}$$

where α_s is the proportion of simple zeros of $\zeta(s)$ on the ½-line.

Lou [14] chose the mollifier $\Psi(s) = \psi(s) + \chi(s)L^2 h\, \psi_1(s)$ where ψ is as in (10) and (19), h is constant, and

$$\psi_1(s) = \sum_{k \leqslant y} \frac{\mu^2(k) b_1(k)}{k^{\frac{1}{2}+a-s}} \frac{\log {}^{y}/k}{\log y}, \quad b_1(k) = \sum_{n|k} \frac{\mu(n) d(n)}{n^{2s-2a}} . \tag{22}$$

This leads to

$$\alpha, \alpha_s \geqslant 0.35 . \tag{23}$$

Levinson [12] suggested new identities for $\zeta(s)$ which generalize (1) and lead to better results. In $[T, T+U]$ we can write Levinson's G as

$$G(s) \sim \zeta(s) + \frac{\zeta'(s)}{L} \tag{24}$$

because of (3). Conrey [4] used identities of the sort

$$H(s)\zeta(s) = H(s)\mathfrak{G}(s) + H(1-s)\mathfrak{G}(1-s) \tag{25}$$

where

$$\mathfrak{G}(s) \sim \sum_n a_n \zeta^{(n)}(s) L^{-n} \tag{26}$$

and the a_n are certain complex numbers. These identities can be derived in a variety of ways. (See Levinson [11] and [12], Bombieri [3], Conrey [4] and [5], and Anderson [1] for different approaches.) Still another method for developing (25) and (26) in a rather simple way is as follows.

Let g_0 be arbitrary, g_{2r} purely imaginary, and g_{2r+1} real for $r = 1, 2, \ldots$. Let

$$\xi(s) = H(s)\zeta(s) \tag{27}$$

so that $\xi(\frac{1}{2} + it)$ is real and $\xi(s) = \xi(1-s)$ is the functional equation for $\zeta(s)$. Then

$$\begin{aligned}(2 \operatorname{Re} g_0)\xi(s) &= g_0\xi(s) + \overline{g}_0\xi(1-s) \\ &= g_0\xi(s) + \sum_r g_r\xi^{(r)}(s)L^{-r} \\ &\quad + \overline{g}_0\xi(1-s) - \sum_r g_r(-1)^r\xi^{(r)}(1-s)L^{-r}\end{aligned} \tag{28}$$

since

$$\xi^{(r)}(s) = (-1)^r\xi^{(r)}(1-s) . \tag{29}$$

This expresses $(2 \operatorname{Re} g_0)\xi(s)$ as a sum of complex conjugates when $s = \frac{1}{2} + it$, just as in (1). We take

$$\mathbb{G}(s) = (g_0\xi(s) + \sum_r g_r\xi^{(r)}(s)L^{-r})/H(s) . \tag{30}$$

We rewrite $\xi^{(r)}(s)$ using (27) and Leibniz's formula. Also

$$\frac{H^{(k)}(s)}{H(s)} \sim (\frac{L}{2})^k \tag{31}$$

for $T \leqslant t \leqslant T + U$. Thus

$$\mathbb{G}(s) \sim g_0\zeta(s) + \sum_r g_r L^{-r} \sum_k \binom{r}{k} \zeta^{(k)}(s)(\frac{L}{2})^{r-k} \tag{32}$$

which is (26) with

$$a_0 = g_0 + \sum_r 2^{-r}g_r \tag{33}$$

and

$$a_k = 2^k \sum_r 2^{-r} \binom{r}{k} g_r \qquad (34)$$

for $k \geqslant 1$. Hence the a_k are restricted only by the conditions on the g_r. Since g_0 is arbitrary, we may take $a_0 = 1$. With this normalization, (13) holds with $\mathfrak{G}$ in place of G.

Conrey [4] also used the more general mollifier coefficients

$$b(n) = \frac{\mu(n)}{n^{\frac{1}{2} - a}} \, P\left(\frac{\log {}^{y}/_{n}}{\log y} \right) \qquad (35)$$

where P is an analytic function with $P(0) = 0$ and $P(1) = 1$. By choosing the a_k and P appropriately this gives [4]

$$\alpha \geqslant 0.3658 \, . \qquad (36)$$

(It is possible to choose P optimally here by calculus of variations.)

With Levinson's original G and the mollifier coefficients (35) one can show [5]

$$\alpha_s \geqslant 0.3485 \qquad (37)$$

which is not as good as (23). However, Anderson [1] pointed out that for counting multiple zeros one may use

$$\mathfrak{G}_1(s) = \zeta(s) + a_1 \zeta'(s) L^{-1} \qquad (38)$$

with a_1 an arbitrary real. He used this and the coefficients (19) and obtained [1]

$$\alpha_s \geqslant 0.3532 \, . \qquad (39)$$

If one uses (38) with the coefficients (35) the result is

$$\alpha_s \geqslant 0.358 \,. \tag{40}$$

Further improvements in the method seem to rely on taking a longer Dirichlet polynomial for the mollifier. That is, we want

$$y = T^{\theta}, \quad \theta > 1/2 \,. \tag{41}$$

Balasubramanian, Conrey, and Heath-Brown [2] have shown (using (10) and (35) with $a = 1/2$) that

$$\frac{1}{T}\int_0^T |\zeta(\tfrac{1}{2} + it)|^2 \, |\psi(\tfrac{1}{2} + it)|^2 dt \sim 1 + \frac{\log T}{\log y}\int_0^1 P'(x)^2 dx \tag{42}$$

for any $\theta < 9/17$. This result needs to be generalized to the integrals

$$\frac{1}{T}\int_0^T \zeta^{(u)}(a + it)\zeta^{(v)}(a - it)\,|\psi(\tfrac{1}{2} + it)|^2 dt$$

in order to evaluate (16). If this is done, one will obtain the result

$$\alpha \geqslant 0.38 \,. \tag{43}$$

If Hooley's conjecture R^* [10] is assumed, then one can prove (42) for any $\theta < 4/7$ which would lead to

$$\alpha \geqslant 0.4077 \,. \tag{44}$$

Next we consider what bounds we can obtain for α_s if we first assume something about the zeros of $\zeta(s)$. Montgomery [15] showed, assuming the Riemann Hypothesis, that

$$(R/H) \qquad \alpha_s \geqslant 2/3 \,. \tag{45}$$

With Taylor (see [16]), he improved this to

$$\text{(RH)} \qquad \alpha_s \geqslant \frac{3}{2} - \frac{\sqrt{2}}{2} \cot \frac{\sqrt{2}}{2} = 0.6725 . \tag{46}$$

Assuming his pair correlation conjecture, Montgomery [15] obtained

$$\text{(RH , PC)} \qquad \alpha_s = 1 . \tag{47}$$

Recently, by a new technique, Conrey, Ghosh, and Gonek [6] showed, assuming the generalized Riemann Hypothesis, that

$$\text{(GRH)} \qquad \alpha_s \geqslant \frac{19}{27} = 0.703\ldots . \tag{48}$$

The assumption can probably be weakened to RH. The method we used is as follows.

Assume the Riemann Hypothesis. Let $\frac{1}{2} + i\gamma$ denote a typical zero of $\zeta(s)$. Then by Cauchy's inequality

$$N_s(T) \geqslant \frac{\left| \sum_{\gamma \leqslant T} \psi\zeta'(\frac{1}{2} + i\gamma) \right|^2}{\sum_{\gamma \leqslant T} \left| \psi\zeta'(\frac{1}{2} + i\gamma) \right|^2} \tag{49}$$

where $N_s(T)$ is the number of simple zeros of $\zeta(s)$ with $0 < \gamma \leqslant T$ and $\psi(s)$ is as in (10), (14), and (35) with $a = \frac{1}{2}$. The expressions on the right side of (49) can be estimated asymptotically by methods similar to, but more complicated than what Gonek [8] used. For a real function $P(x)$ the numerator is

$$\sim \left(\frac{1}{2} + \frac{1}{2} \int_0^1 P(x)\,dx \right)^2 \left(\frac{TL^2}{2\pi} \right)^2 \tag{50}$$

and the denominator is

$$\sim \left(\frac{1}{3} + \frac{1}{4} \left(\int_0^1 P(x)\,dx \right)^2 + \frac{1}{2} \int_0^1 P(x)\,dx + \frac{1}{6} \int_0^1 P'(x)^2\,dx \right) \frac{TL^3}{2\pi} . \tag{51}$$

By the calculus of variations

$$P(x) = -\frac{1}{2}x^2 + \frac{3}{2}x \tag{52}$$

is optimal from which we obtain (48). We needed GRH to obtain the estimate (51) but his could probably be done by another method.

It is interesting that the integral on the left side of (42) arises in the evaluation (51). It may be possible, as with (42), to take $y = T^{\theta}$ with $\theta > 1/2$ in this problem which would lead to an improvement in (48).

The last problem we consider is the existence of small and large gaps between the ordinates of consecutive zeros of $\zeta(s)$. Let γ and γ' denote ordinates of consecutive zeros of $\zeta(s)$ with $0 < \gamma \leqslant \gamma'$. Then the average value of

$$(\gamma' - \gamma)\frac{\log \gamma}{2\pi} \tag{53}$$

is 1. Let

$$\lambda = \limsup\, (\gamma' - \gamma)\frac{\log \gamma}{2\pi} \tag{54}$$

and

$$\mu = \liminf\, (\gamma' - \gamma)\frac{\log \gamma}{2\pi}\,. \tag{55}$$

Selberg [20] remarks that $\mu < 1$ and $\lambda > 1$ can be shown (unconditionally). Montgomery [15] obtained

$$\text{(RH)} \qquad \mu < 0.68 \tag{56}$$

while Mueller [18], using results from Gonek's thesis (see [8]) obtained

$$\text{(RH)} \qquad \lambda > 1.9\,. \tag{57}$$

Montgomery and Odlyzko [17] showed

$$\text{(RH)} \qquad \mu < 0.5179\,, \quad \lambda > 1.9799\,. \tag{58}$$

Recently, Conrey, Ghosh, and Gonek [7] have proven

$$\text{(RH)} \qquad \mu < 0.5172\,, \quad \lambda > 2.337\,. \tag{59}$$

Our idea is based on that of Mueller [18] and has surprising similarity to the method of Montgomery and Odlyzko [17] which appears to be much different at the outset. We assume the Riemann Hypothesis and consider

$$M_1 = \int_{-\beta}^{\beta} \sum_{T < \gamma \leqslant 2T} \left| A(\tfrac{1}{2} + i \;\; + i\alpha)\right|^2 d\alpha \tag{60}$$

where $\beta = \pi b/L$, $L = \log T$, and

$$A(s) = \sum_{n \leqslant N} a(n) n^{-s} \tag{61}$$

is a Dirichlet polynomial of length

$$N = T^{1-\delta} \tag{62}$$

where $\delta > 0$ is small. We compare this to

$$M_2 = \int_{T}^{2T} \left| A(\tfrac{1}{2} + it)\right|^2 dt\,. \tag{63}$$

If for some choice of A,

$$M_1 < M_2 \tag{64}$$

then $\zeta(s)$ has a pair of consecutive zeros with ordinates in $[T, 2T]$ which are farther apart than $2\pi b/L$, that is, farther apart than b times the average. If

$$M_2 < M_1 \tag{65}$$

then $\zeta(s)$ has a pair of consecutive zeros with ordinates in $[T, 2T]$ which are nearer to each other than b times the average. We can carry out the estimation of M_1 and M_2 asymptotically for any arithmetical function $a(n)$ with

$$a(n) \ll_{\varepsilon} n^{\varepsilon} . \tag{66}$$

This leads to a formula which is equivalent to (20) and (21) of Montgomery and Odlyzko [17]. To obtain long gaps we take

$$a(n) = d_r(n) , \tag{67}$$

the coefficient of n^{-s} in the Dirichlet series for

$$\zeta(s)^r \tag{68}$$

with $r = 2.2$. To obtain short gaps we take

$$a(n) = \lambda(n) d_r(n) \tag{69}$$

where λ is Liouville's function and $r = 1.1$. This leads to (59).

School of Mathematics
The Institute for Advanced Study
Princeton, New Jersey 08540

Present address:
Department of Mathematics
Oklahoma State University
Stillwater, Oklahoma 74074

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REFERENCES

1. Anderson, R.J., Simple zeros of the Riemann zeta-function, J. Number Th., 17 (1983), 176-182.
2. Balasubramanian, R., J.B. Conrey, and D.R. Heath-Brown, Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial, J. Reine Angew. Math., to appear.
3. Bombieri, E., A lower bound for the zeros of Riemann's zeta-function on the critical line (following N. Levinson), Seminaire Bourbaki 469 (1975), 176-181.
4. Conrey, J.B., Zeros of derivatives of Riemann's xi-function on the critical line, J. Number Th., 16 (1983), 49-74.
5. ------, Zeros of derivatives of Riemann's xi-function on the critical line II, J. Number Th., 17 (1983), 71-75.
6. ------, A. Ghosh, and S.M. Gonek, Simple zeros of the zeta function, preprint.
7. ------, ------, and ------, A note on gaps between zeros of the zeta function, Bull. London Math. Soc., 16 (1984), 421-424.
8. Gonek, S.M., Mean values of the Riemann zeta-function and its derivatives, Invent. Math., 75 (1984), 123-141.
9. Heath-Brown, D.R., Simple zeros of the Riemann zeta-function on the critical line, Bull. London Math. Soc. 11 (1979), 17-18.
10. Hooley, C., On the greatest prime factor of a cubic polynomial, J. Reine Angew. Math., 303/304 (1978), 21-50.
11. Levinson, N., More than one-third of zeros of Riemann's zeta-function are on $\sigma = 1/2$, Adv. Math. 13 (1974), 383-436.
12. ------, Generalization of recent method giving lower bound for $N_0(T)$ of Riemann's zeta-function, Proc. Nat. Acad. Sci. USA 71, No. 10 (1974), 3984-3987.

13. ------, Deduction of semi-optimal mollifier for obtaining lower bound for $N_0(T)$ for Riemann's zeta-function, Proc. Nat. Acad. Sci. USA 72, No. 1 (1975), 294-297.

14. Lou Shi-Tuo, A lower bound for the number of zeros of Riemann's zeta-function on $\sigma = 1/2$, Recent Progress in Analytic Number Theory, Academic Press, London, 1981; vol. 1, 319-324.

15. Montgomery, H.L., The pair correlation of zeros of the zeta function, Proc. Sympos. Pure Math. 24 (1973), 181-193.

16. ------, Distribution of the zeros of the Riemann zeta function, Proceedings of the International Congress for Mathematicians, Vancouver, B.C., 1974, vol. 1, 379-381.

17. ------ and A. Odlyzko, Gaps between zeros of the zeta function, to appear.

18. Mueller, J.H., On the difference between the consecutive zeros of the Riemann zeta function, J. Number Theory, 14 (1982), 327-331.

19. Selberg, A., On the zeros of Riemann's zeta-function, Skr. Norske, Vid. Acad. Oslo c No. 10 (1942), 1-59.

20. ------, The zeta-function and the Riemann hypothesis, Skandinaviske Mathematiker Knogres 10 (1946), 187-200.

Notes added in proof

We can now give a simpler proof of (16) which avoids the use of the approximate functional equation for $G(s)$. Instead, it relies on the determination of the poles of a function similar to Estermann's $\Sigma d(n)e(nh/k)n^{-s}$ for the main terms and the large sieve for the error terms (see [21]).

An even easier approach to [30] is as follows. Suppose that $\eta(s)$ is entire and is imaginary on the $\frac{1}{2}$-line and that g_0 is a complex number which is not purely imaginary. Then, on the $\frac{1}{2}$- line, $g_0\,\xi(s) + \eta(s)$ is imaginary if and only if $\xi(s) = 0$, since $\xi(\frac{1}{2} + it)$ is real. We take $\eta(s) = \Sigma g_n \xi^{(n)}(s) L^{-n}$ where g_n is real if n is odd and g_n is imaginary if n is even. Then it suffices to bound the change in argument of $(g_0\,\xi(s) + \eta(s))/H(s)$ as in (30).

We have succeeded in proving (48) subject only to RH. The proof of this is similar to Vaughan's proof of Bombieri's theorem in that it uses an analogous identity and large sieve estimates; however, here we must bound mean sixth powers of L-functions at one stage. By a similar method we can show that at least 1/3 of the zeros of the Riemann zeta-function are not zeros of any given Dirichlet L-function (on RH). In conjunction with (48) this implies that a positive proportion (at least 1/54) of the zeros of the Dedekind zeta-function of a quadratic field are simple (on RH). (See [24].)

Combining the techniques of [6] and [7] we can now improve (59) to $\lambda > 2.68$ on RH. To do this we replace $A(s)$ of (61) by $\zeta(s) \Sigma n^{-s}$ where the sum is for $n \leqslant T^{\theta}$ and $\theta < \frac{1}{2}$. (See [23].)

In [22] we give upper and lower bounds for the proportion of gaps between consecutive zeros of the zeta-function which are less than α times the average spacing. These bounds are non-trivial for $0.77 < \alpha < 1.33$.

21. Conrey, J.B., and A. Ghosh, A simpler proof of Levinson's theorem, Proc. Cambridge Phil. Soc., to appear.

22. ------, ------, D. Goldston, S.M. Gonek, and D.R. Heath-Brown, Distribution of gaps between zeros of the zeta-function, Quarterly J. of Math., to appear.

23. ------, ------, and S.M. Gonek, Large gaps between zeros of the zeta-function, preprint.

24. ------, ------, and ------, Simple zeros of the Dedekind zeta-function of a quadratic field, in preparation.