9 Mean Values of the Riemann Zeta-Function with Application to the Distribution of Zeros

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The most precise results about the horizontal distribution of the zeros of the Riemann zeta-function are deduced from mean value theorems that involve the zeta-function multiplied by a Dirichlet polynomial. We are interested here in those results that give information about zeros on or near the critical line. The first result of this sort that required a detailed arithmetic argument involving the coefficients of the Dirichlet polynomial in order to accurately estimate the mean in question is due to Selberg [15] in his proof that a positive proportion of the zeros are on the critical line. His paper also contains a density result

$$N(\sigma, T) \ll \frac{T}{(\sigma - \frac{1}{2})}$$

uniformly in $\sigma > \frac{1}{2}$, where, as usual, $N(\sigma, T)$ is the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ with $0 < \gamma < T$ and $\beta \geq \alpha$; he later [16] strengthened this result to

$$N(\sigma, T) \ll T^{1 - 1/4(\sigma - 1/2)} \log T$$

uniformly for $\frac{1}{2} \leq \sigma \leq 1$.

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In 1973, Levinson [11], relying on the sort of mean value theorem mentioned above but using a different starting point, showed that at least $\frac{1}{3}$ of the zeros of the zeta-function are on the critical line. Levinson called the Dirichlet polynomial he used in his argument a "mollifier" because, as a rough approximation to $1/\zeta(s)$, it succeeded in smoothing the wild behavior of $\zeta(s)$ near the critical line. Improvements in the lower bound for the proportion of zeros on the critical line have depended in part on better choices for the mollifier, which have been found through the use of the calculus of variations.

Further developments in this method of mollifying have yielded lower bounds for the proportion of zeros of $\xi^{(m)}(s)$ on the critical line where $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)$; $\xi(s)$ is entire, real on the critical line, and has the same complex zeros as $\zeta(s)$ does. Most notably, this proportion tends to 1 as m tends to infinity. Lower bounds can also be obtained for the proportion of zeros of $\xi^{(m)}(s)$ that are simple and on the critical line (see Conrey [2] and [3]).

Jutila [10] has used the method of Selberg to improve his density result. He showed that for any $\delta > 0$,

$$N(\sigma, T) \ll_\delta T^{1-(1-\delta)(\sigma-1/2)} \log T.$$

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Here we state a mean value theorem from which many of the above-mentioned results can be deduced. We also give two new corollaries. Then an analogous theorem about a discrete mean value is given, along with some of its consequences. Before stating the theorem, let us introduce some notation. Let $T > 0$ be large and let

$$B(s, P) = \sum_{n \le y} \frac{b(n, P)}{n^s},$$

where $y = T^\theta$, and

$$b(n, P) = \mu(n) P\left(\frac{\log y/n}{\log y}\right),$$

where P is entire with $P(0) = 0$ and μ is the usual Möbius function. Let $L = \log T$, $\alpha = a/L$, $\beta = b/L$ for complex numbers a and b. Let Q_1 and Q_2 be polynomials.

Theorem 1. *If $0 < \theta < \frac{1}{2}$ and α and β tend to 0 as $T \to \infty$ then for fixed c with $\frac{1}{2} \le c < \frac{3}{2} - \theta$,*

$$\frac{1}{i}\int_{c+i}^{c+iT} Q_1\left(\frac{-d}{da}\right)\zeta(s+\alpha)Q_2\left(\frac{-d}{db}\right)\zeta(1-s+\beta)B(s,P_1)B(1-s,P_2)\,ds$$

$$\sim T\left[Q_1(0)Q_2(0)P_1(1)P_2(1) + \frac{\partial}{\partial u}\frac{\partial}{\partial v}\frac{1}{\theta}\int T_aQ_1T_bQ_2\int P_1P_2\bigg|_{u=v=0}\right],$$

where the integrals are $\int_0^1 \ldots dx$, $P_1 = P_1(x+u)$, $P_2 = P_2(x+v)$,

$$T_aQ_1 = e^{-a(x+\theta u)}Q_1(x+\theta u), \qquad \textit{and} \qquad T_bQ_2 = e^{-b(x+\theta v)}Q_2(x+\theta v).$$

For example,

$$\int_1^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^2 \left|B\left(\frac{1}{2}+it, P\right)\right|^2 dt \sim T\left(P(1)^2 + \frac{1}{\theta}\int_0^1 P'(x)^2\,dx\right).$$

We note, also, that Levinson's theorem follows with the choices $P_1(x) = P_2(x) = x$, $Q_1(x) = Q_2(x) = -1-x$, $c = \frac{1}{2}$, $a = -b = -1.3$, and $\theta = \frac{1}{2} - \varepsilon$, $\varepsilon \to 0^+$. Also, Jutila's result follows with the choices $P_1(x) = P_2(x) = x$, $Q_1(x) = Q_2(x) = 1$, $a = b > 0$, $c = \frac{1}{2}$, and $\theta = \frac{1}{2} - \varepsilon$, $\varepsilon \to 0^+$.

Two other applications are as follows. First, let $N^d(T)$ denote the number of distinct zeros of the zeta-function in $0 < t < T$; then

$$N^d(T) \ge (0.628 + o(1))N(T),$$

where $N(T) \sim TL/(2\pi)$. This may be proved as follows. Let $N_r(T)$ denote the number of zeros of the zeta-function in $0 < t < T$ with multiplicity at most r, where zeros are counted according to their multiplicity. Then it is easy to show that

$$N^d(T) \ge \sum_{r=1}^{R} \frac{N_r(T)}{r(r+1)} + \frac{N_{R+1}(T)}{R+1}$$

for any $R \ge 1$. The above-mentioned result on N^d follows from the constants in Conrey [3] that may be deduced from Theorem 1.

As a second application we mention some results on the distribution of zeros of $\zeta^{(k)}(s)$. This topic is of interest because of its connection with the Riemann Hypothesis (see Levinson [11], Levinson and Montgomery [13], and Speiser [17]). In particular, the Riemann Hypothesis is equivalent to the assertion that all complex zeros of $\zeta'(s)$ have real part at least $\frac{1}{2}$. Levinson's method is based on a quantitative version of this. We mention that Levinson and Montgomery [13] have shown that

$$\sum_{\gamma_k < T} (\beta_k - \tfrac{1}{2}) \sim \frac{kT}{2\pi} \log\log T,$$

where $\rho_k = \beta_k + i\gamma_k$ denotes a zero of $\zeta^{(k)}(s)$; the number of terms in the sum is $\sim TL(2\pi)$. On RH at most finitely many terms in the sum are negative. On the other hand, it can be shown that

$$\sum_{0<\gamma_k<T} T^{1/2-\beta_k} \gg T$$

so that, e.g., there exist $R > 0$ and $c > 0$ such that at least cT zeros satisfy $0 < \gamma_k < T$ and $\beta_k < \frac{1}{2} + R/L$. Using Theorem 1 we can show that for any $R > 0$ there is a $c > 0$ such that

$$\sum_{\substack{0<\gamma_k<T \\ \beta_k > (1/2)+R/L}} 1 > cTL$$

for all large T. The question of the precise horizontal distribution of the zeros of $\zeta'(s)$ remains open to conjecture.

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Now we turn to discrete mean value theorems. We use the same notation mentioned before Theorem 1.

Theorem 2. *If $0 < \theta < \frac{1}{2}$ and α and β tend to 0 as $T \to \infty$, then*

$$\sum_{0<\gamma<T} Q_1\Big(-\frac{d}{da}\Big)\zeta(\rho+\alpha)Q_2\Big(-\frac{d}{db}\Big)\zeta(1-\rho+\beta)B(\rho, P_1)B(1-\rho, P_2)$$
$$\sim \frac{TL}{2\pi}\frac{\partial}{\partial u}\frac{\partial}{\partial v}\Bigg\{\Bigg[\frac{1}{\theta}\int P_1P_2 + \int P_1 \int P_2\Bigg]$$
$$\times \Bigg[\int T_aQ_1T_bQ_2 - \int T_aQ_1 \int T_bQ_2\Bigg]$$
$$+ \int P_1 \int P_2\Big(Q_1(0) - \int T_aQ_1\Big)\Big(Q_2(0) - \int T_bQ_2\Big)\Bigg\}\Bigg|_{u=v=0},$$

with the same notation conventions as in Theorem 1.

Theorem 3. *If $0 < \theta < \frac{1}{2}$ and α tends to 0 as $T \to \infty$, then*

$$\sum_{0<\gamma<T} Q\left(\frac{-d}{da}\right)\zeta(\rho+\alpha)B(\rho,P) \sim \frac{-TL}{2\pi}\frac{d}{du}\left(\left(Q(0)-\int T_a Q\right)\int P\right)\bigg|_{u=0},$$

where

$$P = P(x+u) \qquad \text{and} \qquad T_a Q = e^{-a(x+\theta u)}Q(x+\theta u).$$

As a first application of these theorems, we mention the results of Conrey, Ghosh, and Gonek [6] on $N^d(t)$ and on the number of simple zeros $N_1(T)$ in $0 < t < T$; assuming RH, Montgomery [14] proved that

$$N_1(T) \geq \left(\frac{2}{3} + o(1)\right)N(T)$$

using his pair correlation method. In [6] we show that on RH

$$N_1(T) \geq \left(\frac{19}{27} + o(1)\right)N(T);\ N^d(T) \geq \left(\frac{5}{6} + \frac{1}{81} + o(1)\right)N(T).$$

The first inequality is obtained via the Cauchy-Schwarz inequality in the form

$$N_1(T) \geq \frac{\left|\sum_{0<\gamma<T} \zeta'(\rho)B(\rho,P)\right|^2}{\sum_{0<\gamma<T} |\zeta'(\rho)B(\rho,P)|^2}.$$

Clearly, the right side can be evaluated by our theorems: The choice $P(x) = -\theta x^2 + (1+\theta)x$ is optimal (with $\theta \to 1/2^-$). The second inequality may be deduced from the first using Montgomery's theorem [14]

$$\text{(on RH)} \qquad \sum_{\gamma<T} m(\rho) \leq \left(\frac{4}{3} + o(1)\right)N(T),$$

where $m(\rho)$ denotes the multiplicity of the zero ρ.

A second application is to bound $N_2(T)$, the number of simple and double zeros in $0 < t < T$, from below. Again by Cauchy's theorem

$$N_2(T) \geq \frac{\left|\sum_{\gamma<T} (\zeta'(\rho)B(\rho,P_1) + \zeta''(\rho)B(\rho,P_2))\right|^2}{\sum_{\gamma<T} |\zeta'(\rho)B(\rho,P_1) + \zeta''(\rho)B(\rho,P_2)|^2}.$$

The right side may be evaluated using Theorems 2 and 3 and RH. Then using

$$P_1(x) = 0.866x - 0.115x^2 - 0.082x^3$$

and

$$P_2(x) = 0.761x - 0.362x^2 - 0.024x^3,$$

we obtain the new result:

Theorem 4. *Assuming the Riemann Hypothesis,*

$$N_2(T) \geq (0.955 + o(1))N(T).$$

Thus, on RH, fewer than 4.5 % of the zeros have multiplicity three or greater.

We may also use Theorems 2 and 3 to obtain some information about the number $N(T, U)$ of pairs of zeros of the Riemann zeta-function with imaginary parts γ, γ' between 0 and T for which $0 < \gamma' - \gamma \leq U$. Montgomery [14] has conjectured that

$$N(T, U) \sim N(T) \int_0^{UL} \left\{1 - \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2\right\} d\alpha$$

uniformly for $0 < \alpha_0 \leq UL \leq \alpha_1 < \infty$, and Gallagher [9] has shown, on RH, that

$$N(T, U) \leq \left(A + \frac{1}{2\pi^2 A} + O(A^{-2})\right)N(T),$$

where $A = UL$ is a positive integer or half-integer, and that if, in addition, almost all the zeros are simple then

$$N(T, U) \geq \left(A - 1 + \frac{1}{2\pi^2 A} + O(A^{-2})\right)N(T).$$

Gallagher's results allow for the possibility that for some $U \approx 1/L$,

$$N(T, U^+) - N(T, U^-) > \left(1 - O\left(\frac{1}{A^2}\right)\right)N(T).$$

We can apply Theorems 2 and 3 above to deduce:

Theorem 5. *Assuming the Riemann Hypothesis,*

$$N(T, U^+) - N(T, U^-) \le \left(\frac{2}{3} + O\left(\frac{1}{A}\right)\right) N(T)$$

uniformly for $0 < \alpha_0 \le UL = A \le \alpha_1 < \infty$.

To prove this theorem we first of all note that on RH it is equivalent to the following assertion:

$$\sum_{\substack{0<\gamma<T \\ \zeta(\rho+iU)\neq 0}} 1 \ge \left(\frac{1}{3} + O\left(\frac{1}{A}\right)\right) N(T).$$

By the Cauchy Schwarz inequality, the left side is

$$\ge \left| \sum_{\gamma<T} \zeta(\rho+iU) B(\rho+iU, P) \right|^2 \left(\sum_{\gamma<T} |\zeta(\rho+iU) B(\rho+iU, P)|^2 \right)^{-1},$$

which can be evaluated asymptotically via Theorems 2 and 3. (The fact that the argument of B is shifted presents no problem as

$$B(s + a/L, P) = B(s, P_1)$$

where $P_1(x) = e^{-ia\theta(1-x)} P(x)$.) Then, we find that if $P(1) = 1$, then

$$\sum_{\gamma<T} \zeta\left(\rho + \frac{iA}{L}\right) B\left(\rho + \frac{iA}{L}, P\right) \sim -N(T)\Big\{1 - J(iA)$$
$$+ iAJ(iA) \int_0^1 e^{-iA\theta(1-x)} P(x)\, dx\Big\}$$

and

$$\sum_{\gamma<T} \left| \zeta\left(\frac{\rho + iA}{L}\right) B\left(\rho + \frac{iA}{L}, P\right) \right|^2$$
$$\sim N(T)\Big\{1 + (1 - |J(iA)|^2) \frac{1}{\theta} \int_0^1 P'(x)^2\, dx$$
$$+ \left| 1 - iA\theta \int_0^1 e^{-iA\theta(1-x)} P(x)\, dx \right|^2$$
$$- 2\,\mathrm{Re}\left\{ J(iA)\left(1 - iA\theta \int_0^1 e^{-iA\theta(1-x)} P(x)\, dx\right)\right\}\Big\},$$

where $J(r) = \int_0^1 e^{-rx}\,dx$. Now, since $P(0) = 0$ and $P(1) = 1$, we find by an integration by parts that

$$\int_0^1 e^{-iA\theta(1-x)} P(x)\,dx = \frac{1}{iA\theta} + O\left(\frac{1}{A^2}\right)$$

for fixed P. Also $J(iA) \ll 1/A$ so that

$$\sum_{\substack{\gamma < T \\ \zeta(\rho + iU) \neq 0}} 1 \geq N(T)\left(1 + O\left(\frac{1}{A}\right)\right) \bigg/ \left(1 + \frac{1}{\theta}\int_0^1 P'(x)^2\,dx + O\left(\frac{1}{A}\right)\right).$$

The result now follows from the choices $P(x) = x$, $\theta \to 1/2^-$.

Finally, we mention two results of Conrey, Ghosh, and Gonek ([5] and [7]) that do not follow directly from Theorems 2 and 3 but are proven using similar techniques. Firstly, on GRH, a positive proportion of the zeros of the zeta-function of a quadratic number field are simple. (This result does not seem to be accessible via Montgomery's pair correlation method.) Secondly, on GRH, the gaps between consecutive zeros of the zeta-function are infinitely often larger than 2.68 times the average spacing.

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We now give a description of the main steps in the proofs of Theorems 1 and 2. Many of the details are similar to the work done in [6]. Let $\chi(s)$ denote the usual factor from the functional equation for the zeta-function; $\zeta(s) = \chi(s)\zeta(1-s)$. Let c be a fixed number satisfying $1 < c < \frac{3}{2} - \theta$, and let

$$I(\alpha, \beta, P_1, P_2) = \frac{1}{i}\int_{c+i}^{c+iT} \chi(1-s)\zeta(s+\alpha)\zeta(s+\beta)B(s, P_1)B(1-s, P_2)\,ds$$

and

$$I_1(\alpha, \beta, P_1, P_2) = \frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s) \times \frac{\zeta'}{\zeta}(s)\zeta(s+\alpha)\zeta(1-s+\beta)B(s, P_1)B(1-s, P_2)\,ds.$$

Let M and M_1 denote the means in question in Theorems 1 and 2 with $Q_1 = Q_2 \equiv 1$.

As a first step, we use the approximations

$$\chi(1-s+\alpha) = \left(\frac{t}{2\pi}\right)^{-\alpha} \chi((1-s)(1+O(1/|t|)) \sim e^{-a}\chi(1-s)$$

and

$$\frac{\chi'}{\chi}(s) \sim -L$$

for $t \approx T$. This gives

$$M \sim e^{-b} I(\alpha, -\beta, P_1, P_2)$$

and

$$M_1 \sim e^{-b} I_1(\alpha, -\beta, P_1, P_2) + e^{-a}\overline{I_1(-\bar{\alpha}, \bar{\beta}, \bar{P}_2, \bar{P}_1)} + \frac{e^{-a}L}{2\pi}\overline{I(-\bar{\alpha}, \bar{\beta}, \bar{P}_2, \bar{P}_1)}.$$

Next, in view of the relationship

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s) r^{-s}\, ds \sim e(-r),$$

for $0 < r < T/2\pi$ we can show that

$$I(\alpha, \beta, P_1, P_2) \sim \sum_{h,k \le y} \frac{b(h, P_1) b(k, P_2)}{k} \sum_{mn \le (Tk/2\pi h)} m^{-\alpha} n^{-\beta} e\left(-mn\frac{H}{K}\right),$$

where $H = h/(h, k)$ and $K = k/(h, k)$, and that

$$I_1(\alpha, \beta, P_1, P_2) \sim -\sum_{h,k \le y} \frac{b(h, P_1) b(k, P_2)}{k} \sum_{\ell mn \le (Tk/2\pi h)} \Lambda(\ell) m^{-\alpha} n^{-\beta} e\left(\frac{-\ell mnH}{K}\right).$$

Now we estimate the inner sums here using Perron's formula; this requires knowledge of the generating functions. If $(H, K) = 1$, then

$$\sum_{m,n} m^{-s-\alpha} n^{-s-\beta} e\left(\frac{-mnH}{K}\right) - K^{1-\alpha-\beta-2s}\zeta(s+\alpha)\zeta(s+\beta)$$

is an entire function. Also,

$$\sum_{\ell,m,n} \frac{\Lambda(\ell)}{\ell^s} m^{-s-\alpha} n^{-s-\beta} e\left(\frac{-\ell m n H}{K}\right)$$
$$- \frac{\zeta(s+\alpha)\zeta(s+\beta)}{K} \left(\frac{\zeta'}{\zeta}(s) + \sum_{p|K} p^{s+\alpha+\beta-1} \log p\right)$$
$$\dot{\times}\ (\mu^* T_{1-s-\alpha} 1^* T_{1-s-\beta} 1)(K)$$

has poles that are either not $\ll 1/L$ from 1 or are a distance $\ll 1/L$ from 1 but have residues that are small when averaged over h and k. In this formula $*$ denotes Dirichlet convolution, and

$$T_r 1(n) = n^r$$

for any r and n. Thus

$$I \sim \frac{1}{i} \int_{(c)} \sum_{h,k \leq y} \frac{b(h, P_1) b(k, P_2)}{h^s k^s} (h, k)^{1-\alpha-\beta-2s} \left(\frac{T}{2\pi}\right)^s \zeta(s+\alpha)\zeta(s+\beta) \frac{ds}{s}$$

and

$$I_1 \sim \frac{1}{2\pi i} \int_{(c)} \sum_{h,k \leq y} \frac{b(h, P_1) b(k, P_2)}{h^s k^{2-s}} (h, k) \zeta(s+\alpha)\zeta(s+\beta)$$
$$\times \left(\frac{-\zeta'}{\zeta}(s) + \sum_{p|K} p^{s+\alpha+\beta-1} \log p\right)$$
$$\times\ (\mu * T_{1-s-\alpha} 1 * T_{1-s-\beta} 1)(K) \left(\frac{T}{2\pi}\right)^s \frac{ds}{s},$$

where (c) denotes the straight line path from $c - i\infty$ to $c + i\infty$. The main terms arise from the poles of the integrand; using $\zeta(s) \sim 1/(s-1)$ we find

$$I \sim TL \frac{(e^{-a} S(-\alpha, \beta, P_1, P_2) - e^{-b} S(-\beta, \alpha, P_1, P_2))}{b - a}$$

and

$$I_1 \sim \frac{-TL^2}{2\pi} \left(\frac{S_2(\alpha, \beta, P_1, P_2)}{ab}\right.$$
$$+ \frac{e^{-a}}{b-a} \left(\frac{S(-\alpha, \beta, P_1, P_2)}{-a} + \frac{1}{L} S_1(-\alpha, \beta, P_1, P_2)\right)\Bigg)$$
$$- \frac{e^{-b}}{b-a} \left(\frac{S(-\beta, \alpha, P_1, P_2)}{-b} + \frac{1}{L} S_1(-\beta, \alpha, P_1, P_2)\right),$$

where

$$S(\alpha, \beta, P_1, P_2) = \sum_{h,k \le y} \frac{b(h, P_1)b(k, P_2)}{h^{1+\alpha}k^{1+\beta}} (h, k)^{1+\alpha+\beta},$$

$$S_1(\alpha, \beta, P_1, P_2) = \sum_{h,k \le y} \frac{b(h, P_1)b(k, P_2)}{h^{1+\alpha}k^{1+\beta}} (h, k)^{1+\alpha+\beta} \sum_{p|K} p^{\beta} \log p,$$

and

$$S_2(\alpha, \beta, P_1, P_2) = \sum_{h,k \le y} \frac{b(h, P_1)b(k, P_2)}{hk} (h, k)(\mu * T_{-\alpha}1 * T_{-\beta}1)(K).$$

The error terms are estimated using large sieve techniques and a Vaughan type identity.

Now we are to the arithmetic part of the argument. We can show that

$$S(\alpha, \beta, P_1, P_2) \sim \frac{1}{\theta L} \frac{\partial}{\partial u} \frac{\partial}{\partial v} e^{a\theta u + b\theta v} \int_0^1 P_1(x + u)P_2(x + v)\, dx \bigg|_{u=v=0}$$

$$= \frac{1}{\theta L} \int_0^1 (P_1'(x) + a\theta P_1(x))(P_2'(x) + b\theta P_2(x))\, dx,$$

and that

$$S_1(\alpha, \beta, P_1, P_2) \sim -\theta L S(\alpha, \beta, P_1, P_2^{(-1)}),$$

where

$$P_2^{(-1)}(x) = \int_0^x P_2(t)\, dt;$$

also,

$$S_2(\alpha, \beta, P_1, P_2) \sim S(-\alpha, \beta, P_1, P_2) + \frac{a}{L} P_1(1)P_2(1) + \frac{ab\theta}{L} P_1(1)P_2^{(-1)}(1).$$

Moreover,

$$S(\alpha, \beta, P_1, P_2) = S(\beta, \alpha, P_2, P_1)$$

and

$$S_2(\alpha, \beta, P_1, P_2) = S_2(\beta, \alpha, P_1, P_2)$$

so that in view of the above relationship between S and S_2 it follows that

$$S(-\beta, \alpha, P_1, P_2) \sim S(-\alpha, \beta, P_1, P_2) - \frac{b - a}{L} P_1(1)P_2(1).$$

Thus, we are led to

$$I(\alpha, \beta, P_1, P_2) \sim TL\left(\frac{e^{-a} - e^{-b}}{b-a} S(-\alpha, \beta, P_1, P_2) + \frac{e^{-b}}{L} P_1(1)P_2(1)\right)$$
$$\sim TL\left(\frac{e^{-a} - e^{-b}}{b-a} S(-\beta, \alpha, P_1, P_2) + \frac{e^{-a}}{L} P_1(1)P_2(1)\right)$$

and

$$I_1(\alpha, \beta, P_1, P_2) \sim \frac{-TL^2}{2\pi}\Bigg(S(-\alpha, \beta, P_1, P_2)\left(\frac{1}{ab} - \frac{e^{-a}}{a(b-a)} + \frac{e^{-b}}{b(b-a)}\right)$$
$$+ \theta S(-\alpha, \beta, P_1, P_2^{(-1)}) \frac{e^{-b} - e^{-a}}{b-a}$$
$$+ \frac{P_1(1)P_2(1)}{L}\left(\frac{1-e^{-b}}{b}\right)$$
$$+ \frac{\theta P_1(1)P_2^{(-1)}(1)}{L}(1 - e^{-b})\Bigg).$$

Then

$$M \sim T\left(P_1(1)P_2(1) + \frac{\partial}{\partial u}\frac{\partial}{\partial v} e^{-a\theta u - b\theta v} \frac{1}{\theta}\int_0^1 P_1(x+u)P_2(x+v)\,dx\bigg|_{u=v=0}\right.$$

follows, as well as

$$M_1 \sim \frac{TL^2}{2\pi}\Bigg\{S(-\alpha, -\beta, P_1, P_2)\left(\frac{1-e^{-a-b}}{a+b} + \frac{e^{-a}}{ab} - \frac{e^{-a-b}}{b(a+b)} - \frac{1}{a(b+a)}\right.$$
$$\left. + \frac{e^{-b}}{ab} - \frac{e^{-a-b}}{a(b+a)} - \frac{1}{b(b+a)}\right)$$
$$+ \theta S(-\alpha, -\beta, P_1, P_2^{(-1)})\left(\frac{1-e^{-b-a}}{a+b}\right)$$
$$+ \theta S(-\alpha, -\beta, P_1^{(-1)}, P_2)\left(\frac{1-e^{-b-a}}{a+b}\right)$$
$$+ \frac{P_1(1)P_2(1)}{L}\left(1 - \frac{1-e^{-a}}{a} - \frac{1-e^{-b}}{b}\right)$$
$$+ \frac{\theta}{L}\left(P_1(1)P_2^{(-1)}(1)(1-e^{-b}) + P_1^{(-1)}(1)P_2(1)(1-e^{-a})\right)\Bigg\}$$

$$= \frac{TL^2}{2\pi}\Bigg\{S(-\alpha, -\beta, P_1, P_2)(J(a+b) - J(a)J(b))$$

$$+ \theta J(a+b)(S(-\alpha, -\beta, P_1, P_2^{(-1)}) + S(-\alpha, \beta, P_1^{(-1)}, P_2))$$

$$+ \frac{P_1(1)P_2(1)}{L}(1 - J(a) - J(b)) + \frac{\theta}{L}(P_1(1)P_2^{(-1)}(1)bJ(b)$$

$$+ P_1^{(-1)}(1)P_2(1)aJ(a))\Bigg\},$$

where $J(r) = \int_0^1 e^{-rx}\,dx$. Now let

$$f(u, v) = \int_0^1 (P_1^{(-1)}(x+u)P_2(x+u) + P_1(x+u)P_2^{(-1)}(x+v))\,dx$$

$$= P_1^{(-1)}(x+u)P_2^{(-1)}(x+v)\Big|_0^1$$

$$= P_1^{(-1)}(1+u)P_2^{(-1)}(1+v) - P_1^{(-1)}(u)P_2^{(-1)}(v)$$

and let

$$g(u, v) = \int_0^1 P_1(x+u)\,dx \int_0^1 P_2(x+v)\,dx.$$

Then $f(0,0) = g(0,0)$, $f_u(0,0) = g_u(0,0)$, $f_v(0,0) = g_v(0,0)$ and $f_{uv}(0,0) = g_{uv}(0,0)$. We will use this to replace $f(u,v)$ by $g(u,v)$ in the formula for M_1. By our earlier formula for S we now have

$$M_1 \sim \frac{TL}{2\pi}\frac{\partial}{\partial u}\frac{\partial}{\partial v}\Bigg\{e^{-a\theta u - b\theta v}\left[\frac{1}{\theta}\int P_1P_2 + \int P_1 \int P_2\right][J(a+b) - J(a)J(b)]$$

$$+ \int P_1 \int P_2(1 - e^{-a\theta u}J(a))(1 - e^{-b\theta v}J(b))\Bigg\}\Bigg|_{u=v=0},$$

where the integrals are $\int_0^1 \ldots dx$ and $P_1 = P_1(x+u)$, $P_2 = P_2(x+v)$.

Finally, our formulas are uniform in α and β and may be differentiated with respect to these variables (using Cauchy's formula for example). Since $Q(-d/da)e^{-ay} = Q(y)e^{-ay}$, Theorems 1 and 2 follow.

5. Concluding Remarks

In conclusion, we mention possible directions for further development that this work may suggest. The most obvious possibility regards the range of θ in Theorem 1. For the special choice $Q_1 = Q_2 \equiv 1$, $a = b = 0$, the range $(0, \frac{9}{17})$ is admissible for θ, as shown by the work of Balasubramanian, Conrey, and Heath-Brown [1]. There is no reason why this shouldn't work for arbitrary Q_1, Q_2, a, and b. More significant, however, is the work of Iwaniec and Deshouillers [8], which suggests that the range $(0, \frac{4}{7})$ should be admissible for θ.

Next is the question of optimal choices of functions P and Q for various applications. In general, if our application requires that $P_1 = P_2$ and $Q_1 = Q_2$ with P_1 (resp. Q_1) specified, then the optimal choice of Q_1 (resp. P_1) can be determined through the calculus of variations in a straightforward way. This is the situation with the result on simple zeros. However, in the case of the lower bound for N_r with $r \geq 2$ and in the case of the lower bound for the proportion of zeros of ζ on the critical line, the optimal choices have not been completely determined. (See Conrey [4] for a description of what choices of P and Q are admissible in the latter problem).

Finally, it may well be that there are direct applications of these theorems to the questions of how large and how small the gaps between consecutive zeros of the zeta-function can be.

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Note added in proof.

We are unable, at present, to estimate some of the error terms in Theorem 2 without assuming the Generalized Lindelöf Hypothesis. Consequently, the applications of Theorem 2 mentioned here depend on RH and GLH.