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On the Zeros of the Taylor Polynomials Associated with the Exponential Function

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While investigating a certain mean-value associated with the zeros of the nth derivative of the Riemann zeta-function [2] we obtained for each nonnegative integer n a formula with a constant factor

$$\alpha_n := n + 1 - \sum_{\nu=1}^{n} e^{-z_\nu},$$

where the complex numbers z_ν are the roots of the polynomial

$$E_n(z) := 1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}.$$

Thus $\alpha_0 = 1$, $\alpha_1 = 2 - e$, $\alpha_3 = 3 - 2e\cos 1$, and if

$$r = (\sqrt{2} + 1)^{1/3} \quad \text{and} \quad s = (\sqrt{2} - 1)^{1/3},$$

then

$$\alpha_3 = 4 - e^{2+r-s} - 2e^{(1+(s-r)/2)}\cos\frac{\sqrt{3}}{2}(r+s).$$

The numbers z_ν have been well studied, and for a given n most of them have real part smaller than $-kn$ for some positive constant k. Thus it is natural to expect that α_n should grow exponentially with n. We computed the first few α_n and

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quickly changed our expectations:

$$\begin{aligned} \alpha_0 &= 1 & \alpha_8 &= -0.0000062064\ldots \\ \alpha_1 &= -0.7182818284\ldots & \alpha_9 &= -0.0000018672\ldots \\ \alpha_2 &= +0.0626121201\ldots & \alpha_{10} &= -0.0000004703\ldots \\ \alpha_3 &= +0.0120619221\ldots & \alpha_{11} &= -0.0000000989\ldots \\ \alpha_4 &= +0.0019468374\ldots & \alpha_{12} &= -0.0000000153\ldots \\ \alpha_5 &= +0.0002139607\ldots & \alpha_{13} &= -0.0000000004\ldots \\ \alpha_6 &= -0.0000093400\ldots & \alpha_{14} &= +0.0000000009\ldots \\ \alpha_7 &= -0.0000154019\ldots & \alpha_{15} &= +0.0000000005\ldots. \end{aligned}$$

It seems that α_n is approaching 0 rather rapidly! Of course $\alpha_n \neq 0$ by Lindemann's famous theorem [4].

Szegö [**8**] initiated the study of the zeros of $E_n(z)$. It is convenient to scale down by a factor of n and let

$$\zeta_\nu = \frac{z_\nu}{n}.$$

Szegö proved that the ζ_ν cluster around the simple closed curve $\Gamma = \{z\colon |ze^{1-z}| = 1, |z| \leqslant 1\}$ as $n \to \infty$ and that the proportion which cluster along a given arc of Γ is asymptotic to the change in

$$\frac{1}{2\pi}\arg ze^{1-z}$$

as z varies along the arc. We mention that this implies that the proportion of zeros of E_n with negative real part is asymptotically

$$\frac{1}{2} + \frac{1}{\pi e} = 0.617099\ldots$$

since the arc of Γ which lies in the half plane $\operatorname{Re} z \leqslant 0$ has endpoints $z = \pm i/e$.

Buckholtz [**1**] has shown that the ζ_ν all lie strictly outside Γ and are within a distance $2e/n^{1/2}$ of Γ. By the Eneström-Kakeya theorem on polynomials with monotone coefficients (see Polya-Szegö [7, part III, problem 23]) all the ζ_ν are inside the unit circle $|z| = 1$. Moreover, Newman and Rivlin [**5**], [**6**] have established that the region $y^2 \leqslant cx$ has no zeros z_ν (no scaling) if c is any positive number such that $ce^c < \pi/2$; their paper [**5**] also contains a figure showing the location of the zeros of $E_n(z)$ for $n \leqslant 47$. The regular spacing of the z_ν is quite striking as is the parabolic region free of zeros. Saff and Varga considered the existence in general of a "parabolic" region free of zeros of the sections of power series of entire functions and have conjectured a precise relationship between the "width" of such a region and the order of the entire function. In [3], with Edrei, they prove the conjecture for a class of functions; this work also has an extensive bibliography on this and related problems.

The following indicates another aspect of the interesting geometry of the z_ν.

THEOREM. *If β is any positive number for which $\beta < 1 - \log 2 = 0.3068\ldots$, then*

$$|\alpha_n| \leqslant e^{-\beta n}$$

for all sufficiently large n.

Thus, α_n is an exponentially small sum of terms, most of which are exponentially large as functions of n. As a contrast we mention that it is not difficult to prove that $\sum_{\nu=1}^{n} e^{z_\nu}$ does increase exponentially with n.

The proof of the theorem is not difficult. We write

$$e^z = E_n(z) + R_n(z), \tag{1}$$

where

$$R_n(z) = \sum_{k=n+1}^{\infty} \frac{z^k}{k!}. \tag{2}$$

The idea of the proof is roughly as follows. Let $z_\nu = x_\nu + iy_\nu$ be a zero of $E_n(z)$ and consider $\sum e^{-z_\nu}$. If $x_\nu > n(1 - \log 2)$ then e^{-z_ν} is small. If $x_\nu < n(1 - \log 2)$ we use $e^{-z_\nu} = 1/R_n(z_\nu)$. In this case $|z_\nu|$ is not too large because ζ_ν is near Γ as above. Then $1/R_n(z_\nu)$ can be expanded into an absolutely convergent series of increasing powers of z_ν; the first term is $(n+1)!z_\nu^{-n-1}$. Now using the Lagrange interpolation formula we can show that

$$\sum_{\nu=1}^{n} z_\nu^{-m} = \begin{cases} 1/n! & \text{if } m = n+1 \\ 0 & \text{if } 2 \leqslant m \leqslant n \\ -1 & \text{if } m = 1. \end{cases} \tag{3}$$

Thus the $n + 1$ in the definition of α_n arises from $m = n + 1$ here. Then we show that the contribution of terms with "large" x_ν and $m \geqslant -1$ is small and similarly for terms with "small" x_ν and $m < -1$. These estimations require a bound for $n!$, a bound for the coefficients in the expansion of $R_n(z)^{-1}$ and the fact that the ζ_ν are near Γ. Note that the points z of Γ for which $x = 1 - \log 2$ satisfy $|z| = 1/2$.

LEMMA. *With $R_n(z)$ as above,*

$$\frac{1}{R_n(z)} = \frac{(n+1)!}{z^{n+1}}\left(1 + \sum_{k=1}^{\infty} c_k z^k\right),$$

where

$$|c_k| \leqslant \frac{1}{2}\left(\frac{2}{n+2}\right)^k.$$

The series is absolutely convergent for $|z| < (n+2)/2$.

We will prove this lemma and (3) later. Now we give the proof of the theorem. Fix positive numbers $\beta < \gamma^- < \gamma < \gamma^+ < 1 - \log 2$. Define a partition of $\{1, 2, \ldots, n\}$ into $S \cup L$ by $\nu \in S$ if $x_\nu \leqslant n\gamma$ and $\nu \in L$ if $x_\nu > n\gamma$. Note that by Buckholtz's results,

$$\left|\frac{z_\nu}{n}\right| = e^{\frac{x_\nu}{n} - 1} + o(1) \qquad (1 \leqslant \nu \leqslant n). \tag{4}$$

Trivially

$$|e^{-z_\nu}| \leqslant e^{-n\gamma} \qquad (\nu \in L), \tag{5}$$

and by (4),

$$|z_\nu| \geqslant ne^{\gamma^- - 1} \qquad (\nu \in L) \tag{6}$$

and

$$|z_\nu| \leqslant ne^{\gamma^+ - 1} < \frac{n}{2} \qquad (\nu \in S) \tag{7}$$

for sufficiently large n. From (5) we see that

$$\left| \sum_{\nu \in L} e^{-z_\nu} \right| \leqslant ne^{-n\gamma} \leqslant \frac{1}{2} e^{-n\beta} \tag{8}$$

for large n. By (1), (7), and the lemma,

$$\sum_{\nu \in S} e^{-z_\nu} = \sum_{\nu \in S} 1/R_n(z_\nu) = (n+1)! \sum_{\nu \in S} \sum_{k=0}^{\infty} c_k z_\nu^{k-n-1} \tag{9}$$

where $c_0 = 1$. By (3),

$$\sum_{k=0}^{n} \sum_{\nu \in S} c_k z_\nu^{k-n-1} = \frac{1}{n!} - c_n - \sum_{k=0}^{n} \sum_{\nu \in L} c_k z_\nu^{k-n-1}. \tag{10}$$

Then by (9) and (10)

$$\begin{aligned} \sum_{\nu=1}^{n} e^{-z_\nu} = (n+1) + \sum_{\nu \in L} e^{-z_\nu} - (n+1)! \\ \times \left(c_n + \sum_{\nu \in L} \sum_{k=0}^{n} c_k z_\nu^{k-n-1} - \sum_{\nu \in S} \sum_{k=n+1}^{\infty} c_k z_\nu^{k-n-1} \right). \end{aligned} \tag{11}$$

Now by (6), (7), and the lemma, we can bound

$$\left| c_n + \sum_{\nu \in L} \sum_{k=0}^{n} c_k z_\nu^{k-n-1} - \sum_{\nu \in S} \sum_{k=n+1}^{\infty} c_k z_\nu^{k-n-1} \right|$$

from above by

$$\begin{aligned} &\left(\frac{2}{n}\right)^n + n \sum_{k=0}^{n} \left(\frac{2}{n}\right)^k \left(ne^{\gamma^- - 1}\right)^{k-n-1} + n \sum_{k=n+1}^{\infty} \left(\frac{2}{n}\right)^k \left(ne^{\gamma^+ - 1}\right)^{k-n-1} \\ &\leqslant \left(\frac{2}{n}\right)^n + \frac{n^{-n}}{\left(e^{\gamma^- - 1}\right)^{n+1}} \sum_{k=0}^{\infty} \left(2e^{\gamma^+ - 1}\right)^k \\ &= \left(\frac{2}{n}\right)^n + \frac{n^{-n}}{\left(e^{\gamma^- - 1}\right)^{n+1}} \frac{1}{1 - 2e^{\gamma^+ - 1}} \end{aligned} \tag{12}$$

since $e^{\gamma^+ - 1} < 1/2$. It is easy to show that $(n+1)! \leqslant n^2(n/e)^n$. Thus by (5), (11), and (12),

$$\left| \sum_{\nu=1}^{n} e^{-z_\nu} - (n+1) \right| \leqslant \frac{1}{2} e^{-n\beta} + n^2 \left(\frac{2}{e}\right)^n + \frac{n^2 e^{-n\gamma}}{e^{\gamma^- - 1}\left(1 - 2e^{\gamma^+ - 1}\right)} \tag{13}$$

if n is sufficiently large. Since $(2/e) < e^{-\beta}$ and $\gamma^- > \beta$, the right-hand side of (13) is $\leqslant e^{-n\beta}$ if n is sufficiently large.

Now we prove (3) and the lemma. For any polynomial $Q(z)$ of degree $\leqslant n - 1$,

$$Q(z) = E_n(z) \sum_{\nu=1}^{n} \frac{Q(z_\nu)}{E_n'(z_\nu)(z - z_\nu)}$$

since both sides are polynomials of degree $\leqslant n-1$ which agree at the n points $z_1, \ldots, z_n$. We observe that

$$E_n'(z_\nu) = -\frac{z_\nu^n}{n!}.$$

We obtain the first formula in (3) by taking $Q(z) \equiv 1$, $z = 0$. The second formula follows from the choice $Q(z) = z^k$, $z = 0$ for each of $k = 1, 2, \ldots, n-1$. Finally, the third formula of (3) is a consequence of the fact that the numbers z_ν^{-1} are roots of $z^n + z^{n-1}/1! + \cdots + 1/n!$ so that their sum is -1.

To prove the lemma we write

$$f(z) = \frac{z^{n+1}}{(n+1)! R_n(z)} = \left(1 + (n+1)! \sum_{k=1}^{\infty} \frac{z^k}{(n+1+k)!}\right)^{-1}$$

and expand the right-hand side as a geometric series. This is legitimate if $|z| < (n+2)/2$, since then

$$\left|(n+1)! \sum_{k=1}^{\infty} \frac{z^k}{(n+1+k)!}\right| < \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^k = 1.$$

The power series we obtain for $f(z)$ is majorized by

$$1 + \sum_{l=1}^{\infty} \left[(n+1)! \sum_{k=1}^{\infty} \frac{z^k}{(n+1+k)!}\right]^l,$$

which in turn is majorized by

$$1 + \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} \frac{z^k}{(n+2)^k}\right)^l.$$

Since

$$1 + \sum_{l=1}^{\infty} \left(\sum_{k=1}^{\infty} w^k\right)^l = 1 + \sum_{l=1}^{\infty} \left(\frac{w}{1-w}\right)^l = \frac{1-w}{1-2w} = 1 + \frac{w}{1-2w}$$
$$= 1 + \frac{1}{2} \sum_{k=1}^{\infty} 2^k w^k,$$

the lemma follows.

We remark that $R_n(z)$ has no zeros z with $|z| \leqslant n+2$. This fact can be proved exactly as the Eneström-Kakeya theorem mentioned earlier. Thus the series for $z^{n+1}/R_n(z)$ actually converges absolutely for $|z| \leqslant n+2$. Since the z_ν satisfy $|z_\nu| \leqslant n$ we have the formulae

$$\sum_{\nu=1}^{n} e^{-z_\nu} = \sum_{\nu=1}^{n} 1/R_n(z_\nu) = n + 1 - (n+1)! c_n + (n+1)! \sum_{k=0}^{\infty} c_{n+1+k} \sum_{\nu=1}^{n} z_\nu^k \tag{14}$$

and

$$\limsup_{m \to \infty} |c_m|^{1/m} < \frac{1}{n+2}. \tag{15}$$

Thus, it is possibly the case that α_n is asymptotic to $-(n+1)!\, c_n$ but it is not clear how to better estimate c_k for $k \leqslant 2n$ and so prove this.

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Letters to the Editor

Editor:

The method used in Norwegian books (e.g., **[1]**) to prove the reflective property of a parabola seems more direct than that used by Robert Williams **[2]**. Referring to Williams's diagram, let Q by the y-intercept of l_2. Then, since l_2's slope is $x_0/2c$ and the parabola's equation is $y = x^2/4c$, $Q = (0, -y_0)$. It follows that $QFPD$ is a parallellogram (where $D = (x_0, -c)$), whence its diagonal QP bisects $\angle FPD$ and $\alpha = \beta$.

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Editor:

With reference to your recent editorial 'Strings, Substrings and the Nearest Integer Function' (*Amer. Math. Monthly*, 94 (Nov. 87) 855–860), I note that in your second example you express the nth Fibonacci number F_n using the nearest integer function. It may be of interest to note that F_n can be expressed recursively using the nearest integer function; *viz.*, F_n is the nearest integer to the geometric mean of F_{n-1} and F_{n+1}. (See my Lemma 1 on diophantine defining relationals in *Abstracts AMS*, 8 (Oct. 87) 437–438.) Indeed, there exist infinitely many recursions of this type; e.g., F_n is the nearest integer to the geometric mean of F_{n-2}, F_{n-1}, and F_{n+2}.

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