ACTA ARITHMETICA
LII (1989)

On mean values of the zeta-function, II

by

J. B. Conrey and A. Ghosh* (Stillwater, Okla.)

In the study of the finer behavior of the Riemann zeta-function the problem of finding an asymptotic formula for

$$M(T) = \int_1^T |\zeta(1/2+it)|\,dt$$

is of some interest. Ramachandra [6] has shown that $M(T)$ has order of magnitude $T(\log T)^{1/4}$. The present authors [2] have shown, assuming the Riemann Hypothesis, that

$$M(T) \gtrsim T \sum_{n \leqslant T} \frac{d_{1/2}(n)^2}{n}$$

where $d_{1/2}(n)$ is the nth coefficient in the Dirichlet series expansion for $\zeta(s)^{1/2}$ with $\sigma > 1$. Moreover, Heath-Brown's argument in [4] can be adapted to prove that

$$M(T) \lesssim 3.32\, T \sum_{n \leqslant T} \frac{d_{1/2}(n)^2}{n}.$$

We remark that the sum here is easily evaluated as

$$\sum_{n \leqslant T} \frac{d_{1/2}(n)^2}{n} \sim c(\log T)^{1/4}$$

with

$$c = \Gamma(5/4)^{-1} \prod_p \left((1-p^{-1})^{1/4} \sum_{m=0}^{\infty} \left(\frac{\Gamma(m+1/2)}{\Gamma(1/2)\, m!}\right)^2 p^{-m}\right)$$

where the product is over primes p (and is absolutely convergent).

---

* Research of both authors partially supported by NSF grants.

While it is known that

$$\int_1^T |\zeta(\sigma+it)|\, dt \sim T \sum_{n=1}^{\infty} \frac{d_{1/2}(n)^2}{n^{2\sigma}}$$

for $\sigma > 1/2$ (see [7], Section 7.11), the corresponding mean with $\sigma = 1/2$ is elusive. In fact, if $F(s)$ is a function which is representable by a Dirichlet series in some half plane and if $F(s)$ has infinitely many simple zeros on the line $\sigma = \sigma_0$, then no asymptotic formula for

$$\int_1^T |F(\sigma_0+it)|\, dt$$

seems to be known.

In this paper we give an example where we can find upper and lower bounds for such a mean where the constants involved are close.

THEOREM. *Let*

$$I(T) = \int_1^T |\zeta(1/2+it)\, \zeta'(1/2+it)|\, dt.$$

Then, assuming the Riemann Hypothesis,

$$0.53\ldots \lesssim \frac{I(T)}{T \log^2 T} \lesssim 0.57\ldots$$

Remark. The upper constant is $3^{-1/2}$ and it is obtained unconditionally. The lower constant is

$$\left(\frac{1}{4} + \left(\frac{e^2-5}{4\pi}\right)^2\right)^{1/2}.$$

An unconditional lower bound of 1/2 is essentially trivial.

In Conrey [1] the more complicated example

$$J(T) = \int_1^T |\zeta^3(1/2+it)\, \zeta'(1/2+it)|\, dt$$

is considered and upper and lower bounds

$$0.02608\ldots \lesssim \frac{J(T)}{T \log^5 T} \lesssim 0.02616\ldots$$

are obtained.

The theorem follows from two lemmas. For the first lemma we recall the function $Z(t)$ from the theory of the Riemann zeta function. It is a real valued function of a real variable such that

$$|Z(t)| = |\zeta(1/2+it)|.$$

Also, $Z(t)$ changes sign at $t = t_0$ if and only if $1/2+it_0$ is a zero of $\zeta(s)$ with odd multiplicity. These conditions define $Z(t)$ apart from a plus or minus sign which would not be important here.

LEMMA 1. *Assuming the Riemann Hypothesis,*

$$\int_1^T |Z(t)Z'(t)|\,dt \sim \frac{e^2-5}{4\pi} T\log^2 T.$$

Proof. Let γ and γ^+ be successive zeros of $Z(t)$ with $0 < \gamma \leqslant \gamma^+ \leqslant T$. Then, assuming the Riemann Hypothesis, there is a unique number t_γ in $[\gamma, \gamma^+]$ where $Z'(t_\gamma) = 0$. If $\gamma \leqslant t \leqslant t_\gamma$, then $Z(t)Z'(t) \geqslant 0$ while if $t_\gamma \leqslant t \leqslant \gamma^+$, then $Z(t)Z'(t) \leqslant 0$. Therefore, if γ_0 denotes the least positive zero of $Z(t)$ and γ_T denotes the least zero of $Z(t)$ which is $\geqslant T$, then

$$\int_0^T |Z(t)Z'(t)|\,dt = \int_0^{\gamma_0} |ZZ'| + \sum_{0<\gamma\leqslant T} \Big(\int_\gamma^{t_\gamma} ZZ' - \int_{t_\gamma}^{\gamma^+} ZZ'\Big) - \int_T^{\gamma_T} |ZZ'|$$

$$= \sum_{0<\gamma\leqslant T} \left(\frac{Z(t)^2}{2}\Big|_\gamma^{t_\gamma} - \frac{Z(t)^2}{2}\Big|_{t_\gamma}^{\gamma^+}\right) + O(T^{1/3})$$

$$= \sum_{0<\gamma\leqslant T} |Z(t_\gamma)|^2 + O(T^{1/3})$$

since

$$|Z(t)Z'(t)| \ll t^{1/3}$$

and

$$|\gamma_T - T| \ll 1.$$

In Conrey and Ghosh [3] we show that, on RH,

$$\sum_{0<\gamma\leqslant T} |\zeta(1/2+it_\gamma)|^2 \sim \frac{e^2-5}{4\pi} T\log^2 T.$$

Since $|Z(t)| = |\zeta(1/2+it)|$, this completes the proof of the lemma.

LEMMA 2. *Suppose that $f(x)$ and $g(x)$ are real continuous functions on $[a, b]$, and that*

$$\int_a^b f(x)\,dx \geqslant \alpha, \qquad \int_a^b g(x)\,dx \geqslant \beta,$$

where α and β are non-negative. Then

$$\int_a^b |f(x)+ig(x)|\,dx \geqslant (\alpha^2+\beta^2)^{1/2}.$$

Proof. Let

$$F(t)=\int_a^t f(x)\,dx, \quad G(t)=\int_a^t g(t)\,dt$$

and suppose that $F(b)=m$ and $G(b)=n$ where $m\geqslant\alpha$, $n\geqslant\beta$. Then we require a lower bound for

$$\int_a^b (F'(t)^2+G'(t)^2)^{1/2}\,dt$$

where F' and G' are continuous functions with $F(a)=G(a)=0$, $F(b)=m$, $G(b)=n$. But this integral gives the arc length of the path $p(t)=(F(t), G(t))$ in the plane from $(0, 0)$ to (m, n). This arc length is clearly not less than the length of the straight line path, which is $(m^2+n^2)^{1/2}$. The lemma follows.

Proof of theorem. For the upper bound we have by the Cauchy–Schwarz inequality (see Ingham [5] for the second moment of ζ'),

$$I(T)\leqslant\Big(\int_1^T|\zeta(1/2+it)|^2\,dt\Big)^{1/2}\Big(\int_1^T|\zeta'(1/2+it)|^2\,dt\Big)^{1/2}$$

$$\sim(T\log T)^{1/2}(\tfrac{1}{3}T\log^2 T)^{1/2}=3^{-1/2}T\log^2 T.$$

For the lower bound, we make use of the fact that

$$\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+it)\sim-\tfrac{1}{2}\log t,$$

whence follows easily

$$\int_1^T|\zeta(1/2+it)|^2\left|\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+it)\right|dt\sim\frac{T}{2}\log^2 T. \tag{1}$$

This leads to the "trivial" lower bound, since

$$I(T)=\int_1^T|\zeta(1/2+it)|^2\left|\operatorname{Re}\frac{\zeta'}{\zeta}(1/2+it)+i\operatorname{Im}\frac{\zeta'}{\zeta}(1/2+it)\right|dt \tag{2}$$

is clearly greater than or equal to the integral in (1). We observe that by the properties of $Z(t)$,

$$\zeta(1/2+it)=Z(t)e^{i\vartheta(t)}$$

where $\vartheta(t)$ is a real valued function of t. Then

$$i\frac{\zeta'}{\zeta}(1/2+it)=\frac{Z'}{Z}(t)+i\vartheta'(t)$$

so that

$$\frac{Z'}{Z}(t) = -\operatorname{Im}\frac{\zeta'}{\zeta}(1/2+it).$$

Thus,

$$\int_1^T |\zeta(1/2+it)|^2 \left|\operatorname{Im}\frac{\zeta'}{\zeta}(1/2+it)\right| dt = \int_1^T |Z(t)Z'(t)|\, dt \sim \frac{e^2-5}{4\pi} T\log^2 T \tag{3}$$

by Lemma 1.

The theorem now follows from (1), (2), (3) and Lemma 2.

References

[1] J. B. Conrey, *The fourth moment of derivatives of the Riemann zeta-function*, Quart. J. Math. Oxford (2) 39 (1988), 21–36.

[2] J. B. Conrey and A. Ghosh, *On mean values of the zeta-function*, Mathematika 31 (1984), 159–161.

[3] — — *A mean value theorem for the Riemann zeta-function at its relative extrema on the critical line*, J. London Math. Soc. (2) 32 (1985), 193–202.

[4] D. R. Heath-Brown, *Fractional moments of the Riemann zeta-function*, J. London Math. Soc. (2) 24 (1981), 65–78.

[5] A. E. Ingham, *Mean-value theorems in the theory of the Riemann zeta-function*, Proc. London Math. Soc. (2) 27 (1926), 273–300.

[6] K. Ramachandra, *Some remarks on the mean value of the Riemann zeta function and other Dirichlet series. III*. Ann. Acad. Sci. Fenn. AI Math. 5 (1980), 145–158.

[7] E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, 2nd ed., Clarendon Press, Oxford 1986.

OKLAHOMA STATE UNIVERSITY
Stillwater, OK 74078, U.S.A.

Current Address:
J. B. Conrey
INSTITUTE FOR ADVANCED STUDY
Princeton, NJ 08540, U.S.A.

Received on 29.9.1987
and in revised form on 6.1.1988 (1753)