Simple Zeros Of The Zeta - Function Of A Quadratic Number Field, II

by

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1. Introduction.

Let K be a fixed quadratic extension of Q and write $\zeta_K(s)$ for the Dedekind zeta - function of K, where $s = \sigma + i\,t$. It is well-known, and easy to prove, that the number $N_K(T)$ of zeros of $\zeta_K(s)$ in the region $o < \sigma < 1$, $o < t \leq T$ satisfies

(1.1) $$N_K(T) \sim \frac{I}{\Pi} \log T$$

as $T \to \infty$. On the other hand, not much is known about the number of these zeros, $N_K^*(T)$, that are simple. Indeed, it was only recently that the present authors [2] should that

$$N_K^*(T) \gg T^{6/11}$$

and, if the Lindelof hypotheses is true, that

$$N_K^*(T) \gg_{\varepsilon} T^{1-\varepsilon}$$

for any $\varepsilon > o$. Before this, it was not even known whether $\zeta_K(s)$ has infinitely many zeros in $o < \sigma < 1$. In this paper we shall prove that if the Riemann hypothesis (RH) is true for $\zeta(s)$, Riemann zeta-function, then a positive proporation of the zeros of $\zeta(s)$, are simple. More precisely we have

THEOREWM 1. Assume that RH is true for $\zeta(s)$. Then

$$N_K^*(T) \geq \left(\frac{1}{54} + o(1)\right) N_K(T)$$

as $T \to \infty$.

Remark. As we shall see below, if we assume the Riemann hypothesis for $\zeta_K(s)$, the constant 1/54 can be replaced by 1/27.

In the case of the Riemann zeta-function, there are two known methods for proving that a positive proportion of the zeros are simple. One is the pair correlation method of Montgomery [11], the other is the method of Corey, Ghosh, and Gonek [1]. Let N(T) and N*(T) denote the number of zeros and simple zeros, respectively, of $\zeta(s)$ in $o < \sigma < 1$, $o < t \leq T$. Then

(1.2) $$N(T) \sim \frac{T}{2P} \log T$$

and, if RH is true, one can show that

(1.3) $$N^*(T) \geq (.68 + O(1))\, N(T)$$

by Montgomery's method, and that

(1.4) $$N^*(T) \geq (\frac{19}{27} + o(1))\, N(T)$$

by the authors' method. However, neither approach seems to work for $\zeta_K(s)$. In both cases the cause of failure is due to the presence of a $\Gamma(s)$ factor in the functional equation for $\zeta_K(s)$ as apposed to a Γ (5/2) factor in the equation for the zeta-function.

The present method overcomes this difficulty by exploriting the factorization

(1.5) $$\zeta_K(s) = \zeta(s)\, L(s,\chi);$$

here χ is the quadratic (Kronecker) character of the field K and $L(s,\chi)$ is thee associated Dirichlet L-function. Unfortunateely, our approach ahs the drawback that it will not apply to functions like the Dinchlet series associated with wsp forms, for; although these functions also have a $\Gamma(s)$ term in their functional equations, they do not factor as a product of two "natural" Dirichlet series.

To establish Theorem 1 we shall require the following result which is of interest in its own right.

THEOREM 2. Assume RH for $\zeta(s)$ and let $\rho = \frac{1}{2} + i\gamma$ denote the typical nontrivial zero of $\zeta(s)$. Then if χ is any nonprincipal character (not neccessarily quadratic), we have

$$\sum_{\substack{o < \gamma \leq T \\ L(\rho,\chi) = 0}} 1 \leq (2/3 + O(1))\, N(T)$$

s $T \to \infty$. That is, at most two-thirds of the zeros of $\zeta(s)$ are also zeros of $L(s,\chi)$.

With a lot more work, we could actually show that any two L-functions with inequivalent characters have at most two-thirds of their zeros in common, provided the Riemann hypothesis holds for one of them. A result of this type has also been given by A. Fujii [5] using a method different from ours. Moreover, his result is unconditional. However, his constant (which he does not calculate) is presumably quite small and would therefore not serve to prove Theeorem 1.

To provee Theoreem 1 we first observee from (1.5) that a zero of $\chi_K(s)$ is simple if and only if it is

1) a simple zero of $\chi(s)$ and not a zero of $L(s,\chi)$ or

2) a simple zero of $L(s,\chi)$ and not a zero of $\zeta(s)$. Furthermore, these two conditions are mutually exclusive. Now by (1.4) and Theorem 2, the number of zeros satisfying 1) is $\geq$ (19/27 - 2/3 + O(1)) N(T) = (1/27 + O(1) N(T). But N(T) $\frac{1}{2} N_K(T)$ by (1.1) and (1.2). Hence, Theorem 1 follows.

Notice that we could have appealed to (1.3) instead of (1.4) with some loss in the constant. Also notice that we have assumed RH only for $\zeta(s)$ and not for $L(a,\chi)$. If one assumes it for both functions (or, equivalently, for $\zeta_K(s)$), it can be shown by the method in [1] that 19/27ths of the zeros of $L(s,\chi)$ are simple, and by the method in this paper that at most 2/3rds of the zeros of $L(s,\chi)$ are zeros of $\zeta(s)$. In this way one can count the simple zeros of $\zeta_K(s)$ of type 2) above, thereby doubling the constant in Theorem 1.

Theorem 2 also has an application to the Harwitz zeta-function defined for $o < \alpha \leq 1$ by

$$\zeta(s,\alpha) = \sum_{n=1}^{\infty} (n+\alpha)^{-s} \qquad (\sigma > 1).$$

The meromorphic continuation of $\zeta(s.\alpha)$ has $- \frac{I}{2\Pi}\log T$ zeros in the strips $o \leq t < T$ and, if $\alpha \neq 1/2$ or 1, the number of these zeros to the right of the line $\sigma = 1/2 + \varepsilon$ $(\varepsilon > o)$ is $O(T)$ (see for example [7] or [15]. Although this means the Riemann hypothesis is generally false for $\zeta(s,\alpha)$, it could still be the case that there are $\frac{I}{2\Pi}\log T$ zeros on the line $\sigma = 1/2 + \varepsilon(\varepsilon>0)$ is

$O(T)$ (see for example [7] or [15]). Although this means the Riemann hypothesis is generally false for $\zeta(s,\alpha)$, it could still be the case that there are $\frac{I}{2\,\Pi}$ log T zeros in the strip $o < t \leq T$ and, if $\alpha \neq 1/2$ or 1, the number of these zeros to the right of the line $\sigma = 1/2 + \varepsilon$ $(\varepsilon > o)$ is $O(T)$ (see for example [7] or [15]). Although this means the Riemann hypothesis is generally false for $\zeta(s,\alpha)$, it could still be the case that there are $\frac{I}{2\,\Pi}$ log T zeros on the line $\sigma = 1/2$ (up to height T). However, S.M. Gonek [8] has shown that when α is a reduced fraction with denominator 3,4, or 6, there exists a positive constant c such that at least $c\,\frac{I}{2\Pi}\log T$ zeros are off the line

$\sigma = 1/2$. This is unconditional but the constant is amall (it is the constant in Fujii´s resultmentioned previously). Using Theorem 2 and the method of [8], one may easily deduce THEOREM 3. assume RH. If $\alpha = 1/3, 2/3, 1/4, 3/4, 1/6,$ or $5/6$, then at least one-third of the zeros of $\zeta(s,\alpha)$ lie off the line $\sigma = 1/2$.

In the next section we begin the proof of Theorem 2.

2. Beginning of the proof of Theorem 2

Throughout, T is large, $L = \log T$, and ε ισ αν αθβιτθαθιωυ σμαωω ροσιτιχε νξμβεθ τηοξλη νοτ νεψεσσαθιωυ τηε σαμε ονε ατ εαψη οψψξθενψε˜ +στιματεσ φερενφινλ ιμρωιψιτωυ ον ε will be denoted by O_ε or $\ll\varepsilon$.

Since it clearly suffices to prove Theorem 2 for a primitive character χ and its modulus q will be fixed from now on. Consequently, the constants implied by the symbols 0 and << may depend on q and χ.

Let

$$A(s,\chi) = \sum_{k \le y} a(k)\, k^{-s},$$

where

$$a(k) = \mu(k)\, \chi(k) \left(1 - \frac{\log k}{\log y}\right)$$

and

$$y = T\eta;$$

here μ is the Mobius function and $\eta \in (0, 1/2)$ will be selected near the end of the proof. By the Canchy-Schwarz inequality we have

$$\left(\sum_{0 < \gamma \le T} \left| L(1/2 + i\gamma, \chi)\, A(1/2 + i\gamma, \chi) \right| \right)^2$$

$$\le \sum_{0 < \gamma \le T} 1 \sum_{0 < \gamma \le T} \left| L(1/2 + i\gamma, \chi)\, A(1/2 + i\gamma, \chi) \right|^2 ,$$

with γ running through the ordinates of the zeros of $\zeta(s)$. The purpose of $A(s,\chi)$ here is to mollify $L(s,\chi)$ and thereby sharpen the inequality. We rewrite this as

$$(2.1) \qquad \frac{|\underline{N}|^2}{\underline{D}} \le \sum_{\substack{0 < \gamma \le T \\ L(1/2 + i\gamma,\, \chi) \neq 0}} 1$$

where

$$(2.2) \qquad \underline{N} = \sum_{0 < \gamma \le T} L(1/2 + i\gamma,\, \chi) \; A(1/2 + i\gamma,\, \chi)$$

and

$$(2.3) \qquad \underline{D} = \sum_{0 < \gamma \le T} \left| L(1/2 + i\gamma,\, \chi) \; A(1/2 + i\gamma,\, \chi) \right|^2 .$$

The remainder of the paper is concerned with the estimation of these two expressions. We shall show that on RH, if $y = T^{1/2\,\varepsilon}$, then

(2.4) $$\underline{N} \sim \frac{I}{2\Pi} L$$

and

(2.5) $$\underline{D} \sim 3\frac{T}{2\Pi} L$$

as $T \to \infty$. Combined with (2.1) and (1.2), these estimates imply the result.

The first step in treating $\underline{N}$ and $\underline{D}$ is to express them as contour integrals by means of Canchy's residue theorem. To this end let TT denote a sequence of numbers T_n such that

$$n < T_n \leq n + 1 \qquad (n = 3, 4, \ldots .)$$

and

(2.6) $$\frac{\zeta^1}{\zeta} (\sigma + i\, T_n) \ll \log^2 T_n$$

uniformly for $-1 \leq \sigma \leq 2$ (confer Davenport [4; p.108]). In particular, T_n

is not the ordinate of any zero of $\zeta(s)$. <u>Until the very end of the paper, we shall always assume that $T \in TT$.</u>

Next set

$$a = 1 + L^{-1}$$

and let $\mathcal{R}$ be a positively oriented rectangle with vertices at $a + i$, $a + iT$, $1-a + iT$, and $1-a + i$. Then on RH we have

$$(2.7) \qquad \underline{N} = \frac{1i}{2\pi i} \int_{\mathcal{R}} \frac{\zeta'}{\zeta}(s)\, L(s,\chi)\, A(s,\chi)\, ds$$

and

$$(2.8) \qquad \underline{D} = \frac{1}{2\pi i} \int_{\mathcal{R}} \frac{\zeta'}{\zeta}(s)\; L(s,\chi)\; L(1-s,\bar{\chi})\; A(s,\chi)\; A(1-s,\bar{\chi})\, ds \, .$$

Let us consider $\underline{N}$ first. As it happens, it is easier to work with

$$\underline{N} = \frac{-1}{2\pi i} \int_{\mathcal{R}} \frac{\zeta'}{\zeta}(1-s)\; Ls,\chi)\; As,\chi)\, ds \, .$$

This is equivalent to (2.7) because $\frac{\zeta'}{\zeta}(s)$ and $-\frac{\zeta'}{\zeta}(1-s)$ have the same poles and residues inside $\mathcal{R}$. Now for s inside or on $\mathcal{R}$,

$$(2.9) \qquad A(s,\chi) \ll_{\varepsilon} y^{1-\sigma+\varepsilon}$$

and

$$(2,10) \qquad L(s,\chi) \ll_\varepsilon T^{\frac{1}{2}(1-\sigma)+\varepsilon} .$$

The first bound is trivial; the scond follows in the same way as the analogous bound for $\zeta(s)$ (see Titchmarsh [13; pp. 81-82]). These borinds and (2.6) imply that the top and bottom edges of R contribute

$$\ll_\varepsilon y\, T^{1/2+\varepsilon}$$

to N.

For the left edge of $\mathcal{R}$ we replace s by 1-s and find that

$$\frac{-1}{2\pi i}\int_{1-a+iT}^{1-iT} \frac{\zeta'}{\zeta}(1-s)\, L(s,\chi)\, A(s,\chi)\, ds$$

$$= \frac{-1}{2\pi i}\int_{a-i}^{a-iT} \frac{\zeta'}{\zeta}(s)\, L(1-s,\chi)\, A(1-s,\chi)\, ds$$

$$= \frac{1}{2\pi i}\int_{a+1}^{a+iT} \frac{\zeta'}{\zeta}(s)\, L(1-s,\bar{\chi})\, A(1-s,\bar{\chi})\, ds.$$

For the right-hand side of $\mathcal{R}$ we use the identity

$$(2.11) \qquad -\frac{\zeta'}{\zeta}(1-s) = \frac{\zeta'}{\zeta}(s) - \frac{X'}{X}(1-s)$$

This follows from the functional equation

$$(2.12) \qquad \zeta(1-s) = X(1-s)\ \zeta(s) ,$$

where

$$(2.13) \qquad X(1-s) = \pi^{1/2-s}\ \Gamma(s/2)\ /\ \Gamma(\tfrac{1-s}{2}) .$$

We then find that the integral along this side is

$$\frac{1}{2\pi i} \int_{a+i}^{a+iT} \left(\frac{\zeta'}{\zeta}(s) - \frac{X'}{X}(1-s)\right) L,(s,\chi)\ A(s,\chi)\ ds .$$

Thus, we may write

$$(2.14) \qquad \underline{N} = \underline{\bar{N}}_1 + \underline{N}_2 - \underline{N}_3 + O_\varepsilon(yT^{1/2+\varepsilon})$$

with

(2.15) $$\frac{1}{2\pi i}\int_{a+i}^{a+iT} \left(-\frac{\zeta'}{\zeta}(s)\right) L(1-s,\bar{\chi})\ A(1-s,\bar{\chi})\ ds\ ,$$

(2.16) $$\underline{N}_2 = \frac{1}{2\pi i}\int_{a+1}^{a+iT} \frac{\zeta'}{\zeta}(s)\ L(s,\chi)\ A(s,\chi)\ ds$$

and

(2.17) $$\underline{N}_3 = \frac{1}{2\pi i}\int_{a+i}^{a+iT} \frac{X'}{X}(1-s)\ L(s,\chi)\ A(s,\chi)\ ds\ .$$

We now come to $\underline{D}$. The top and bottom edges of R contribute

$$\ll_\varepsilon y\ T^{1/2+\varepsilon}$$

to $\underline{D}$ by (2.6), (2.9), and (2.10). Replacing s by 1-s and use (2.11), we find that the contribution of the left edge of R equals

$$\frac{1}{2\pi i}\int_{a-i}^{a-iT} \frac{\zeta'}{\zeta}(1-s)\ L(1-s,\chi)\ L(s,\bar{\chi})\ A(1-s,\chi)\ A(s,\bar{\chi})\ ds$$

$$= \frac{1}{2\pi i}\int_{a-i}^{a-iT} \left(-\frac{\zeta'}{\zeta}(s) + \frac{X'}{X}(1-s)\right)\ L(1-s,\chi-\ L(s,\bar{\chi})\ A(1-s,\chi)\ A(s,\bar{\chi})\ ds.$$

If we write

$$(2.18)\quad \underline{D}_1 = \frac{1}{2\pi i}\int_{z+i}^{a+iT} \frac{\zeta'}{\zeta}(s)\ L(s,\chi)\ L(1-s,\chi)\ a(s,\chi)\ A(1-s,\bar{\chi})\ ds$$

and

$$(2.19)\quad \underline{D}_2 = \frac{1}{2\pi i}\int_{a+i}^{a+iT} \frac{X'}{X}(1-s)\ L(s,\chi)\ L(1-s,\bar{\chi})\ A(s,\chi)\ A(1-s,\bar{\chi})\ ds$$

the integral above equals

$$\bar{\underline{D}}_1 - \bar{\underline{D}}_2 .$$

Notice that $\underline{D}_1$ is also the contribution of the right-hand side of R to $\underline{D}$. Hence, on combining these results, we obtain

$$(2.20)\qquad \underline{D} = 2\ \mathrm{Re}\ \underline{D}_1 - \bar{\underline{D}}_2 + O_\varepsilon\ (uT^{1/2+\varepsilon}).$$

We conclude this section by introducing some useful notation and formulae.

For convenience we use the symbol $\int_{(c)}$ to denote $\int_{c-i\infty}^{c+i\infty}$ and $\sum_{m=1}^{r}{}^{1}$ to denote $\sum_{\substack{m=1\\(m,r)=1}}^{r}$. As usual we write e(x) instead of $e^{2\pi ix}$.

Ramonujon´s sum is

$$c_r(a) = \sum_{m=1}^{r}{}^{1}\ e(\frac{ma}{r})$$

and one shows (see[4; pp. 148-149]) that

$$c_r(a) = \sum_{\substack{d|r \\ d|a}} d \;\mu\left(\frac{r}{d}\right). \tag{2.21}$$

Unalogously, we define

$$C_x(a) = \sum_{m=1}^{q} \chi(m)\, e\left(\frac{ma}{q}\right)$$

so that Ganss' sum is

$$\tau(x) = C_x(1).$$

It is not difficult to show (confer [12; p.358]) that

$$c_x(a) = \begin{cases} \bar{\chi}(a)\,\tau(\chi) & \text{if } (a,q) = 1, \\ 0 & \text{if } (a,q) > 1. \end{cases} \tag{2.22}$$

We shall write the functional equation for $L(s,\chi)$ in the form

$$L(1-s),\bar{\chi}) = X(1-s,\chi)\;\; L(s,\chi), \tag{2.23}$$

where

$$(2.24)\qquad X(1-s,\chi) = \frac{i}{\tau(\chi)}\, q^{s}\, \pi^{1/2-s}\, \Gamma\left(\frac{s+\alpha}{2}\right)/\Gamma\left(\frac{1-s+\alpha}{2}\right)$$

and

$$\alpha = \begin{cases} 0 & \text{if } \chi(-1) = 1, \\ 1 & \text{if } \chi(-1) = -1. \end{cases}$$

Observe that if $q = 1$, χ is principal and

$$X(1-s,\chi) = X(1-s),$$

where $X(1-s)$ is the factor in the functional equation for $\zeta(s)$ (see (2.12) and (2.13)).

Finally, we adopt the notation

$$F_n(s) = \prod_{p/n} (1-p^{-s}),$$

$$F_n(s,\chi) = \prod_{p/n} (1-\chi(p)\, p^{-s}).$$

The next section contains the necessary lemmas for estimating the $\underline{N}_i$ and $\underline{D}_i$.

3. Lemmas.

Lemma 1. Let r be a positive real number and suppose that $X(1-s,\chi)$ is given by (2.24). Then for $a = 1 + L^{-1}$ and T large, we have

$$\frac{1}{2\pi i}\int_{a+i}^{a+iT} X(1-s,\chi)\, r^{-s}\, ds$$

$$= \begin{cases} \dfrac{X(-1)}{\tau(x)} \; e(\dfrac{-r}{q}) + \dfrac{(q/r)^a}{\tau(x)} \; E(\dfrac{r}{q}, T) & \text{if } r \leq \dfrac{qT}{2\Pi}, \\ \dfrac{(q/r)^a}{\tau(x)} \; E(r/q, T) & \text{if } r > \dfrac{qT}{2\Pi}, \end{cases}$$

where

$$E(r/q, T) \ll T^{1/2} + \frac{T^{3/2}}{|T-2\Pi r/q| + T^{1/2}}.$$

PROF. When $q = 1$, $X(1-s,\chi) = X(1-s)$ and, except for minor modifications, the result is essentially Lemma 2 of Gonek [9]. Suppose them that $q > 1$. From (2.24)

(3.1) $$X(1-s,\chi) = \frac{i\alpha}{\tau(\chi)} \; q^s \; \pi^{1/2 s} \; \frac{\Gamma(\frac{s+\alpha}{2})}{\Gamma(\frac{1-s+\alpha}{2})}.$$

Id $\alpha = o$ this is just

$$X(1-s, \chi) = \frac{1}{\tau(\chi)} \; q^s \; X(1-s).$$

If $\alpha = 1$ we use the formula

$$\Gamma(s) \; \Gamma(1-s) - \frac{\Pi}{\sin \pi s}$$

and find that

$$\Gamma(\frac{s+1}{2}) = \frac{\Pi}{\cos \frac{\pi s}{2} \quad \Gamma(\frac{1-5}{2})}$$

$$\Gamma (1- \frac{s}{2}) = \frac{\Pi}{\sin \frac{\pi s}{2} \quad \Gamma(\frac{s}{2})} .$$

Hence

$$\Pi^{1/2 - s} \frac{\Gamma(\frac{s+1}{2})}{\Gamma(1- \frac{s}{2})} + \text{ton} \frac{\pi s}{2} X(1-s) = X(1-s) (i + 0(e^{-\pi t}))$$

for $t > 0$ and bounded σ. Inserting this in (3.1), we see that

$$X(1-s,\chi) + \frac{-1}{\tau(\chi)} q^s X(1-s) (1 + 0(e^{-\pi t})) .$$

Thus, in either case

$$X(1-s,\chi) = \frac{\chi(-1)}{\tau(\chi)} q^s X(1-s) (1 + 0(e^{-\pi t})). .$$

Using this and the case $q = 1$ of the lemma, we easily obtain the result.

LEMMA 2. Let $d(n)$, $B(n)$ be arithmetic functions such that $d(n) \ll 1$ and $B(n) \ll d_r(n) \log^l n$, where $d_r(n)$ is the coefficient of n^{-s} in $\zeta^r(s)$ and l is a non-negative integer. Also let $a = 1 + L^{-1}$. Then if $1 < x \leq T$,

$$\frac{1}{2\pi i}\int_{a+i}^{a+iT} X(1-s,\chi)\ \Big({}_{k\leq x}\ \alpha(k)\,k^{s-1}\Big)\ \Big({}_{j=1}^{\infty}\ B(j)\ j^{-s}\Big)\ ds$$

$$= \frac{\chi(-1)}{\tau(\chi)}\ {}_{k\leq x}\ \frac{\alpha(k)}{k}\ {}_{j\leq\frac{qkT}{2\pi}}\ B(j)\ e\Big(\frac{1j}{qk}\Big)$$

$$+\ O_{\varepsilon}\ (x\ T^{1/2+\varepsilon}).$$

PROOF. The integral equals

$${}_{k\leq x}\ \frac{\alpha(k)}{k}\ {}_{j=1}^{\infty}\ B(j)\ \Big(\frac{1}{2\pi i}\int_{a+i}^{a+iT} X(1-s,\chi)\ \Big(\frac{j}{k}\Big)^{-s}\ ds\Big),$$

which by Lemma 1 is equal to

$$\frac{\chi(-1)}{\tau(\chi)}\ {}_{k\leq y}\ \frac{\alpha(k)}{k}\ {}_{j\leq\frac{qkT}{2\pi}}\ B(j)\ e\Big(\frac{-j}{qk}\Big)$$

$$+\ O\Big({}_{k\leq x}\ \frac{|\alpha(k)|}{k}\ {}_{j=1}^{\infty}\ |B(j)|\ \Big(\frac{j}{qk}\Big)^{-a}\ E\Big(\frac{j}{qk},\ T\Big)\Big).$$

Since k^a <8 k for $k \leq x$, the error term is

$$\ll \sum_{k\le x} \sum_{j=1}^{\infty} |B(j)|\, j^{-a} \left(T^{1/2} + \frac{T^{3/2}}{|T - \frac{2\pi j}{qk}| + T^{1/2}}\right)$$

$$\ll xT^{1/2} \sum_{j=1}^{\infty} \frac{d_r(j) \log^{1} j}{j^a}$$

$$+ T^{3/2} \left|\left(\frac{d^1}{ds^1}\right) \zeta^r(s) \right|_{s=a} \ll xT^{1/2} L^{1=r} \ll xT^{1/2+\varepsilon} .$$

We write the second term as

$$T^{3/2} (\quad , + \ldots + \quad _s) ,$$

where in each $_i$ we sum over $k \le x$, and the j sum is over $(o, qkT/4\pi]$ in $_1$,

$(\frac{qkT}{4\pi}, \frac{qk}{2\pi}(T-T^{1/2})]$ in $_2$, $(\frac{qk}{2\pi}(T-T^{1/2}), \frac{qk}{2\pi}(T+T^{1/2})]$ in $_\varepsilon$, and

$(\frac{qk}{2\pi}(T+T^{1/2}), \frac{3qkT}{4\pi}]$ in $_4$, and $(\frac{3qkT}{4\pi}, \infty)$ in $_s$. First, treating $_1$ and $_s$ together, we have

$$_1 + \quad _s \ll T^{-1} \sum_{k\le x} \left(\sum_{j \le \frac{qkT}{4\pi}} + \sum_{j > \frac{3qkT}{4\pi}} \right) \frac{d_r(j) \log^1 j}{ja}$$

$$\ll xT^{-1} \sum_{j=1}^{\infty} \frac{d_r(j) \log^1 j}{ja} = xT^{-1} \left| \left(\frac{d^1}{ds^1}\right) \zeta^r(s) \right|_{s=a}$$

$$\ll \; xT^{-1} L^{r+1} \; \ll \; xT^{-1+\varepsilon} \; .$$

Next, since $d_r(j) \; \log^1 j \; \ll_\varepsilon j^\varepsilon$,

$$2 \ll_\varepsilon T^{\varepsilon-1} \quad {}_{k \leq x} \; \frac{1}{k} \quad {}_{\frac{qkT}{4\pi} < j \leq \frac{qk}{2\pi}(T-T^{1/2})} \; \frac{1}{(T - 2\pi j/qk)}$$

$$\ll_\varepsilon T^{\varepsilon-1} \quad {}_{k \leq c} \quad {}_{\frac{qkT}{4\pi} < j \leq \frac{qk}{2\pi}(T-T^{1/2})} \; \frac{1}{(qkT/2\pi - j)}$$

$$\ll_\varepsilon T^{\varepsilon-1} \quad {}_{k \leq x} \; \int_{\frac{qkT}{4\pi}}^{\frac{qk}{2\pi}(T-T^{1/2})} \frac{dn}{\frac{qkT}{2\pi} - n}$$

$$\ll_\varepsilon T^{\varepsilon-1} \quad {}_{k \leq x} \; \log T^{1/2} \; \ll_\varepsilon \; \chi T^{\varepsilon-1} \; .$$

Similarly, one obtains the same bound for 4. Finnally,

$$\varepsilon \ll_\varepsilon T^{-1/2+\varepsilon} \quad {}_{k \leq x} \quad {}_{\frac{qk}{2\pi}(T-T^{1/2}) < j \leq \frac{qk}{2\pi}(T+T^{1/2})}$$

$$\ll_\varepsilon \chi T^{1/2+\varepsilon} \; \log\left(\frac{T+T^{1/2}}{T-T^{1/2}}\right) \; \ll_\varepsilon \; xT^{-1+\varepsilon} \; .$$

Combining these bounds, we obtain the result.

LEMMA 3. Let x. be a primitive character mod q and suppose that (H,K) = (K,q)

= 1. Set

$$L(s,\chi,\frac{-H}{qK}) = \sum_{n=1}^{\infty} \chi(n)\, e(\frac{-nH}{qK})\, n^{-s} \qquad (\sigma > 1) .$$

Then L has an analytic contimation to the whole plane except for a possible pole at s=1. At this point it has the same principal part as

$$\zeta(K)\, \bar{\chi}(-H)\, \tau(\chi)\, q^{-s}\, \zeta(s),$$

where $\zeta(K) = 1$ if $K=1$ and is o otherwise.

PROOF. Let

$$\zeta(s,a,qK) = \sum_{n \equiv a \bmod qK} n^{-s} \qquad (\sigma > 1).$$

Then

$$\zeta(s,a,qK) = (qk)^{-s}\, \zeta(s,\frac{a}{qK}),$$

where $\zeta(s, \frac{a}{qK})$ is Hurwitz´s zeta-function.

Tus $\zeta(s,a,qK)$ is everywhere regular except at s=1 where it has a pole with the same principal part as $(qK)^{-s}\, \zeta(s)$.

Now

$$L(s,\chi,\frac{-H}{qK}) = \sum_{a=1}^{qK} \chi(a)\ e(\frac{-aH}{qK})\ \zeta(s,a,qK),$$

so L is also regular everywhere except possibly at s=1. Writing the sum as

$$\sum_{a=1}^{qK} x(a)\ e(\frac{-aH}{qk})\ (\zeta(s,a,qK) = (qK)^{-s}\ \zeta(s))$$

$$+ (qK)^{-s}\ \zeta(s) \sum_{a=1}^{qK} \chi(a)\ e(\frac{-aH}{qk}),$$

we observe that the first term is entire. Hence, setting a = qn + r in the second term, we see that L has the same prinicipal part at s=1 as

$$(qK)^{-s}\ \zeta(s) \sum_{r=1}^{q} \chi(r)\ e(\frac{-rH}{qK}) \sum_{n=0}^{K-1} e(\frac{-nH}{qk})$$

$$= \begin{cases} q^{-s}\ \zeta(s)\ c_\chi(-H) & \text{if } K=1, \\ 0 & \text{if } K>1. \end{cases}$$

By (2.22), $C_x(-H) = \bar{x}(-H)\ \tau(\chi)$. Thus, the principal part is identivcal to that of

$$\zeta(K)\ \bar{\chi}(-H)\ \tau(\chi)\ q^{-s}\ \zeta(s)$$

as required.

LEMMA 4. Let x be a primitive character modulo q and write

$$R(s,\chi,\frac{1}{qk}) = - \sum_{m,n=1}^{\infty} \frac{\Lambda(m)\ \chi(n)}{(mn)^{s}} \ e(\frac{-mn}{qK}) \qquad (\sigma > 1).$$

If (K,q) = 1 and K is square-free, then R has a meromorphic continuation to the entire complex plane. Its only pole in $\sigma \geq 1$ is at s=1 where it has a pole with the same principal part as

$$\frac{n(qK)}{\phi(qK)} \ L(1,\chi) \ F_{K}(o,x) \ \frac{\zeta}{\zeta}^{1}(s) + \frac{\chi(-1)\ \tau(\chi)}{qK} (\zeta(K) \ \frac{L}{L}^{1}(1,\bar{\chi}) - \frac{\Lambda(K)}{1-\bar{x}(K)\ |K} \ \zeta(s),$$

where $\zeta(K) = 1$ if K=1 and is o otherwise.

PROOF. For $\sigma > 1$ we see that

$$(3.2) \quad R(s,\chi,-\frac{1}{qK}) = - \sum_{a=1}^{qK} L(s,\chi,\frac{-a}{qK}) \ \Lambda(s,a,qK)$$
$$= - \sum_{d/K} \ \sum_{b=1}^{aK/d1} L(s,\chi,\frac{-6}{aK/d}) \ \Lambda(s,\ bd,\ qK),$$

where $L(s,\chi,.)$ is as in Lemma 3 and

$$\Lambda(s,a,k) = \sum_{n \equiv a \bmod k} \Lambda(n)\ n^{-s} \qquad (\sigma > 1).$$

By Lemma 3, $L(s,\chi i)$ is regular in the whole plane except for a possible simple pole at s=1 with the same principal part as

$$\zeta(\tfrac{K}{d}) \quad \bar{\chi}(-6) \quad \tau(\chi)\ q^{-s} \quad \zeta(s).$$

Furthermore, it is well-know that $\Lambda(s,a,k)$ has a mesomorphic continuation to the whole plane with a simple pole at $s=1$ if and only if $(a,k) = 1$. The principal part at the pole in this use is identical to that of

$$-\frac{1}{\phi(k)} \ \frac{\zeta^{1}}{\zeta}(s)$$

From these remarks and (3.2), we see that $R(s,\chi,.)$ is meromorphic and that with the possible exception of the point $s=1$ it has no poles in $\sigma \geq 1$. To find the principal part at $s=1$ we write

$$R(s,\chi,\frac{1}{qK}) = - \quad {}_{d/K} \quad {}^{qK/d\,'}_{b=1} \quad [L(s,\chi,\frac{-6}{qK/d}) - \zeta(\tfrac{K}{d})\ \bar{\chi}(-6)\ \tau(\chi)\ q^{-s}\ \zeta(s)] \cdot [\Lambda(s,bd,qK) + \frac{\zeta(d)}{\phi(qK)} \frac{\zeta^{1}}{\zeta}(s)]$$

$$+ \quad {}_{d/K} \quad {}^{qK/d\,'}_{b=1} \quad \frac{\zeta(d)}{\phi(qK)} \ \frac{\zeta^{1}}{\zeta}(s) \quad L(s,\chi,\frac{-6}{qK/d})$$

$$- \quad {}_{d/K} \quad {}^{aK/d\,'}_{b=1} \quad \zeta(\tfrac{K}{d})\ \bar{\chi}(-6)\ \tau(\chi)\ q^{-s}\ \zeta(s)\ \Lambda(s,bd,qK)$$

$$- \quad {}_{d/K} \quad {}^{qK/d\,'}_{b=1} \quad \frac{\zeta(d)\ \zeta(K/d)}{\phi(qK)} \quad \bar{\chi}(-6) \quad \tau(\chi)\ q^{-s}\ \zeta^{1}(s)$$

$$- \ {}_{1} + \ {}_{2} - \ {}_{3} - \ {}_{4},$$

say. Clearly $\quad_1$ is regular at s=1 For $\quad_2$ we have

$$_2 = \frac{1}{\phi(qK)} \frac{\zeta^1}{\zeta}(s) \quad {}_{b=1}^{qK}{}^{1} \; L(s,\chi, -\frac{6}{qK})$$

$$= \frac{1}{\phi(q(K)} \frac{\zeta^1}{\zeta}(s) \quad {}_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad {}_{b=1}^{qK}{}^{1} \quad e(\frac{-bn}{qK})$$

$$= \frac{1}{\phi(qK)} \frac{\zeta^1}{\zeta}(s) \quad {}_{n=1}^{\infty} \frac{{}^{x}(n) \; {}^{C}{}_{qK}(n)}{n^s}$$

For $\sigma > 1$. Now by (2.21), the sum equals

$${}_{n=1}^{\infty} \frac{\chi(n)}{n^s} \quad {}_{d/qK}^{d/n} \quad d \; \eta(\frac{qK}{d})$$

$$= \quad {}_{d/qK} \quad d^{1-s} \quad \eta(\frac{qK}{d}) \quad \chi(d) \quad {}_{m=1}^{\infty} \frac{\chi(m)}{m^s}$$

$$L(s,\chi) \quad \eta(q) \quad {}_{d/K} \quad d^{1-s} \quad \eta(\frac{K}{d}) \quad \chi(d)$$

Since we may clearly assume that (d,g) = 1 and, by hypothesis, (q,K) = 1. Recalling that K is square-free and setting s=1, we see that this becomes

$$L(1,\chi) \; \eta(qK) \; \prod_{p/K}(1-\chi(p))$$

$$=L(1,\chi) \; \eta(qK) \; F_K(o,\chi).$$

Hence, $\quad_2$ has the same principal part as

$$\frac{\eta(qK)}{\phi(qK)} \quad L(1,\chi) \quad F_{\chi}(o,\chi) \quad \frac{\zeta^{1}}{\zeta}(s).$$

Next, for $\sigma > 1$

$$_{3} \quad = \quad \sum_{b=1}^{q_{1}} \quad \bar{\chi}(-b) \quad \tau(\chi) \quad q^{-s} \quad \zeta(s) \quad \Lambda(s, bK, qK)$$

$$= \tau(\chi) \quad q^{-s} \quad \zeta(s) \quad \sum_{b=1}^{q} \quad \bar{\chi}(-b) \quad \sum_{n \equiv bK (\mathrm{mod}\ qK)} \frac{\Lambda(n)}{n^{s}}$$

$$= \chi(-1) \quad \tau(\chi) \quad (qK)^{-s} \quad \zeta(s) \quad \sum_{m=1}^{\infty} \quad \frac{\Lambda(mK) \ \bar{\chi}(m)}{m^{s}} \quad .$$

The sum here is

$$\begin{cases} -\frac{L^{1}}{L}(s,\bar{\chi}) & \text{if } K=1, \\ \frac{\log p}{1 - \bar{\chi}(p)/p^{s}} & \text{if } K=p, \\ 0 & \text{otherwise} \end{cases}$$

If we now set $s=1$ everywhere except in the $\zeta(s)$ term and recall that K is square-free, we have that the principal part of $_{3}$ is the same as that of

$$\chi(-1) \quad \tau(\chi) \quad (qK)^{-1} \quad \zeta(s) \quad \left(-\zeta(K) \ \frac{L^{1}}{L}(1,\bar{\chi}) \ + \ \frac{\Lambda(K)}{1 - \bar{\chi}(K)/K}\right).$$

Finally,

$$4 = \frac{\tau(\chi)}{\phi\phi(k)} q^{-s} \quad \zeta^{1}(s) \quad {}_{b=1}^{q} \; \bar{\chi}(-b)$$
$$= 0.$$

The principal part of $R(s,\chi,.)$ is therefore the same as that of

$$\frac{\eta(qK)}{\phi(qK)} L(1.\chi) \; F_K(o,\chi) \; \frac{\zeta^1}{\zeta}(s) + \chi(-1) \frac{\tau(\chi)}{qK} \left(\zeta(K) \frac{L^1}{L}(1,\bar{\chi}) - \frac{\Lambda(K)}{1-\bar{\chi}(K)/K}\right) \zeta(s),$$

as was to be shown.

LEMMA 5. Let χ be a primitive character mod q and set

$$D(s,\chi,\frac{-H}{qK}) = {}_{n=1}^{\infty} \; d(n)\chi(n) \; e\left(\frac{-nH}{qK}\right) \; n^{-s} \qquad (\sigma > 1).$$

If K is square-free and $(H,K) = (K,q) = 1$, then D has an analytic continuation to the whole plane except for a pole at $s=1$. At this point it has the same principal part as

$$(3.3) \qquad \bar{\chi}(H) \; \chi(K) \; \tau(\chi) \; (qK)^{-s} \; \zeta^2(s) \; (F_q(s0 \; (1+\chi(-1) \; K^{1-s}) - \phi(q) \; q^{-s}).$$

PROOF. We have

$$D(s,\chi,\frac{1H}{qK}) = {}_{m,n=1}^{\infty} \; \frac{\chi(m) \; \chi(n)}{(mn)^{s}} \; e\left(\frac{-mnH}{qK}\right)$$

$$= {}_{a=1}^{qK} \; \chi(a) \; {}_{n \equiv a \bmod qK} n^{-s} \; {}_{m=1}^{\infty} \; \chi(m) \; e\left(\frac{-amH}{qK}\right) \; m^{-s}$$

$$= {}_{a=1}^{qK} \quad \chi(a) \quad \zeta(s, a, qK) \quad L(s, \chi, \frac{-aH}{qK}),$$

wher ζ and L are as in Lemma 3 and its proof. By Lemma 3 and the last line, we see that D may be analytically continued to the whole plane except possibly for the point s=1 where it may have a pole. We now determine the principal part. Write

$$D(s, \chi, \frac{-H}{qK}) =$$

$${}_{a=1}^{qK}\chi(a)[\zeta(w,a,qK)-(qK)^{-s}\zeta(s)][L(s,\chi,\frac{-aH}{qK})-\zeta(\frac{K}{(a,qK)}) \bar{\chi}(\frac{aH}{(a,qK)}) \tau(\chi) q^{-s}\zeta(s)]$$

$$+ (qK)^{-s} \quad \zeta(s) \quad {}_{a=1}^{qK} \quad \zeta(\frac{K}{(a,qK)}) \chi(a) L(s, \chi, \frac{-aH}{qK})$$

$$+ \quad q^{-s} \tau(\chi) \zeta(s) \quad {}_{a=1}^{qK} \quad \zeta(\frac{K}{(a,qK)}) \chi(a) \bar{\chi}(\frac{aH}{(a,qK)}) \quad \zeta(s, a, qK)$$

$$- (q^2K)^{-s} \quad \tau(\chi) \quad \zeta(^2(s) \quad {}_{a=1}^{qK} \quad \zeta(\frac{K}{(a,qK)}) \quad \chi(a) \bar{\chi}(\frac{aH}{(a,qK)})$$

$$= \quad {}_1 \quad + \quad {}_2 \quad + \quad {}_3 \quad - \quad {}_4 \,;$$

here $\zeta(r) = 1$ if r=1, o otherwise. By Lemma 3, ${}_1$ is regular at s=1. Next for $\sigma > 1$

$${}_2 = (qK)^{-s} \quad \zeta(s) \quad {}_{n=1}^{\infty} \quad \chi(n)n^{-s} \quad (\quad {}_{a=1}^{qK} \quad \chi(a) \; e(\frac{-naH}{qK})).$$

Setting a = qb + r, we may write the sum over a as

$$\sum_{r=1}^{q} \chi(r) \; e(\frac{-nrH}{qK}) \sum_{b=0}^{K-1} e(\frac{nbH}{K}).$$

the sum over b is K if K|nH, i.e. K@n; otherwise it is 0. Thus, by (2.22), the abovwe is

$$K \sum_{r=1}^{q} \chi(r) \; e(\frac{-nrH}{qK}) = K\, C_x\, (-H\, \frac{n}{K}) = K\, \bar{X}(-H\, \frac{n}{K})\, \tau(\chi)$$

if K|n, 0 otherwise. Hence, writing n=mK, we have

$$\Sigma_2 = q^{-s}\, K^{1-25}\, \zeta(s)\; \tau(\chi)\; \chi(K)\; \bar{\chi}(-H) \sum_{m=1}^{\infty} |\chi(m)|^2\, m^{-s}$$

$$\bar{\chi}(-H)\; \chi(K)\; \tau(\chi)\; q^{-s}\, K^{1-25}\; f_q(s)\; \zeta^2(s).$$

In order for a summand in Σ_3 to be nonzero, it is necessary that (a,qK) =K or, since we may evidently assume that (a,q) =1, that K|a.

Thus,

$$\Sigma_3 = \tau(\chi)\; q^{-s}\; \zeta(s) \sum_{b=1}^{q} \chi(bK)\; \bar{\chi}(bH)\; \zeta(s, bK, qK)$$

$$= \bar{\chi}(H)\; \chi(K)\; \tau(\chi)\; (qK)^{-s}\; \zeta(s) \sum_{b=1}^{q}{}' \; \zeta(s, b, q).$$

The sum equals

$$\sum_{\substack{n=1 \\ (n,\, q)=1}}^{\infty} n^{-s} = f_q(s)\; \zeta(s),$$

so

$$ {}_3 = \bar{\chi}(H)\ \chi(K)\ \tau(\chi)\ \ (qK)^{-s}\ F_q(s)\ \ \zeta^2(s). $$

Similarly, we see that

$$ {}_4 = \tau(\chi)\ \ (q^2K)^{-s}\ \ \zeta^2(s)\ \ {}_{b=1}^{\;q}\ \ \chi(bK)\ \ \bar{\chi}(bH) $$

$$ = \bar{\chi}(H)\ \ \chi K)\ \ \tau(\chi)\ \ \phi(q)(q^2K)^{-s}\ \ \zeta^2(s). $$

Combining these results, we find that the principal part of D(s,.,1) at s=1 equals that of

$$ \bar{\chi}(H)\ \ \chi(K)\ \ \tau(\chi)\ \ (qK)^{-s}\ \ \zeta^2(s)(F_q(s)\ \ (1 + \chi(-1)K^{1-s}) - \phi(q)\ q^{-s}), $$

as was to be shown.

LEMMA 6. Let χ be a primitive character modulo q and set

$$ Q(s,\ \chi,\ \frac{-H}{qK}\ ) \ = - \ {}_{m,n=1}^{\;\infty} \frac{\Lambda(m)\ \ d(n)\ \ \chi(n)}{(mn)^s}\ \ \ e(-\frac{mnH}{qK})\ \ \ (\ \sigma > 1). $$

Then if H, K, and q are pairwise coprime and K is square-free, Q has a meromorphic contineration to the whole plane. Its only pole in $\sigma \geq 1$ is at s=1 where it has a pole whose principal part is the same as that of

$$ \bar{\chi}(H)\ \chi(K)\ \tau(\chi)\ (qK)^{-s}\ \zeta^2(s)\ \ G_K(s,\ \chi) + \frac{\eta(qK)\ \ g(K)}{\phi(qK)}\ \ L^2(1,\chi)\ \frac{\zeta}{\zeta}^{1}(s), $$

where

(3.4) $\qquad g(K) = \Pi_{p/K}\ \ (1-2\chi(p) + \frac{\chi^2(p)}{\phi})$

and

$$(3.5)\qquad G_K(s) = \left(\frac{L^1}{L}(s,\bar{\chi}) + \frac{F_K^1}{F_K}(s,\bar{\chi})\right)\left(F_q(s)\,(1 + \chi(-1)\,K^{1-s}) - \phi(q)\,q^{-s}\right)$$

$$- \frac{\bar{\chi}(p)\log p}{1-\bar{\chi}(p)/p^s}\left(F_q(s)\,\left(1+ \chi(-1)\left(\frac{K}{p}\right)^{1-s}\right) - \phi(q)\,q^{-s}\right).$$

PROOF. For $\sigma > 1$ [illegible]ave

$$(3.6)\qquad Q(s, \chi, \frac{-H}{qK}) \quad \sum_{a=1}^{qK} D(s,\chi,\frac{-aH}{qK})\ \Lambda(s, a, qK)$$

$$= \sum_{d/K} \sum_{[illegible]} D(s,\chi,\frac{-bH}{qK/d})\ \Lambda(s,bd, qK),$$

where $D(s, 1, 1)$ is [illegible]n Lemma 5,

$$[illegible]a,k) = \sum_{n\equiv a(\mathrm{mod}\ k)} \Lambda(n)\, n^{-s} \qquad (\sigma > 1).$$

It is well known that [illegible]s,a,k) has a meromorphic continuation to the whole plane with a simple p[illegible] at $s=1$ if and only if $(a,k) = 1$. Also $D(s,\chi, .)$ is regular everywhere exc[illegible] for a possible double pole at $s=1$ by Lemma 5. Thus, $Q(s, \chi, .)$ is meromorp[illegible] in the plane and has no poles in $\sigma \geq 1$ except possibly at $s=1$.

To find the principal p[illegible] at this point, first note that $\Lambda(s,,k)$ has the same principal part at $s=1$ a[illegible]$((a,k))\ \frac{1}{\phi(k)}\ \frac{\zeta^1}{\zeta}(s)$. Thus, if we call the expression in (3.3) [illegible]$\chi,\frac{-H}{qK})$, then by (3.6)

$$A(s,\chi,\frac{-H}{d/K}) = - \quad {}_{d/K} \quad \sum_{b=1}^{qK/D} [D(s,\chi,\frac{-bH}{qK/d}) - P(s,\chi,\frac{-bH}{qK/d})]$$

$$\cdot\ [\ \Lambda(s,bd,\ qK) + \zeta(d)\ \frac{1}{\phi(qK)}\ \frac{\zeta'}{\zeta}(s)]$$

$$- \quad {}_{d/K} \quad \sum_{b=1}^{qK/d} P(s,\chi,\frac{-bH}{qK/d})\quad \Lambda(s,\ bd,\ qK)$$

$$+ \quad {}_{d/K} \quad \sum_{b=1}^{qK/d} \quad \frac{-bH}{\phi(qK)})\quad \frac{\zeta'}{\zeta}(s)$$

$$- \quad {}_{d/K} \quad \sum_{b=1}^{qK/d} \frac{\zeta(d)}{\phi(qK)} \quad P(s,\ \chi,\ \frac{-bH}{qK/d})\quad \frac{\zeta'}{\zeta}(s)$$

$$= - \quad {}_1 \quad - \quad {}_2 \quad + \quad {}_3 \quad - \quad {}_4,$$

with ${}_1$ regular at s=1. The principal part of ${}_2$ is the same as that of

$$\bar{\chi}(H)\ \tau(\chi)\ (qK)^{-s}\ \zeta^2(s) \quad {}_{d/J} \quad \chi(\frac{K}{d})\ d^s[F^q(s)\ (1+\chi(-1)\ (\frac{K}{d})^{1-s} - \phi(q)\ q^{-s}]$$

$$\sum_{b=1}^{qK/d} \bar{\chi}(b)\ \Lambda(s,bd,qK).$$

The sum over b equals

$$\sum_{b=1}^{qK/d} \bar{\chi}9b) \quad {}_{n\equiv b(\text{mod } q\ K/d)} \frac{\Lambda(dn)}{(dn)^s} = d^{-s} \quad \sum_{n=1}^{\infty} \frac{?(n)\ \Lambda(nd)}{n^s},$$

and the last sum is zero unless d=1 or p(recall that K is square-free). In any case we may write it as

$$\zeta(d)\ (-\frac{L'}{L}(s,\bar{\chi}) - \frac{F_K'}{F_K}\ (s,\bar{\chi}) \quad + \frac{\Lambda(d)}{d^s - \bar{\chi}(d)}\ .$$

Thus,

$$_{2} = \bar{\chi}(H)\ \chi(K)\ \tau(\chi)\ (qK)^{-s}\ \zeta^2(s)\ \{(-\ \frac{L^1}{L}(s,\bar{\chi})\ -\ \frac{F_K^{\ 1}}{F_K}(s,\bar{\chi})).$$

$$(F_q(s)(1+X(-1)K^{1-s})\ -\ \phi(q)q^{-s})$$

$$+\ _{p/K}\ \frac{\log\ p\ \bar{\chi}(p)\ p^s}{p^s - \bar{\chi}(p)}\ (F_q(s)\ \ (1+\chi(-1)(\frac{K}{p})^{1-s})\ -\ \phi(q)\ q^{-s})\}.$$

$$=\ -\ ?(H)\ \chi(K)\ \tau(\chi)\ (qK)^{-s}\ \zeta^2(s)\ \ G_K(s,\chi).$$

Next,

$$_{3} = \frac{1}{\phi(qK)}\ \ \frac{\zeta^1}{\zeta}(s)\ \ _{b=1}^{qK_1}\ \ D(s,x,\frac{-bH}{qK})$$

$$= \frac{1}{\phi(qK)}\ \frac{\zeta^1}{\zeta}(s)\ \ _{n=1}^{\infty}\ \frac{d(n)\ \chi(n)}{n^s}\ (\ _{b=1}^{qK_1}\ \ \ e(\frac{-bnH}{qK}))$$

for $\sigma > 1/$ Since $(H,qK) = 1$, the sum over b equals

$$_{a=1}^{qK_1}\ \ e(\frac{an}{qK})\ \ =\ c_{qK}(n).$$

Thus by (2.21) we have

$$_{n=1}^{\infty}\ \frac{d(n)\ x(n)}{n^s}\ \ c_{qK}(n)\ \ =\ _{n,m=1}^{\infty}\ \frac{\chi(mn)}{(mn)^s}\ \ \ _{\substack{d/qK\\ d/mn}}\ d\ \ \mu(\frac{qK}{d})$$

$$=\ _{d/qK}\ \ d\eta(\frac{qK}{d})\ \ \ _{n=1}^{\infty}\ \frac{x(n)}{n^s}\ \ \ _{\substack{m=1\\ \frac{d}{(n,d)}\ |\ m}}^{\infty}\ \frac{\chi(m)}{m^s}$$

$$= L(s,\chi) \;{}_{d/qK}\; d^{1-s}\,\eta\left(\frac{qK}{d}\right)\chi(d) \;{}_{n=1}^{\infty}\; \frac{\chi(n/(n,d))}{(n/(n,d))^{s}}$$

$$= L(s,x) \;{}_{d/qK}\; d^{1-s}\,\eta\left(\frac{qK}{d}\right) \;{}_{e/d}\; {}_{(j,d/e)=1}^{j=1,\;\infty}\; \frac{x(j)}{j^{s}}$$

$$= L^{2}(s,\chi) \;{}_{d/qK}\; d^{1-s}\,\eta\left(\frac{qK}{d}\right)\chi(d) \;{}_{e/d}\; F_{d/e}(s,\chi).$$

Since this function is regular at s=1, $_3$ has the same principal part as

$$\frac{1}{\phi(qK)}\,\frac{\zeta^{1}}{\zeta}(s)\; L^{2}(1,\chi) \;{}_{d/qK}\; \eta\left(\frac{qK}{d}\right)\chi(d) \;{}_{e/d}\; F_{e}(1,\chi).$$

Evidently we may restrict the first sum to one over d/K. Also, since (q,K) =1 and K is square-free, the double sum equals

$$\eta(qK) \;{}_{d/K}\; \eta(d)\;\chi(d) \prod_{p/d}(1 + F_{\phi}(1,\chi))$$

$$= \eta(qK) \prod_{p/K} (1- \chi(p)\;(\;1+ F_{\phi}(1,\chi))$$

$$= \eta(qK) \prod_{p/K} (1-2\;\chi(p) + \frac{\chi^{2}(p)}{p}$$

$$= \eta(qK)\;g(K).$$

Thus, the principal part of $_3$ is identical to that of

$$\frac{\eta(qK)}{\phi(qK)}\quad g(K)\;L^{2}(1,\chi)\;\frac{\zeta^{1}}{\zeta}(s).$$

Finally,

$$4 \frac{1}{\phi(qK)} \frac{\zeta'}{\zeta}(s) \quad \bar{\chi}(H) \quad \chi(K) \quad \tau(\chi)(qK)^{-s} \zeta^2(s)$$

$$\cdot (F_q(s)\ (1+ \chi(-1)\ K^{1-s}) - \phi(q)\ q^{-s}) \quad \sum_{b=1}^{qK} \quad \bar{\chi}(b)$$

$$= 0.$$

Collecting these results, we find that $Q(s, \chi, -\frac{H}{qK})$ has the same principal part as

$$\bar{\chi}(H) \ \chi(K) \ \tau(\chi)(qK)^{-s} \ \zeta^2(s) \ G_K(s,\chi) + \frac{\eta(q,K)g(K)}{\phi(qK)} \quad L^2(1,\chi) \ \frac{\zeta'}{\zeta}(s).$$

This completes the proof.

LEMMA 7. suppose that

$$c_1(j) = - \sum_{mn=j} \Lambda(m) \ \chi(n),$$

$$b_1(j) = - \sum_{\substack{hmn=j \\ h \le y}} a(h) \ \Lambda(m) \ \chi(n) \quad d(n),$$

and

$$b_2(j) = \sum_{\substack{hn=j \\ H \le y}} a(h) \ \chi(n) \ d(n).$$

Then if $y = T^{\eta}$ with $\eta < 1/2$,

$$(3.7)\quad \sum_{k\le y} \frac{\bar{a}(k)}{k} \sum_{j<\frac{qkT}{2\pi}} c_1(j)\, e\left(\frac{-j}{qk}\right) = \sum_{k\le y} \frac{\bar{a}(k)}{k} \operatorname*{res}_{s=1} \left(\frac{R(s,\chi,\frac{-1}{qk})}{s} \left(\frac{qkT}{2\pi}\right)^{s}\right) + O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) + O(TL^{-1}),$$

$$(3.8)\quad \sum_{k\le y} \bar{a}(k) \sum_{j<\frac{qkT}{2\pi}} b_1(j)\, e\left(\frac{-j}{qk}\right) = \sum_{h,k\le y} \frac{a(h)\,\bar{a}(k)}{k} \operatorname*{res}_{s=1} \left(\frac{Q(s,\chi,\frac{-H}{qK})}{s} \left(\frac{qKT}{2\pi H}\right)^{s}\right) + O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) + O(TL^{-1}),$$

and

$$(3.9)\quad \sum_{k\le y} \frac{\bar{a}(k)}{k} \sum_{j<\frac{qkT}{2\pi}} b_2(j)\, e\left(\frac{-j}{qk}\right) = \sum_{h,k\le y} \frac{a(h)\,\bar{a}(k)}{k} \operatorname*{res}_{s=1} \left(\frac{d(s,\chi,\frac{-H}{qK})}{s} \left(\frac{qKt}{2\pi H}\right)^{s}\right) + O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) + O(TL^{-1}),$$

where $\tau \le T$ in (.39) and R, Q, and D are as in Lemmas 4, 6, 5.

PROOF. All three formulae are proved by the method used to estimate the sum M_2 in Conrey, Ghosh, and Gonek [1;§5]. Since the method is complicated and quite lengthy, we shall only indicate the idea of the proof of (3.8) here; the interested reader is referred to sections 5-7 of the afore mentioned paper for details. It should be pointed out that had we assumed GRH, the lemma could be established with considerably less work; the reader may wish to consult Lemma 6 in [3] for the proof of a similar result on GRH.

First we set

$$B\left(s,\frac{-j}{qk}\right) = \sum_{j=1}^{\infty} b_1(j)\, e\left(\frac{-j}{qk}\right) j^{-s} \quad (\sigma > 1).$$

Then we have

$$(3.10) \quad \sum_{k\leq y} \frac{\bar{a}(k)}{k} \sum_{j\leq \frac{qkT}{2\Pi}} b_1(j)\, e(\tfrac{-j}{qk}) = \sum_{k\leq y} \frac{\bar{a}(k)}{k} \left(\frac{1}{2\pi i} \int_c B(s,\tfrac{-j}{qk}) \left(\tfrac{jkT}{2\Pi}\right)^s \frac{ds}{s}\right),$$

where c depends on T and $c > 1$. Now by the definitions of $b_1(j)$ and $Q(s,x,.)$, we see that

$$(3.11) \quad B(s,\tfrac{-j}{qk}) = \sum_{h\leq y} a(h)\; Q(s,x,\tfrac{-H}{qK}),$$

where $H=\frac{h}{(h,k)}$ and $K=\frac{k}{(h,k)}$. From this and Lemma 6 it follows that $B(s,\frac{-j}{qk})$ is a meromorphic function whose only pole in $\sigma \geq 1$ is at $s=1$. Inserting (3.11) into (3.10), we see that this pole should give rise to the main term

$$(3.12) \quad \sum_{h,k\leq y} \frac{a(h)\; \bar{a}(k)}{k} \operatorname*{res}_{s=1} \left(Q(s,\chi,\tfrac{-H}{qK}) \frac{\left(\frac{qKT}{2\Pi H}\right)^s}{s}\right).$$

To prove that this is the case we need to replace the exponential in $B(s,\frac{-j}{qk})$ by a character sum. We may then procede as in the proofs of the Bombieri-Vinogradov theorem given by Vaughan [14] and Gallagher[6].

By [5.12) in [1] we find that

$$e(\tfrac{-j}{qk}) = \sum_{q^1/qk} \sum^{*}_{\psi \bmod q^1} \tau(\bar{\psi}) \sum_{\substack{d/qk \\ d/j}} \psi(\tfrac{j}{d})\, \zeta(q^1,qk,d,\psi),$$

where

$$\zeta(q^1,qk,d,\psi) = \frac{\psi\left(\frac{d}{(d,qk/q^1)}\right) \bar{\psi}^r\left(\frac{-k}{(d,qk/q^1)q^1}\right) \eta\left(\frac{qk(d,qk/q^1)}{q^1}\right)}{\phi\left(\frac{qk}{(d,qk/q^1)}\right)}.$$

Evidently, we may suppose that $(k,q) = 1$ (otherwise $a(k) = 0$ in (3.8)). Hence, the divisors q^1 and d split as $q^1 q_1 q_2$ and $d = d_1 d_2$ with $q_1 | q$, $q_2 | k$, d_1®q, $d_2 | k$, and $(q_1, q_2) = (d_1, d_2) = 1$. Also, since ψ mod $q_1 q_2$ is primi.ive, there is a unique pair of primitive characters ψ_1, mod q_1, ψ_2 mod q_2 such that $\psi = \psi_1 \psi_2$. Forms this and the coprimality of q_1 and q_2 it is easy to show that

$$\tau(\bar{\psi}) = \bar{\psi}_1 (q_2)\, \bar{\psi}_2(q_1)\, \tau(\bar{\psi}_1)\, \tau(\bar{\psi}_2).$$

Using these factorizations for q^1, d, ψ, and $\tau(\bar{\psi})$, we may now write

$$e(\tfrac{-j}{qk}) = \quad q_1 | q \quad d_1 | q \quad \psi \bmod q_1 \overset{*}{} \quad \tau(\bar{\psi}_1) \quad q_2 | k \quad \psi_2 \bmod q_2 \overset{*}{} \quad \tau(\bar{\psi}_2)\, \bar{\psi}_1(q_2)\, \bar{\psi}_2(q_1)$$

$$\cdot \quad \underset{d_1 d_2 | j}{d | k} \quad \psi_1 \psi_2 (\tfrac{j}{d_1 d_2}) \quad \zeta(q_1 q_2, qk, d_1 d_2, \psi_1 \psi_2).$$

Substituting this for the exponential in the definition of $B(s, \frac{-j}{qk})$ and using the result in (3.10), we find after rearranging the sums that the right-hand side of (3.10) equals

$$d_1 | q \quad q_1 | q \quad \psi_1 \bmod q_1 \overset{*}{} \quad \tau(\bar{\psi}_1) \quad k \le y \; \frac{1}{1} \; [\quad q_2 \le y/k \quad \frac{\bar{a}(q_2 k)}{q_2} \quad \psi_2 \bmod q_2 \overset{*}{} \quad \tau(\bar{\psi}_2)$$

$$\cdot \quad d_2 | q_2 k \qquad \zeta(q_1 q_2, q\, q_2\, k, d_1 d_2, \psi_1 \psi_2)\, \bar{\psi}_1(q_2)\, \bar{\psi}_2(q_1)$$

$$1\left(\frac{1}{2\pi i}\right) \left(\int_{(c)}\right) \left(\sum_{m=1}^{\infty} \frac{b_1(d_1 d_2 m)\, \psi_1(m)\, \psi_2(m)}{m^s}\right) \left(\frac{qkT}{2\pi d_1 d_2}\right)^s \frac{ds}{s})]$$

The expression inside the brackets is analogous to E_2 in (5.15) of [1] and is treated in precisely the sameway. That is, we distinguish between the cases $q_2 \leq L^a$ for some $A > o$, and $L^A < q_2 \leq y/k$. The integrand above has a pole at $s=1$ if and only if $q_1 = q_2 = 1$, so the contribution of this term must be identical to (3.12). For $q_2 \leq L^A$ we pull the contour to the left and use Siegel's theorem as in the proof of the prime number theorem for arithmetic progressions. For the remaining cases we use a Vaughan-type identity and the large sieve.

LEMMA 8. Let $\sigma_{-1/2}(m) = \sum_{d|m} d^{-1/2}$. Then

i) $\frac{m}{\phi(m)} \ll \sigma_{-1/2}(m)$,

and for $x \geq 1$

ii) $\sum_{m \leq x} \frac{\sigma_{-1/2}^2(m)}{m} = c \log x + O(1)$,

where $c = \zeta(2) \sum_{(d,e)=1}^{\infty} (de)^{-3/2}$

PROOF. Note that $\frac{m}{\phi(m)} = \prod_{p|m}(1-\frac{1}{p})^{-1}$ and

$$\sigma_{-1/2}(m) = \prod_{p^{\alpha}\|m} (1+p^{-1/2}+ \ldots +p^{-\alpha/2}) \geq \prod_{p|m} (1+p^{-1/2}).$$

The first assertion therefore follows from the inequality

$$(1-\frac{1}{p})^{-1} \leq 1+p^{-1/2}$$

valid for ≥ 3.

For ii) we have

$$\sum_{m\leq x} \frac{\sigma^{2}_{-1/2}(m)}{m} = \sum_{m\leq x} \frac{1}{m} \sum_{\substack{d|m \\ e|m}} d^{-1/2} e^{-1/2}$$

$$= \sum_{d,e\leq x} d^{-1/2} e^{-1/2} \sum_{\substack{m\leq x \\ [d,e]|m}} \frac{1}{m} \quad ,$$

where [d,e] denotes the least common multile of d and e. This in turn equals

$$\sum_{d,e\leq x} d^{-1/2} e^{-1/2} [d.e]^{-1} \sum_{r\leq \frac{x}{[d.e]}} \frac{1}{r}$$

$$= \log x \sum_{d,e\leq x} d^{-1/2} e^{-1/2} [d,e]^{-1} + O\Big(\sum_{d,e\leq x} d^{-1/2} e^{-1/2} [d,e]^{-1} \log 2\,[d,e]\Big).$$

If we write $g=(d,e)$ and then replace d by kg and l by lg, we obtain

$$\log x \sum_{g\le x} \frac{1}{g^2} \sum_{\substack{k,l\le x/g\\(k,l)=1}} (kl)^{-3/2} + O\Big(\sum_{g\le x} \frac{1}{g^2} \sum_{\substack{k,l\le x/g\\(k,l)=1}} (kl)^{-3/2} \log 2g\,kl\Big).$$

The error term is clearly $O(1)$. The first term equals

$$\log x \sum_{g\le x} \frac{1}{g^2} \sum_{\substack{k,l=1\\(k,l)=1}}^{\infty} (kl)^{-3/2} + O\Big(\frac{\log^2 2x}{x}\Big) .$$

The proof is now completed by noting that

$$\sum_{g\le x} \frac{1}{g^2} = \zeta(2) + O\Big(\frac{1}{x}\Big).$$

LEMMA 9. For a fixed character x mod q, m a positive integer, and $x \ge 1$, we have

$$\text{i)} \quad \sum_{p|m} \frac{\log^j p}{p} \ll \begin{cases} \log\log\log 30m & \text{if } j=0, \\ (\log\log 3m)^j & \text{if } j=1,2, \end{cases}$$

and

ii) $$\underset{\substack{p\le x\\ p\nmid m}}{}\ \frac{\chi(p)}{p}\log^j p \ll \begin{cases}\log\log\log 30m & \text{if } j=0\\ (\log\log 3m)^j & \text{if } j=1,2.\end{cases}$$

PROOF. To prove i) it suffices to assume that m is square-free and m>1. Let $p_1, p_2, \ldots$ denote the primes listed in increasing order and let r denote the unique positive integer for which

$$p_1 p_2 \cdots p_r \le m \le p_1 p_2 \cdots p_{r+1}.$$

On the one hand, we see that $\log m \ge \underset{i=1}{\overset{r}{}} \log p_i \gg p_r$.

On the other hand,

$$\underset{p\mid m}{}\ \frac{\log^j p}{p} \le \underset{i=1}{\overset{r}{}}\ \frac{\log^j p_i}{p_i} \ll \begin{cases}\log\log 3p_r & \text{if } j=0,\\ \log^j p_r & \text{if } j=1,2.\end{cases}$$

Combing these, we obtain i).

We write the sum in ii) as

$$\underset{p\le u}{}\ \chi(p)\ \log p \ll \frac{u}{\log^3 su}$$

(see Davenport [4;p.132]), we find by partial summation that the first sum above is bounded. For the O-term we use the estimate in i), whereupon ii) Follows.

LEMMA 10. Let $G_k(s,x)$ be as in (3.5) with x a fixed character mod q and k a positive integer.

Then

$$G_k(1,x) = -\chi(-1)\frac{\phi(q)}{q}\sum_{p|k}\bar{X}(p)\log p + O(\log\log 3k)$$

and

$$G^1_k(1,\chi) = \chi(-1)\frac{\phi(q)}{q}\sum_{p|k}\bar{\chi}(p)\log p\log\frac{k}{p} + O(\log 2k\log\log 3k)$$

PROOF. From the definition of $f_k(s,\bar{\chi})$ and Lemma 9 i), we have

$$(3.13)\qquad \frac{F^1_k}{F_k}(1,\bar{\chi}) = \sum_{p|k}\frac{\bar{X}(p)\log p}{p-m\bar{X}(p)} \ll \log\log 3k$$

$$(3.14)\qquad \left(\frac{F^1_k}{F_k}\right)^1(1,\bar{\chi}) = \sum_{p|k}\frac{\bar{\chi}(p)\log^2 p}{p(1-\bar{\chi}(p)/p)^2} \ll (\log\log 3k)^2.$$

By (3.5) with s=1 we have that

$$G_k(1,\chi) = \chi(-1)\frac{\phi(q)}{q}\left(\frac{L^1}{L}(1,\bar{\chi}) + \frac{F^1_k}{F_k}(1,\bar{\chi}) - \sum_{\phi|k}\frac{\bar{\chi}(-)\log p}{1-\bar{\chi}(p)/p}\right).$$

Thus, using (3.13) and Lemma 9 i), we obtain

$$G_k(1.\chi) = -\chi(-1)\,\frac{\phi(q)}{q} \sum_{p|k} \bar{\chi}(p) \log p + O(\log\log 3k)$$

as required.

Again by (3.5) we see that

$$G_k'(1,\chi) = \chi(-1)\,\frac{\phi(q)}{q}\Big(\Big(\frac{L'}{L}\Big)'(1,\bar{\chi}) + \Big(\frac{F_K'}{F_k}\Big)'(1,\bar{\chi}) + \sum_{p|k} \frac{\bar{\chi}^2(p)\log^2 p}{p(1-\bar{\chi}(p)/p)^2}$$

$$+\Big(\frac{L'}{L}(1,\bar{\chi}) + \frac{F_k'}{F_k}(1,\bar{\chi})\Big)\Big(\frac{F_q'}{F_q}(1)\,(1+\chi(-1)) + x(-1)\log q - \log k\Big)$$

$$- \sum_{p|k} \frac{\bar{\chi}(p)\log p}{1-\bar{\chi}(p)/p}\Big(\frac{F_q'}{F_q}(1)\,(1+\chi(-1) + x(-1)\log q - \log \frac{k}{\phi}\Big)\Big).$$

Of these terms, the only one we cannot afford to estimate trivially is

$$\chi(-1)\,\frac{\phi(q)}{q} \sum_{p|k} \frac{\bar{\chi}(p)\log p}{1-\bar{\chi}(p)/p} \log \frac{k}{p}$$

$$= \chi(-1)\,\frac{\phi(q)}{q} \sum_{p|k} \bar{\chi}(p)\log p \log\frac{k}{p} + O(\log 2k \log\log 3k),$$

by Lemma 9i).

By (3.13), (3.14), and Lemma 9i), the other terms are seen to be at most $O(\log 2k \log\log 3k)$. Hence the result follows.

LEMMA 11. For $x \geq 1$ and q fixed,

$$\sum_{\substack{m \leq x \\ (m,q)=1}} \frac{\eta^2(m)}{\phi(m)} = \frac{\phi(q)}{q} \log x + O(1)$$

PROOF. The proof is standard so we will merely sketch it. The generating function for the sum is

$$J(s) = \sum_{\substack{m=1 \\ (m,q)=1}}^{\infty} \frac{\eta^2(m)}{\phi(m)\, m^s} = \prod_{p \nmid q} \left(1+ \frac{1}{\phi(p)p^s}\right)$$

$$= \prod_{p \mid q} \left(1+\frac{1}{\phi(p)p^s}\right)^{-1} \prod_p \left(1+\frac{1}{p^{s+1}}\right) P(s)$$

$$= \prod_{p \mid q} \left(1+\frac{1}{\phi(p)p^s}\right)^{-1} \frac{\zeta(s+1)}{\zeta(2s+2)} P(s),$$

where

$$P(s) = \prod_p \left(1+\frac{1}{\phi(p)p^s}\right) \left(1+\frac{1}{p^{s+1}}\right)^{-1}.$$

The product for P(s) is absolutely convengent for $\sigma > 1$, hence P(s) is uniformly bounded in $\sigma > \frac{1}{2}$ say. Also,

$$P(o) = \prod_p \left(1+\frac{1}{p-1}\right)\left(1+\frac{1}{p}\right)^{-1}$$

$$= \prod_p \left(1+ \frac{1}{p^2}\right)^{-1} = \zeta(2).$$

Thus, applying Perron's formaulu in the usual way, we fir.d that

$$\sum_{\substack{m\le x\\(\overline{m},q)=1}} \frac{\eta^2(m)}{\phi(m)} = \operatorname*{res}_{s=0} J(s)\frac{x^s}{s} + O(1).$$

$$=\frac{\phi(q)}{q}\log \chi + O(1)/$$

LEMMA 12. Let y, a(h), and $F_h(s,x)$ be as usual, and let g(h) be as in (3.4). Then

i) $$\sum_{h\le y/m} \frac{a(mh)}{h} \ll \sigma_{-1/2}(m),$$

ii) $$\sum_{h\le y/m} \frac{\eta(h)\,\bar{a}(h)\,F_h(o,x)}{\phi(h)} \ll L^{-1},$$

iii) $$\sum_{h\le y/m} \frac{\eta(h)\,\bar{a}(mh)\,g(h)}{h\,\phi(h)} \ll \sigma_{-1/2}(m)\,L^{-1},$$

where $\phi_{-1/2}(m) = \sum_{d|m} d^{-1/2}$.

PROOF. The proofs of all three assertions are similar so we will only prove the most involved, namely iii).

The sum in iii) equals

(3.15) $$\frac{\eta(m)\,\bar{\chi}(m)}{\log y} \sum_{\substack{h\le y/m\\(h,m)=1}} \frac{\eta^2(h)\,\bar{\chi}(h)\,g(h)}{h\phi(h)} \log y/mh$$

$$= \frac{\eta(m)\ \bar{\chi}(m)}{\log y} ,$$

say. Now the generating function for is

$$H(s) = \sum_{h=1}^{\infty} \frac{\eta^2(h)\ \bar{\chi}(h)\ g(h)}{\phi(h)\ h^s} = \prod_{p \nmid m} \left(1 + \frac{\bar{\chi}(p)\ g(p)}{\phi(p)\ p^s}\right)$$

$$= \prod_{p|m} \left(1 + \frac{\bar{\chi}(p)\ g(p)}{\phi(p)\ p^s}\right)^{-1} \prod_p \left(1 + \frac{\bar{\chi}(p)}{p^{s+1}}\right)\left(1 - \frac{2|\chi(p)|^2}{p^{s+1}}\right) \quad P(s)$$

$$= \prod_{p|m} \left(1 + \frac{\bar{\chi}(p)\ g(p)}{\phi(p)\ p^s}\right)^{-1} \prod_{p|q} \left(1 - \frac{2}{p^{s+1}}\right)^{-1} L(s+1,\bar{\chi})\ L^{-1}(2s+2,\bar{\chi}^2)\ \zeta^{-2}(s+1)\ P(s),$$

where

$$P(s) = \prod_p \left(1 + \frac{\bar{\chi}(p)\ g(p)}{\phi(p)\ p^s}\right)\left(1 + \frac{\bar{\chi}(p)}{p^{s+1}}\right. \quad \left(1 - 2x(p)\ \frac{-2|\chi(p)|^2}{p^{s+1}}\right)\Big)^{-1}.$$

The product for $P(s)$ is absolutely convergent for $\sigma > -1/2$ so $P(s)$ is uniformly bounded in $\sigma \geq -1/4$, say. For the other fctors in $H(s)$ we have in the half-plane $\sigma \geq L^{-1}$,

$$L^{-1}(2s+2,\bar{\chi}^2), \quad \prod_{p|q}\left(1 - \frac{2}{p^{s+1}}\right) \ll 1,$$

$$L(s+1,\bar{\chi}) \ll \log (H\ 1+2),$$

$$\zeta^{-2}(s+1) \ll \min\left(|s|^2, \log^2(H\ 1+2)\right),$$

and

$$\prod_{p|m} \left(1+\frac{\bar{\chi}(p)\, g(p)}{\phi(p)\, p^s}\right)^{-1} \ll \sigma_{-1/2}(m).$$

These estimates are all standard except for the last, which follows from

$$\left|1+\frac{\bar{\chi}(p)\, g(p)}{\phi(p)\, p^s}\right| > \begin{cases} 1-\frac{4}{p} \geq (1+p^{-1/2})^{-1} & \text{if } p\geq 27, \\ 0 & \text{if } p<27. \end{cases}$$

Combining these, we find that

$$H(s) \ll \sigma_{-1/2}(m) \log (H\ 1+2) \min (|s|^2, \log^2 (H\ 1+2))$$

for $\sigma \geq L^{-1}$.

Consequently, since $(y/m)^{1/L} \ll 1$,

$$= \frac{}{2\pi i} \int_{(L^{-1})} H(s) \frac{(y/m)^s}{s^2}\, ds$$

$$\ll \sigma_{-1/2}(m) \left(\int_0^1 dt + \int_1^\infty \log^3 (H\ 1+2) \frac{dt}{t^2}\right)$$

$$\ll \sigma_{-1/2}(m).$$

This and (3.15) give iii).

LEMMA 13. Let y and a(h) be as usual. Then

i) $${}_{h\le y/m} \frac{a(mh)\,\bar\chi(h)}{h} = \frac{q\;m\;\eta(m)\;\chi(m)}{\phi(q)\;\phi(m)\;\log y} + O\left(\frac{\sigma_{-1/2}(m)}{\log y\;\log^4 2y/m}\right)$$

and

ii) $${}_{h\le y/m} \frac{a(mh)\,\bar\chi(h)\,\log h}{h} = \frac{qm\;\eta(m)\;\chi(m)\;\log y/m}{\phi(w)\;\phi(m)\;\log y} + O\left(\frac{\sigma_{-1/2}(m)\;\log L}{\log y}\right),$$

where $\sigma_{-1/2}(m) = {}_{d|m}\; d^{-1/2}$.

PROOF. We base the proof on the formula

$${}_{\substack{k\le x\\(k,r)=1}} \frac{\eta(k)}{k}\log\frac{x}{k} = \frac{r}{\phi(r)} + O(\sigma_{-1/2}(r)\,\log^{-4} 2x),$$

(see Graham [10;p.]).

For i) we have

$${}_{h\le y/m} \frac{a(mh)\,\bar\chi(h)}{h} = \frac{\eta(m)\;\chi(m)}{\log y}\;{}_{\substack{h\le y/m\\(h,mq)=1}} \frac{\eta(h)}{h}\;\log\frac{y}{mh}$$

$$= \frac{\eta(m)\;\chi(m)\;mq}{\phi(mq)\;\log y} + O\left(\frac{\sigma_{-1/2}(mq)}{\log y\;\log^4 2y/m}\right).$$

The original sum vanishes if $(m,q) > 1$ so the result follows from the multiplicativity of ϕ and $\sigma_{-1/2}$.

Now consider ii). Since we may clearly assume that h is square-free, we have

$$k \leq y/m \quad \frac{a(mh)\ \bar{\chi}(h)\ \log h}{h} = k \leq y/m \quad \frac{a(mh)\ \bar{\chi}(h)}{h} \quad p|h \quad \log p$$

$$p \leq y/m \quad \frac{\bar{\chi}(p)\ \log p}{p} \quad k \leq y/mp^{1} \quad \frac{a(mpk)\ \bar{\chi}(k)}{k}.$$

Evidently, we may assume here thatm,p, and q are pairwise coprime. Thus by i), the last line equals

$$\frac{q\ m\ \eta(m)\ \chi(m)}{\phi(q)\ \phi(m)\ \log y} \quad \underset{p+mq}{p \leq y/m} \quad \frac{\log p}{\phi(p)} + O(\frac{\sigma_{1/2}(m)}{\log y} \quad p \leq y/m \quad \frac{\log p}{p \log^4 2y/mp})$$

The sum in the main term equals

$$p \leq y/m \quad \frac{\log p}{p} + O(\quad p|qm \quad \frac{\log p}{p}) + O(1)$$

$$= \log y/m + O(\log \log 3qm)$$

$$= \log y/m + O(\log \log 3m),$$

by Lemma 9 i). By the prime number theorem, the sum in the error term is

$$\int_2^{y/m} \frac{du}{n \log^4 2y/mn} + \int_2^{y/m} \frac{d\ E(u)}{n \log^4 2y/mn},$$

where $E(u) \ll n/\log^4 n$. One easily sees that both these integrals are bounded. Hence, the left-hand side of ii) equals

$$\frac{-qm\ \eta(m)\ \chi(m)\ \log y/m}{\phi(q)\ \phi(m)\ \log y} + O(\frac{m \log\log 3m}{\phi(m) \log y}) + O(\frac{\sigma_{-1/2}(m)}{\log y}).$$

The result now follows from Lemma 8 i) and the fact that $m \leq y$.

4. The estimation of $\underline{\underline{N}}$.

Recall from (2.14) that

$$(4.1) \qquad \underline{\underline{N}} = \underline{\underline{N}}_1 + \underline{\underline{N}}_2 - \underline{\underline{N}}_3 + O_\varepsilon(y\, T^{1/2+\varepsilon}),$$

where the $\underline{\underline{N}}i$ are given by (2.15) - 2.17).

We first consider $\underline{\underline{N}}_1$. Using the functional equation 2.23) in (2.15), we have

$$\underline{\underline{N}}_1 = \frac{1}{2\pi i} \int_{a+i}^{a+iT} \frac{\zeta'}{\zeta}(s)\ L(s,\chi)\ A(1-s,\bar{\chi})\ X(1-s,\chi)\ ds.$$

Setting

$$c_1(j) = -\sum_{mn=j} \Lambda(m)\ \chi(n)$$

and using Lemma 2, we then obtain

$$\underline{\underline{N}}_1 = \frac{\chi(-1)}{\tau(\chi)} \sum_{k\leq y} \frac{\bar{a}(k)}{k} \sum_{j\leq \frac{qkT}{2\pi}} c_1(j)\ e(\frac{-j}{qk}) + O_\varepsilon(y\, T^{1/2+\varepsilon}).$$

Now by (3.7) we find that

$$\underline{N}_{-1} = \frac{\chi(-1)}{\tau(\chi)} \sum_{k\leq y} \frac{\bar{a}(k)}{k} \operatorname*{res}_{s=1} \left(\frac{R(s,\chi,\frac{-1}{qk})}{s} \left(\frac{qkT}{2\Pi}\right)^{s} \right)$$

$$+ O_{\varepsilon}(y^{1/2}T^{3/4+\varepsilon}) + O(TL^{-1})$$

Here the sum may be taken over square-free k coprime to q (lect $\bar{a}(k) = 0$), hence he residue may be computed by means of Lemma 4. The result is, after simplification,

$$\underline{N}_{-1} = \frac{T}{2\Pi}\left(\frac{L'}{L}(1,\chi) - \frac{\chi(-1)}{\tau(\chi)} \frac{\eta(q)\, q}{\phi(q)} L(1,\chi) \sum_{k\leq y} \frac{\eta(k)\, \bar{a}(k)\, f_{k}(0,\chi)}{\phi(k)}\right.$$

$$\left. - \sum_{p\leq y} \frac{\bar{a}(p) \log p}{p-\bar{\chi}(p)}\right) + O(y^{1/2}T^{3/4+\varepsilon}) + O(TL^{-1}).$$

By Lemma 12 ii) the sum over k is $\ll L^{-1}$. The sum over p equals

$$\frac{-1}{\log y} \sum_{p\leq y} \frac{\bar{\chi}(p) \log p \log y/p}{p} + O\left(\sum_{p\leq y} \frac{\log p}{p^{2}}\right).$$

The error term is clearly bounded and, by Lemma 9 ii) (with m=1), so is the first term. Hence

$$(4.2) \qquad \underline{N}_{1} = O_{\varepsilon}(y^{1/2}T^{3/4+\varepsilon}) + O(T).$$

Next, by (2.16), we have

$$\underline{N}_1 = \frac{1}{2\pi i} \int_{a+i}^{a+iT} \frac{\zeta'}{\zeta}(s)\ L(s,\chi)\ A(s,\chi)\ ds$$

$$= \sum_{n=2}^{\infty} c_2(n)\ n^{-a}\left(\frac{1}{2\pi} \int_1^T n^{-it}\ dt\right)$$

$$\ll \sum_{n=2}^{\infty} \frac{|c_2(n)|}{n^a \log n},$$

where

$$c_2(n) = -\sum_{hjk=n} \Lambda(h)\ \chi(j)\ a(k).$$

But

$$|c_2(n)| \leq \log n \sum_{hjk=n} 1 = d_3(n) \log n,$$

so

(4.3) $$\underline{N}_2 \leq \sum_{n=2}^{\infty} d_3(n)\ n^{-a} < \zeta^3(a) \ll L^3.$$

Finally we come to $\underline{N}_3$. Taking the logarithmic derivative of (2.13) and using the formula

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(|s|^{-1})$$

for $|s| \to \infty$ in the region $|\arg s| < \pi - \zeta$ $(\zeta>0)$ (e.g. see Whittaker and Watson [; Chs. 12 and 13]), we see that

(4.4) $$\frac{X'}{X}(1-s) = -\log\frac{t}{2\Pi} + O(\frac{1}{t})$$

for $t \geq 1$, $o \leq \sigma \leq 2$, say. Inserting this into (2,17), we obtain

$$\underline{N}_3 = \frac{-1}{2\Pi}\int_1^T L(a+it,\chi)\; A(a+it,\chi)\log\frac{t}{2\Pi}\,dt$$

$$+ O(\int_1^T |L(a+it,\chi)\; A(a+it,\chi)|\frac{dt}{t}).$$

Clearly

$$L(a+it,\chi)\; A(a+it,\chi) \ll \zeta^2(a) \ll L^2,$$

so the error term is $\ll L^3$.

The first term is

$$-\sum_{n=1}^{\infty} c_3(n)\, N^{-a}\left(\frac{1}{2\Pi}\int_1^T n^{-it}\log\frac{t}{2\Pi}\,dt\right),$$

with

$$c_3(n) = {}_{hk=n} \chi(h)\ a(k).$$

In particular, $c_3(1) = 1$, hence this term contributes

$$\frac{T}{2\pi} L + O(T)$$

to $\underline{N}_3$. Since $|c_3(n)| \leq {}_{hk=n} 1 = d(n)$,

The remaining terms contribute

$$\ll L \ {}_{n=2}^{\infty} \frac{d(n)}{n^a \log n} \ll L\, \zeta^2(a) \ll L^3.$$

Therefore

$$\underline{N}_3 = -\frac{T}{2\pi} L + O(T).$$

Combining this with (4.1)-(4.3) we see that

$$\text{(4.5)} \qquad \underline{N} = \frac{T}{2\pi} L + O(T) + O_\varepsilon(y^{1/2} T^{3/4+\varepsilon}).$$

5. The estimation of $\underline{D}_1$

We now turn to the first term $\underline{D}_1$ in the denominator $\underline{D}$ (see (2.08), (2.18), and (2.20)). By the functional equation (2.23) we have

$$\underline{D}_1 = \frac{1}{2\pi i} \int_{a+i}^{a+iT} \frac{\zeta'}{\zeta}(s)\; L^2(s,\chi)\; A(s,\chi)\; A(1-s,\bar{\chi})\; X(1-s,\chi)\, ds,$$

where a 1 + L^{-1}. We set

$$\sum_{j=1}^{\infty} b_1(j)\, j^{-s} = \frac{\zeta'}{\zeta} 9s)\;\; 1^2(s,\chi)\;\; A(s,\chi)\quad (\sigma > 1)$$

so that

$$b_1(j) = - \sum_{\substack{hmn=j \\ h\leq y}} a(h)\; \Lambda(m)\; d(n)\; \chi(n).$$

Then by Lemma 2,

$$\underline{D}_1 = \frac{\chi(-1)}{\tau(\chi)} \sum_{k\leq y} \frac{\bar{a}(k)}{k} \sum_{j\leq \frac{qkT}{2\pi}} b_1(j)\; e\left(\frac{j}{qk}\right) + O_\varepsilon(yT^{1/2+\varepsilon}).$$

To estimate this we use (3.8) of Lemma 7 and find that

$$\underline{D}_1 = \frac{\chi(-1)}{\tau(\chi)} \sum_{h,k\leq y} \frac{a(h)\; \bar{a}(k)}{k} \operatorname*{res}_{s=1} \left(\frac{\left(\frac{qkT}{2\pi H}\right)^s}{s}\; Q\left(s,\chi,\frac{-H}{qk}\right)\right)$$

$$+\; O_\varepsilon(y^{1/2}T^{3/4\,+\,\varepsilon}) \;+\; O(TL^{-1}),$$

where

$$H = \frac{h}{(h,k)} \quad \text{and } K = \frac{k}{(h,k)} \quad .$$

Observe that in the seyal above we may suppose that both H and K are square-free and that $(H,q) = (K,q) = 1$. Lemma 6 is therefore applicable and we may write the residue as

$$\bar{\chi}(H)\ \chi(K)\ \tau(\chi)\ \underset{s=1}{\text{res}}\ (\frac{(\frac{T}{2\pi H})^{s}}{}\ G_K(s,\chi)\ \zeta^2(s))$$

$$+\ \frac{\eta(qK)\ q(K)}{\phi(qK)}\ \ L^2(1,\chi)\ \underset{s=1}{\text{res}}\ (\frac{(\frac{qKT}{2\pi H})^{s}}{s}\ \ \frac{\zeta^1}{\zeta}(s)).$$

If we use the expansion $\zeta(s) = \frac{1}{s-1} + \gamma + \ldots$ near s=1 to evaluate theses residues and insert the result into $\underline{D}_1$, we obtain

$$\underline{D}_1 = \chi(-1)\ \frac{T}{2\pi}\ \sum_{h,k\le y} \frac{a(h)\ \bar{a}(k)\ \bar{\chi}(h)\ \chi(k)(h,k)}{hk}\ (G_K(1,\chi)\log \frac{Te^{2\gamma-1}}{2\pi H}\ + G_K^1(1,\chi))$$

$$-\ \frac{\chi(-1)}{\tau(\chi)}\ \frac{\eta(q)\ \phi(q)}{q}\ L^2(1,\chi)\ \frac{T}{2\pi}\ \sum_{h,k\le y} \frac{a(h)\ \bar{a}(k)\ \eta(\frac{k}{(h,k)})\ g(\frac{k}{(h,k)})}{hk\ \phi(\frac{k}{(h,k)})}$$

$$+\ O_\varepsilon(y^{1/2}\,T^{3/4\,+\,\varepsilon})\ +\ O(TL^{-1}).$$

We next apply the Möbius inversion formula in the form

$$f((h,k)) = \sum_{\substack{m|h \\ m|k}} \sum_{h|m} \eta(n)\, f\left(\frac{m}{n}\right) .$$

This leads to

$$\underline{D}_1 = \chi(-1)\,\frac{T}{2\pi} \sum_{h,k\leq y} \frac{a(h)\,\bar{a}(k)\,\bar{\chi}(h)\,\chi(k)}{hk} \sum_{\substack{m|h \\ m|k}} \sum_{n|m} \frac{\eta(n)\,m}{n}$$

$$\cdot\left(G_{\frac{kn}{m}}(1,\chi)\,\log\frac{Te^{2\gamma-1}m}{2\pi hn} + G^{1}_{\frac{kn}{m}}(1,\chi)\right)$$

$$\frac{\chi(-1)}{\tau(\chi)}\,\frac{\eta(q)\,\phi(q)}{q}\,L^2(1,\chi)\,\frac{T}{2\pi} \sum_{h,k\leq y} \frac{a(h)\,\bar{a}(k)}{hk} \sum_{\substack{m|h \\ m|k}} \sum_{n|m} \frac{\eta(n)\,\eta\left(\frac{kn}{m}\right)\, g\left(\frac{kn}{m}\right)}{\phi\left(\frac{kn}{m}\right)}$$

$$+\, O_{\varepsilon}\left(y^{1/2}\,T^{3/4+\varepsilon}\right) + O(TL^{-1}).$$

Interchanging the order of summation and replacing h by hm, k by km, we see that

$$\underline{D}_1 = \chi(-1)\,\frac{T}{2\pi} \sum_{m\leq y} \frac{1}{m} \sum_{n|m} \frac{\eta(n)}{n} \sum_{h,k\leq y/m} \frac{a(mh)\,\bar{a}(mk)\,\bar{\chi}(h)\,\chi(h)}{hk}$$

$$\cdot\left(G_{kn}(1,\chi)\,\log\frac{Te^{2\gamma-1}}{2\pi hn} + G^{1}_{kn}(1,\chi)\right)$$

$$-\frac{\chi(-1)}{\tau(\chi)}\ \frac{\eta(q)\ \phi(q)}{q}\ L^2(1,\chi)\ \frac{T}{2\Pi}\ {}_{m\leq y}\ \frac{1}{m}\ {}_{h|m}\ \eta(n)_{h,k\leq y/m}\ \frac{a(mh)\ \overline{a(mk)}\ \eta(nk)\ g(nk)}{hk\ \phi(nk)}$$

$$+\ O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) \quad + O(TL^{-1}),$$

or

$$(5.1) \qquad \underline{D}_1 = \underline{D}_{11} - \underline{D}_{12} + O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) + O(TL^{-1}).$$

We first treat $\underline{D}_{11}$. By Lemma 10, the expression in parentheses equals

$$G_k(1,\chi)\log\frac{Te^{2\gamma-1}}{2\Pi hn} + G_k(1,\chi) + O(L\log\log 3kn)$$

$$= -\chi(-1)\ \frac{\phi(q)}{q}\ (\log\frac{T}{h}\ {}_{p|k}\ \overline{\chi}(p)\log p - {}_{p|k}\overline{\chi}(p)\log p\log\frac{k}{p})$$

$$+\ O(L\log\log 3kn) + O(\log 2k\log 2n).$$

Since neither n nor k is greater than y the error terms here are $<< L\log 2n$ L. Hence, using the identity ${}_{n|m}\ \frac{\eta(n)}{n} = \frac{\phi(m)}{m}$, we have

$$\underline{D}_{11} - \frac{\phi(q)}{q}\ \frac{T}{2\Pi}\ {}_{m\leq y}\ \frac{\phi(m)}{m^2}\ {}_{h,k\leq y/m}\ \frac{a(mh)\ \overline{a}(mk)\ \overline{\chi}(h)\ \chi(k)}{hk}$$

$$\cdot\ {}_{p|k}\ \overline{\chi}(p)\log p\log\frac{Tp}{hk}$$

$$+ O\Big(TL \sum_{m\le y} \frac{1}{m} \sum_{n|m} \frac{\log 2nL}{n} \Big| \sum_{h\le y/m} \frac{a(mh)\,\bar{\chi}(h)}{h} \Big|^2 \Big).$$

Notice that in each sum over m we may assume that m is square-free. With this in mind, we see from Lemmas 13i) and 8 that the O-term is

$$\ll \frac{TL}{\log^2 y} \sum_{\substack{m\le y \\ m \text{ square-free}}} \frac{\sigma^2_{-1/2}(m)}{m} \sum_{n|m} \frac{\log 2nL}{n}$$

$$\ll \frac{TL}{\log^2 y} \sum_{n\le y} \frac{\sigma^2_{-1/2}(n)}{n^2} \log 2nL \sum_{r\le y/n} \frac{\sigma^2_{-1/2}(r)}{r}$$

$$\ll \frac{TL}{\log y} \sum_{n\le y} \frac{\sigma^2_{-1/2}(n)}{n} \log 2nL$$

$$\ll \frac{TL \log L}{\log y} \ll T \log L.$$

We may therefore rewrite $\underline{D}_{11}$ as

$$\underline{D}_{11} = \frac{\phi(q)}{q} \frac{T}{2\Pi} \sum_{m\le y} \frac{\phi(m)}{m^2} \sum_{\substack{p\le y/m \\ (p,q)=1}} \frac{\log p}{p} \Big(L \sum_{h\le y/m} \frac{\bar{a}(mh)\,\bar{\chi}(h)}{h} \sum_{l\le y/mp} \frac{\bar{a}(mpl)\,\chi(l)}{l}$$

$$- \sum_{k\le y/m} \frac{a(mh)\,\bar{\chi}(h)\log h}{h} \sum_{l\le y/mp} \frac{\bar{a}(mpl)\,\chi(l)}{l}$$

$$- \sum_{h\le y/m} \frac{a(mh)\,\bar{\chi}(h)}{h} \sum_{l\le y/mp} \frac{\bar{a}(mpl)\,\chi(l)\log l}{l} \Big)$$

$+ O(T \log L)$.

Using Lemma 13 to estimate the sums over h and l and noting that we may suppose that $(p,m) = (q,m) = 1$, we find that the expression in parenthesis is

$$\frac{\eta^2(m)\, m^2\, q^2}{\phi^2(m)\, \phi^2(q) \log^2 y} \frac{p\, \bar{\chi}(p)}{\phi(p)} (L + \log y/m + \log y/mp)$$

$$+ O\left(\frac{\sigma^2_{1/2}(m)}{\log y \log^4 \frac{2y}{mp}}\right) + O\left(\frac{\sigma^2_{1/2}(m) \log L}{\log^2 y}\right).$$

Thus,

$$(5.2)\quad \underline{D}_{11} = \frac{q}{\phi(q) \log^2 y} \frac{T}{2\pi} \sum_{\substack{m \le y \\ (m,q)=1}} \frac{\eta^2(m)}{\phi(m)} \sum_{\substack{p \le y/m \\ (p,m)=1}} \frac{\bar{\chi}(p)}{\phi(p)} (L + 2 \log \frac{y}{m} - \log p)$$

$$+ O\left(\frac{T}{\log y} \sum_{m \le y} \frac{\phi(m)\, \sigma^2_{1/2}(m)}{m^2} \sum_{p \le y/m} \frac{\log p}{p \log^4 \frac{2y}{mp}}\right)$$

$$+ O\left(\frac{T \log L}{\log^2 y} \sum_{m \le y} \frac{\phi(m)\, \sigma^2_{1/2}(m)}{m^2} \sum_{p \le y/m} \frac{\log p}{p}\right)$$

$$+ O(T \log L).$$

The first)-term is

where $w(n) \; \prod_{p|n} 1$. Hence,

$$\underline{D}_{12} \ll T \sum_{n\leq y} \frac{\eta^2(n)\, d^4(n)}{n\, \phi(n)} \ll T.$$

It follows from this, (5.1), and (5.3) that

(5.4) $\underline{D}_1 = O_\varepsilon(y^{1/2} T^{3/4+\varepsilon}) + O(T \log L)$.

6. The estimation of $\underline{D}_2$.

We shall see in this section that the main term in $\underline{D}$ comes from $\underline{D}_2$. Recall from (2.19) that

$$\underline{D}_2 = \frac{1}{2\pi i} \int_{a+i}^{a+iT} \frac{\chi'}{\chi}(1-s)\, L(s,\chi)\, L(1-s,\bar{\chi})\, A(s,\chi)\, A(1-s,\bar{\chi})\, ds,$$

where $a = 1 + L^{-1}$. By (4.4),

$$\frac{\chi'}{\chi}(1-s) = -\log \frac{t}{2\pi} + O\left(\frac{1}{t}\right)$$

for $t \geq 1$ and $0 \leq \sigma \leq 2$. Hence, moving the line of integration to $\sigma = 1/2$ and using (2.9) and (2.10), we obtain

$$\ll \frac{T}{\log y} \sum_{m \le y} \frac{\sigma^2_{-1/2}(m)}{m} \int_1^{y/m} \frac{}{u \log^4 \frac{2y}{mu}}$$

by the prime number theorem. The integral is easily seen to be $\ll 1$. So by Lemma 8ii) this is $\ll T$.

The second error term is

$$\ll \frac{T \text{ lgo } L}{\log y} \sum_{m \le y} \frac{\sigma^2_{-1/2}(m)}{m} \ll T \log L.$$

Finally, by Lemmas9 and 11, we see that the first term on the right-hand side of

(5.2) is

$$\ll \frac{T}{\log^2 y} \sum_{m \le y} \frac{\eta^2(m)}{\phi(m)} (L \log\log\log 30\, m + \log\log 3m)$$

$$\ll \frac{T}{\log y} (L \log\log L + \log L) \ll T \log\log L.$$

Thus,

(5.3) $\quad \underline{D}_{11} \ll T \log L.$

We now turn to $\underline{D}_{12}$. We have

$$\underline{D}_{12} = \frac{\chi(-1)}{\tau(\chi)} \frac{\eta(q)\ \phi(q)}{q} L^2(1,\chi) \frac{T}{2\Pi} \sum_{m\le y} \frac{1}{m} \sum_{n|m} \frac{\eta^2(n)\ g(n)}{\phi(n)}$$

$$\sum_{h\le y/m} \frac{a(mh)}{h} \sum_{k\le y/m} \frac{\bar{a}(mk)\ \eta(k)\ g(k)}{k\ \phi(k)} ,$$

since we may obviously asume that $(n,k)=1$.

By Lemma 12i) and iii) and Lemma 8ii), this is

$$\ll \frac{T}{L} \sum_{\substack{m\le y \\ m \text{ square-free}}} \frac{\sigma^2_{1/2}(m)}{m} \sum_{n|m} \frac{\eta^2(n)\ |g(n)|}{\phi(n)}$$

$$\ll \frac{T}{L} \sum_{n\le y} \frac{\eta^2(n)\ \sigma^2_{1/2}(n)\ |g(n)|}{n\phi(n)} \sum_{r\le y/n} \frac{\sigma^2_{1/2}(r)}{r}$$

$$\ll T \sum_{n\le y} \frac{\eta^2 9n)\ \sigma^2_{1/2}(n)\ |g(n)|}{n\ \phi(n)} .$$

Now for square-free n,

$$|g(n0| = \prod_{p|n} \left|1-2\ \chi(p) + \frac{\chi^2(p)}{p}\right| \le 4^{w(n)} = d^2(n)$$

and

$$\sigma^2_{1/2}(n) = \prod_{p|n} (1+ p^{-1/2})^2 \le 2^{2w(n)} = d^2(n),$$

$$(6.1)\quad \underline{D}_2 = \frac{1}{2\Pi} \int_1^T |\, L(1/2 + it,\chi)\, A(1/2 + it,\chi)\,|^2 \log \frac{t}{2\Pi}\, dt$$

$$+ O\Big(\int_1^T |L(1/2 + it,\chi)\ A(1/2 + it,\, \chi)|^2 \frac{dt}{t} \Big)$$

$$+ O_\varepsilon\, (y\, T^{1/2 + \varepsilon}].$$

To estimate this, we need to first estimate

$$\underline{D}_2^*\,(\tau) = \int_1^T |L(1/2 + iu,\chi)\ A(1/2 + iu,\chi)|^2\ du.$$

We shall prove that for $1 \leq \tau \leq T$,

$$(6.2)\quad \underline{D}_2^*(\tau) = \tau\Big(1 + \frac{\log \tau}{\log y}\Big) + O_\varepsilon(y^{1/2}\, T^{3/4 + \varepsilon}) + O\Big(T\, \frac{\log L}{\log y}\Big).$$

First observe that by standard methods one has easily that

$$\underline{D}_2^*\ (\tau)\ \ll_\varepsilon \tau^{1+\varepsilon} \qquad (\ \leq \tau \leq T).$$

If $\tau \leq T^{1/2}$ this is consistent with (6.2), so from now on we assume that $T^{1/2} < \tau \leq T$.

We write the integral for $D_2^*(\tau)$ as

$$D_2^*(\tau) = \frac{1}{i} \int_{1/2 - i}^{1/2 + it} L(s,\chi)\ \ L(1-s,\bar{\chi})\ \ A(s,\chi)\ \ A(1-s,\bar{\chi})\ ds$$

and move the line of integration to $\sigma = a^* = 1 + \dfrac{1}{\log \tau}$. In doing so we introduce an error term of $O_\varepsilon(y\, T^{1/2+\varepsilon})$; this follows from the bounds in (2.9) and (2.10) which, although stated only for $s \in R$, are clearly valid in a slightly larger region. Thus

$$D_2^*(\tau) \; \frac{1}{i} \int_{a^*+i}^{a^*+i\tau} L(s,\chi) \;\; L(1-s,\bar{\chi}) \;\; A(s,\chi) \;\; A(1-s,\bar{\chi}) \; ds$$

$$+ \; O_\varepsilon(yT^{1/2+\varepsilon}).$$

Next we replace $L(1-s,\bar{\chi})$ by $x(1-s,\chi)$ $L(s,\chi)$ (from the functional equation (2.23)) and set

$$B_2\,(j) \;=\; \sum_{\substack{hn=j\\ h\le y}} a(h)\; d(n)\; \chi(n).$$

Because $y< T^{1/2}$ and $T^{1/2}<\tau$, Lemma 2 is applicable with T replaced by τ and x by y. It then follows that

$$D_2^*(\tau) \;=\; \frac{2\Pi\; \chi(-1)}{\tau(\chi)} \sum_{k\le y} \frac{\bar{a}(k)}{k} \sum_{j<\frac{qk\tau}{2\Pi}} b_2(j) \;\; e(\frac{-j}{qk})$$

$$+ \; O_\varepsilon(yT^{1/2+\varepsilon}) \;.$$

By (3.9) , this may be written as

$$D_2^*(\tau) \;=\; \frac{2\Pi\; \chi(-1)}{\tau(\chi)} \sum_{h,k\le y} \frac{a(h)\;\bar{a}(k)}{k} \;\; \operatorname*{res}_{s=1} \; \Big(\frac{(\frac{qk\tau}{2\Pi H})^s}{s} \;\;\; D(s,\chi,\tfrac{-H}{qK})\Big)$$

$$+ \; O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) \;\; + \;\; O(TL^{-1}),$$

where

$$H = \frac{h}{(h,k)} \quad \text{and} \quad K = \frac{k}{(h,k)} \; .$$

Notice that in the sum over k we may assume that K is square-free and $(K,q)=1$ since otherwise $a(k) = 0$. We may therefore use Lemma 5 to compute the residue. We find that it equals

$$\tau(\chi)\ \bar{\chi}(H)\ \chi(K)\ \operatorname*{res}_{s=1}\ \left(\frac{\zeta^2(s)}{s}\left(\frac{\tau}{2\Pi H}\right)^{s}\ \left(F_1(s)\ (1+\chi(-1)\ K^{1-s}) - \phi(q)\ q^{-s}\right)\right).$$

Since $\zeta^2(s) = \frac{1}{(s-1)^2} + \frac{2\gamma}{s-1} + \ldots$ near $s=1$, this in turn equals

$$\tau(\chi)\ \bar{\chi}(h)\ \chi(k)\ \frac{\phi(q)}{q}\ \frac{\tau}{2\Pi}\ \frac{(h,k)}{h}\ \left(\chi(-1)\ \log \frac{\tau e^{2\gamma-1}(h,k)^2}{2\Pi\ hk} + \log q + (1+\chi(-1))\ \frac{F'_q}{F_q}(1)\right)$$

. The expression in parentheses is

$$\chi(-1)\ \log \frac{\tau(h,k)^2}{h\ k} + O\ (1),$$

so we have

$$D_2^{*}(\tau)\ + \tau\ \frac{\phi(q)}{q}\ \sum_{h,k\leq y}\ \frac{a(h)\ \bar{a}(k)\ \bar{\chi}(h)\ \chi(k)\ (h,k)}{h\ k}\ \left(\log \frac{\tau(h,k)^2}{hk} + O(1)\right)$$

$$+ O_{\varepsilon}(y^{1/2}\tau^{3/4+\varepsilon}) + O(\tau L^{-1}).$$

We now apply trhe Môbius inversion formula

$$f((h,k)) = \sum_{\substack{m|h\\ m|k}} \sum_{n|m} \eta(n)\, f(\tfrac{m}{n})$$

and

$$D_2^*(\tau) = \tau \frac{\phi(q)}{q} \sum_{h,k\leq y} \frac{a(h)\,\bar{a}(k)\,\bar{\chi}(h)\,\chi(k)}{h\,k} \sum_{\substack{m|h\\ m|k}} \sum_{n|m} \eta(n)\,\frac{m}{n}\left(\log \frac{\tau m^2}{hkn^2} + O(1)\right)$$

$$+ O_\varepsilon(Y^{1/2}T^{3/4+\varepsilon}) + O(TL^{-1}).$$

Interchanging the order of summation and replacing h by hm, k by km, we see that

$$D_2^*(\tau) = \tau\frac{\phi(q)}{q} \sum_{m\leq y} \frac{1}{m} \sum_{n|m} \frac{\eta(n)}{n} \sum_{h,k\leq y/m} \frac{a(mh)\,\bar{a}(mk)\,\bar{\chi}(h)\,\chi(k)}{h\,k}$$

$$\cdot\left(\log \frac{\tau}{hk} + O(\log 2n)\right)$$

$$+ O_\varepsilon(Y^{1/2}T^{3/4+\varepsilon}) + O(TL^{-1}).$$

Now $\sum_{n|m} \frac{\eta(n)}{n} = \frac{\phi(m)}{m}$, so we may rewrite this as

$$D_2^*(\tau) = \tau \text{ lof } \tau \frac{\phi(q)}{q} \sum_{m\leq y} \frac{\phi(m)}{m^2} \left| \sum_{h\leq y/m} \frac{a(mh)\,\bar{\chi}(h)}{h} \right|^2$$

$$- 2\tau \frac{\phi(q)}{q} \operatorname{Re} \sum_{m\leq y} \frac{\phi(m)}{m^2} \left(\sum_{h\leq y/m} \frac{a(mh)\,\bar{\chi}(h)\log h}{h}\right)\left(\sum_{k\leq y/m} \frac{\bar{a}(mk)\,\bar{\chi}(k)}{k}\right)$$

$$+ O(\tau \sum_{m\le y} \frac{1}{m} \sum_{n|m} \frac{\log 2n}{n} \Big| \sum_{h\le y/m} \frac{a(mh)\,\bar{\chi}(h)}{h} \Big|^2)$$

$$+ O_\varepsilon (y^{1/2}T^{3/4 = \varepsilon}) + O(TL^{-1}),$$

or

$$D_2^*(\tau) = \tau \frac{\phi(q)}{q} (\log \tau \sum_1 - 2 \text{ Re} \sum_2) + O(\tau \sum_3)$$

(6.3)

$$+ O_\varepsilon (y^{1/2}T^{3/4 + \varepsilon}) + O(TL^{-1})$$

for short.

By Lemmas 13i) and 8i) we have

$$\sum_1 = \frac{q^2}{\phi(q)^2 \log^2 y} \sum_{\substack{m\le y \\ (m,q)=1}} \frac{\eta^2(m)}{\phi(m)} + O(\frac{1}{\log^2 y} \sum_{m\le y} \frac{\sigma^2_{-1/2}(m)}{m \log^4 \frac{2y}{m}}) .$$

Set $s(u) = \sum_{m\le n} \frac{\sigma^2_{-1/2}(m)}{m}$. Then by Lemma 8ii),

$$S(u) = c \log u + E(u),$$

where $E(n) \ll 1$. The sum in the error term therfore equals

$$c \int_1^y \log^{-4} \frac{2y}{u} \frac{du}{u} + \int_1^y \log^{-4} \frac{2y}{u} \, d\, E*u)$$

$$= \frac{c}{3} \log^{-3} \frac{2y}{u} \Big|_1^y + E(u) \log^{-4} \frac{2y}{u} \Big|_1^y - 4 \int_1^y E(u) \log^{-5} \frac{2y}{u} \frac{du}{u}$$

$$\ll 1.$$

By Lemma 11, the main term is

$$\frac{q}{\phi(q) \log y} + O(\frac{1}{\log^2 y}) .$$

Hence

(6.4) $$1 = \frac{q}{\phi(q) \log y} + O(\frac{1}{\log^2 y}) .$$

Similarly, by Lemmas 13 and 8i),

$$2 = - \frac{q^2}{\phi(q)^2 \log^2 y} \quad \underset{(m,q)=1}{m \leq y} \quad \frac{\eta^2(m)}{\phi(m)} \log \frac{y}{m} + O(\frac{1}{\log y} \quad m \leq y \quad \frac{\sigma^2_{-1/2}(m)}{m \log^4 \frac{2y}{m}})$$

$$+ O(\frac{\log L}{\log^2 y} \quad m \leq y \quad \frac{\sigma^2_{-1/2}(m)}{m}) .$$

The main term is

$$- \frac{q}{2 \phi(q)} + O(\frac{1}{\log y})$$

by Lemma 11 and partial summation. The sum in the first error term was estimated above and found to be $\ll 1$. Thus the entire error term is

$$\ll \frac{1}{\log y} .$$

By Lemma 8 ii) , the sum in the remaining error term i $\ll$ log y, so whole term is

$$\ll \frac{\log L}{\log y} .$$

Thus

$$(6.5) \qquad {}_2 = - \frac{-q}{2\,\phi(q)} + O\left(\frac{\log L}{\log y}\right) .$$

We now turn to ${}_3$. We may disregard those m which are not square-free, so by Lemmas 13 and 8i) we obtain

$$ {}_3 \ll \frac{1}{\log^2 y} \underset{\substack{m \le y \\ m \text{ square-free}}}{} \frac{\sigma^2_{-1/2}(m)}{m} \underset{n|m}{} \frac{\log 2n}{n}$$

$$\ll \frac{1}{\log^2 y} \underset{n \le y}{} \frac{\sigma^2_{-1/2}(n) \log 2n}{n^2} \underset{r \le y/n}{} \frac{\sigma^2_{-1/2}(r)}{r} .$$

By Lemma 8ii) this is

$$\ll \frac{1}{\log y} \underset{n \le y}{} \frac{\sigma^2_{-1/2}(n)}{n^{z-\varepsilon}} \ll \frac{1}{\log y} .$$

Combining this with (6.3) – (6.5), we finally obtain

$$D_2^*(\tau) = \tau \left(1 + \frac{\log \tau}{\log y}\right) + O_\varepsilon\left(y^{1/2} T^{3/4 + \varepsilon}\right) + O\left(T \frac{\log L}{\log y}\right)$$

for $T^{1/2} < \tau \leq T$. This establishes (6.2) for $1 \leq \tau \leq T$.

We can now estimate $\underline{D}_2$. Using (6.2) and integration by parts, we find that the first term on the right-hand side of (6.1) is

$$-\frac{1}{2\pi}\int_1^T \log\frac{t}{2\pi}\, d\,\underline{D}_2^*(t)$$

$$= \frac{-1}{2\pi}\log\frac{t}{2\pi}\, k\underline{D}_2^*(t)\Big|_1^T + \frac{1}{2\pi}\int_1^T D_2^*(t)\frac{dt}{t}$$

$$= -\frac{T}{2\pi}L\left(1+\frac{L}{\log y}\right) + O_\varepsilon\left(y^{1/2}T^{3/4+\varepsilon}\right) + O(T\log L).$$

Simlarly, we find that the second term in (6.1) is

$$\ll L.$$

Hence,

$$(6.6)\qquad \underline{D}_2 = -\frac{T}{2\pi}L\left(1+\frac{L}{\log y}\right) + O_\varepsilon\left(y^{1/2}T^{3/4+\varepsilon}\right) + O(T\log L).$$

7. Completion of the proof.

By (4.5) we see that

(7.1) $\underline{N} = \frac{T}{2\Pi} L + O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) + O(T).$

Also, from (5.4) we have

$$\underline{D}_1 = O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) + O(T \log L),$$

and from (6.6) that

$$\underline{D}_2 = -\frac{T}{2\Pi} L(1+\frac{L}{\log y}) + O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) + O(T \log L).$$

Thus, by (2.20) it follows that

(7.2) $\underline{D} = \frac{T}{2\Pi} L(1+\frac{L}{\log y}) + O_\varepsilon(y^{1/2}T^{3/4+\varepsilon}) + O(T \log L).$

We now take $y=T^{1/2-2\varepsilon}$ in (7.1) and (7.2) and find that

(7.3) $\underline{N} = \frac{T}{2\Pi} L + O_\varepsilon(T)$

and

(7.4) $\underline{D} = (3+O(\varepsilon)) \frac{T}{2\Pi} L.$

This establishes (2.4) and (2.5) and therefore Theorem 2, provided that T is in the sequence defined in §2 (preceding (2.6)). To remove this restriction first note that every positive T is within O(1) of some element of and that increasing T by O(1) in (2.2) introduces at most O(L) new terms into the sum. However, by (2.9) and (2.10) each of these terms is

$$\ll_\varepsilon y^{1/2+\varepsilon/2} T^{1/4+\varepsilon/2} = T^{1/2-\varepsilon/4-\varepsilon^2}$$

if $y = T^{1/2-2\varepsilon}$. Thus (7.3) is valid for all large T. Similarly, increasing T by O(1) introduces at most O(L) new terms into the sum for $\underline{\underline{D}}$ in (2.3). Each of these is

$$\ll_\varepsilon T^{1-\varepsilon/2-2\varepsilon^2},$$

so (7.4) is also valid for all large T. This completes the proof of Theorem 2.

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