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Zeros of Derivatives Of the Riemann Zeta-Function Near the Critical Line

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To Paul Bateman on the occasion of his seventieth birthday

1. Introduction

The question of the horizontal distribution of the zeros of derivatives of Riemann's zeta-function is an interesting one in view of its connection with the Riemann Hypothesis. Indeed, Speiser [9] showed that the Riemann Hypothesis is equivalent to the assertion that no non-real zero of $\zeta'(s)$ is to the left of the critical line $\sigma = \Re s = 1/2$. Levinson and Montgomery [7] proved a quantitative version of this, namely that $\zeta(s)$ and $\zeta'(s)$ have essentially the same number of zeros to the left of $\sigma = 1/2$. More precisely, if $N_k(T)$ denotes the number of zeros of $\zeta^{(k)}(s)$ in the region $0 < t \le T$, then

$$N_k(T) = \frac{T}{2\pi}\log\frac{T}{2\pi e} + O_k(\log T); \tag{1}$$

Montgomery and Levinson proved that up to a height T the difference between the number of zeros of ζ in $\sigma < 1/2$ and the number of zeros of ζ' there is $\ll \log T$. Moreover, they showed that $\zeta'(s)$ vanishes on $\sigma = 1/2$ only at a multiple zero of $\zeta(s)$ (hence probably never) and that

$$\sum_{\substack{0<\gamma_1<T\\ \beta_1>1/2}} (\beta_1 - 1/2) = \frac{T}{2\pi}\log\log\frac{T}{2\pi} + O(T)$$

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where $\rho_k = \beta_k + i\gamma_k$ denotes a zero of $\zeta^{(k)}(s)$ so that, on average at least, the zeros of $\zeta'(s)$ in $0 < t < T$ are a distance $(\log\log T)/(\log T)$ from the critical line. By contrast the consecutive ordinates of zeros of $\zeta(s)$ in $|t| < T$ differ by $\ll 1/(\log T)$ on average. Thus, zeros of ζ' are rather far from the critical line on average. These observations probably led Levinson to believe that $\zeta'(s)$ does not behave as "erratically" as $\zeta(s)$ in the immediate vicinity of the critical line (i.e. at a distance $\ll 1/(\log t)$ from the critical line) and $\zeta'(s)$ can be "mollified" or smoothed more efficiently near the critical line. Thus, he used Littlewood's lemma and an efficient mollifer to show that $\zeta'(s)$ does not have too many zeros to the left of $\sigma = 1/2$, whence the same is true of $\zeta(s)$. Of course, since the zeros of $\zeta(s)$ are symmetric about $\sigma = 1/2$, this implied that $\zeta(s)$ had zeros on the line $\sigma = 1/2$, specifically, at least 1/3 of the zeros of $\zeta(s)$ must be on $\sigma = 1/2$. This result was a quantitative improvement over Selberg's result that a positive proportion of zeros of $\zeta(s)$ are on the critical line. Of course the methods of Selberg and Levinson are different, but much of the success of Levinson's method should be attributed to the fact that a smoothing of $\zeta(s)$ on the critical line was replaced by a smoothing of $\zeta'(s)$ (near the critical line). Indeed, when smoothing (on $\sigma = 1/2$) with a Dirichlet polynomial

$$B(s) = \sum_{n \le T^\theta} b(n) n^{-s} \tag{2}$$

with $b(1) = 1$, $\theta < 1/2$, the best known result for $\zeta(s)$ is with

$$b(n) = \mu(n)(1 - (\log n)/(\theta \log T))$$

which leads to (as $\theta \to 1/2$)

$$\int_l^T |\zeta(1/2+it)B(1/2+it)|^2\, dt \sim 3T, \tag{3}$$

while with $\zeta'(s)$ the same choice of B leads to

$$\int_2^T \left| \frac{\zeta'(1/2+it)}{\log t} B(1/2+it) \right|^2 dt \sim 4T/3. \tag{4}$$

In fact, a more elaborate choice of B allows the "4/3" in (4) to be replaced by

$$1/2 + \frac{\sqrt{3}}{3} \coth \frac{\sqrt{3}}{2} = 1.3255.... \tag{5}$$

It seems that $\zeta'(s)$ can be smoothed better than $\zeta(s)$ because the presence of zeros of $\zeta(s)$ on the critical line makes the smoothing more difficult.

We see more evidence for this relationship between good "smoothing" and absence of zeros when we consider zeros of higher derivatives of $\zeta(s)$. Thus, Levinson and Montgomery have shown that

$$2\pi \sum_{0<\gamma_k<T} (\beta_k - 1/2) = kT \log\log T + T(\frac{1}{2}\log 2 - k\log\log 2) - 2\pi k \operatorname{li}\left(\frac{T}{2\pi}\right) + O(\log T) \tag{6}$$

and that if the Riemann Hypothesis is true, then only finitely many of the ρ_k satisfy $\beta_k < 1/2$. Thus, on average in $0 < t < T$ the β_k are $1/2 + k(\log\log T)/(\log T)$. Of course the "average" situation may never take place. Nevertheless, there seems to be a definite migration of zeros of higher derivatives of ζ away from the critical line. (For an interesting chart on the location of zeros of $\zeta''(s)$ compared to zeros of $\zeta'(s)$, see Spira [10] where, for small ordinates, the ordinates of zeros of ζ' and ζ'' agree to a surprising degree, while the abscissa of a zero of ζ'' is larger than that of the "corresponding" zero of ζ'.) Thus, as k increases, the zeros of $\zeta^{(k)}(s)$ seem to move farther to the right of the critical line. As far as smoothing goes, we can show that with B as in (2) there is a choice of θ and $b(n)$ which leads to

$$\int_2^T \left| \frac{\zeta^{(k)}(1/2+it)}{\log^k t} B(1/2+it) \right|^2 dt \sim c_k T \tag{7}$$

where

$$c_k = 1/2 + \frac{\coth\left(\frac{k}{2}\sqrt{\frac{1+1/(2k)}{1-1/(2k)}}\right)}{2\sqrt{1-1/(4k^2)}} = 1 + O(1/k^2).$$

Thus, as k increases, $\zeta^{(k)}(s)$ can be smoothed more efficiently as well. (The presence of the $\log^{-k} t$ factor in this formula is inevitable because near the $1/2$ -line $\zeta^{(k)}(s)$ on average has an order of magnitude which is greater than that of $\zeta(s)$ by a factor of $\log^k t$.) We would like to know the precise horizontal distribution of zeros of $\zeta^{(k)}$. In particular, we would like to know whether in Levinson's method there is a loss due to the presence of zeros of ζ' in the region $\sigma < 1/2 + c/\log t$ for all $c > 0$. Unfortunately, we cannot answer this question. However, Theorem 2 below indicates that there probably is some loss. We would conclude that while Selberg's method cannot detect zeros on the critical line which have small gaps between them, Levinson's method cannot detect the zeros of ζ' too near the critical line (and we believe that such zeros exist).

In our statements k is fixed and $T \to \infty$. From Levinson and Montgomery [7] we can say that

(i) Almost all zeros of $\zeta^{(k)}(s)$ are in

$$1/2 - \frac{\phi(t)\log\log t}{\log t} \leq \sigma < 1/2 + \frac{\phi(t)\log\log t}{\log t}$$

where ϕ is any function which goes to infinity with t; on RH the lower bound $1/2$ holds for $t > t_0$;

(ii) a positive proportion of the zeros of $\zeta^{(k)}(s)$ are in the region

$$\sigma < 1/2 + (k+\epsilon)\frac{\log\log T}{\log T}, \qquad 0 < t < T$$

for any $\epsilon > 0$; (this follows from (6) and (12))

(iii) there are $\gg_\epsilon T \log\log T$ zeros in the region

$$\sigma > 1/2 + (k-\epsilon)\frac{\log\log T}{\log T}, \qquad 0 < t < T$$

for any $\epsilon > 0$; (this also follows from (6) and (12)).

We add to these by proving

Theorem 1. *With the above notation:*

(a) *Almost all the zeros of* $\zeta^{(k)}(s)$ *are in the region*

$$\sigma > 1/2 - \frac{\phi(t)}{\log t}$$

for any $\phi(t)$ *which goes to infinity with* t*;*

(b) *for any* $c > 0$*, a positive proportion of zeros of* $\zeta^{(k)}(s)$ *are in the region*

$$\sigma \geq 1/2 + c/\log t;$$

(c) *assuming the Riemann Hypothesis, there are* $\gg_\epsilon T$ *zeros of*

$$\zeta^{(k)}(s)$$

in the region

$$1/2 \leq \sigma < 1/2 + \frac{(1+\epsilon)\log\log T}{\log T}, \qquad 0 < t < T$$

for any $\epsilon > 0$.

We remark that (a) and (c) give new information only when $k > 1$ while (b) is new for all $k \geq 1$. The first two results are a consequence of

Lemma 1. *Let T be large and $L = \log T$. Let*

$$G(s) = Q(\frac{1}{L}\frac{d}{ds})\zeta(s)$$

for some polynomial Q. Let B be as in (2) with

$$b(n) = \mu(n)P(1 - \frac{\log n}{\log y})$$

where P is real analytic with $P(0) = 0$ and $P(1) = 1$. Define

$$I = I(a, P, Q) := \frac{1}{T}\int_1^T |GB(a+it)|^2\, dt.$$

Then for $0 < \theta < 1/2$ and $a = a(T)$ satisfying $|a - 1/2| = o(1)$ as $T \to \infty$ we have

$$I \sim \frac{T^{1-2a}Q(1)^2 + Q(0)^2}{2} + \theta\int_0^1\int_0^1 (\frac{d}{dx}(T^{(1/2-a)x}Q(x)))^2 P(y)^2\, dxdy$$
$$+ \frac{1}{\theta}\int_0^1\int_0^1 T^{(1-2a)x}Q(x)^2 P'(y)^2\, dxdy.$$

This result is essentially contained in Conrey [1].

The result (c) follows from

Theorem 2. *Assuming the Riemann Hypothesis,*

$$\sum_{0<\gamma_k<T} \chi(\rho_k) \sim \alpha_k \frac{T}{2\pi}$$

where $\chi(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin(\pi s/2)$ is the usual factor from the functional equation for $\zeta(s)$ and

$$\alpha_k = k + 1 - \sum_{\nu=1}^{k} e^{-z_\nu}$$

where the z_ν are roots of $f_k(z) = \sum_{j=0}^k \frac{z^j}{j!}$.

Remark. As a function of k we can show that $0 < \alpha_k \ll_\epsilon e^{-(b-\epsilon)k}$ for any $\epsilon > 0$ where $b = 1 - \log 2$ (see Conrey - Ghosh [4].) While (c) of Theorem 1 is all that we can conclude from Theorem 2, it seems that we can speculate more. The χ-function oscillates a lot – its argument at height t is essentially $t\log(t/2\pi e)$. However, the deduction of (c) ignores this fact altogether. Thus, it seems that the proper interpretation of Theorem 2 might be that a positive proportion of zeros of $\zeta^{(k)}$ are within $c/\log t$ of the critical line for any $c > 0$.

We will first show how to deduce the results (a) - (c) from Lemma 1 and Theorem 2 and then we will prove Theorem 2.

2. Deduction of results

As mentioned earlier, k is thought of as fixed. It is well known that

$$\chi(s) = \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma} \exp(-it\log\frac{t}{2\pi e} + \pi/4)(1 + O(1/|t|)).$$

for $s = \sigma + it$. Then,

$$|\chi(s)| = \left(\frac{|t|}{2\pi}\right)^{1/2-\sigma} (1 + O(1/|t|)). \tag{8}$$

Thus, (c) follows directly from Theorem 2 and the theorem of Conrey - Ghosh [4] which gives the bound for α_k: for if $\sigma > 1/2 + ((1+\epsilon)\log\log t)/(\log t)$, then $|\chi(s)| \ll (\log|t|)^{-1-\epsilon}$.

To prove (a) and (b) we take

$$P(x) = \frac{\sinh\theta\Lambda x}{\sinh\theta\Lambda}$$

in Lemma 1 where if we let

$$v(x) = T^{(1/2-a)x} Q(x),$$

then Λ is defined by

$$\Lambda^2 = \frac{\int_0^1 v'(x)^2\,dx}{\int_0^1 v(x)^2\,dx}.$$

Then it is not hard to verify that

$$I = \frac{v(0)^2 + v(1)^2}{2} + \left(\int_0^1 v(x)^2\,dx \int_0^1 v'(x)^2\,dx\right)^{1/2} \coth\theta\Lambda. \tag{9}$$

We remark that (9) can be used to verify (3)-(5) and (7). Now take $Q(x) = x^k$, $k \geq 1$; then $v(0)^2 = 0$, $v(1)^2 = T^{1-2a}$, and if $a \neq 1/2$, then $\int_0^1 v(x)^2\,dx$

$$= \int_0^1 T^{(1-2a)x} x^{2k}\,dx$$
$$= \frac{T^{(1-2a)}}{(1-2a)L}\left(1 - \frac{2k}{(1-2a)L} + \frac{2k(2k-1)}{((1-2a)L)^2} - + \cdots + \frac{(2k)!}{((1-2a)L)^{2k}}\right)$$
$$- \frac{(2k)!}{((1-2a)L)^{2k+1}}$$

where $L = \log T$. Thus

$$\int_0^1 v(x)^2\,dx = \begin{cases} \frac{T^{1-2a}}{(1-2a)L}(1+\mathrm{O}(\frac{1}{(1-2a)L})) & \text{if } (1-2a)L \to \infty \\ \frac{-(2k)!}{((1-2a)L)^{2k+1}}(1+\mathrm{O}(\frac{T^{1-2a}}{|1-2a|L})) & \text{if } (1-2a)L \to -\infty \\ \approx 1 & \text{if } |1-2a|L \ll 1; \end{cases}$$

the last formula follows from an integration by parts. Similarly,

$$\int_0^1 v'(x)^2\,dx = \int_0^1 T^{(1-2a)x} x^{2k-2}((1/2-a)Lx+k)^2\,dx$$

so that $\int_0^1 v'(x)^2\,dx$

$$= \begin{cases} \frac{T^{1-2a}}{4}(1-2a)L(1+\mathrm{O}(\frac{1}{1-2a)L})) & \text{if } (1-2a)L \to \infty \\ \frac{-(2k)!}{4((1-2a)L)^{2k-1}}(1+\mathrm{O}(\frac{T^{1-2a}}{|1-2a|L})) & \text{if } (1-2a)L \to -\infty \\ \approx 1 & \text{if } |1-2a|L \ll 1 \end{cases}$$

Thus,

$$I(a,x^k) = \begin{cases} T^{1-2a}(1+\mathrm{o}(1)) & \text{if } (1-2a)L \to \infty \\ \frac{2(k)!}{2((1-2a)L^{2k}}(1+\mathrm{o}(1)) & \text{if } (1-2a)L \to -\infty \\ \approx 1 & \text{if } |1-2a|L \ll 1. \end{cases} \tag{10}$$

Now let a be such that $|1/2-a| = \mathrm{o}(1)$ as $T \to \infty$. We apply Littlewood's lemma to $\zeta^{(k)}(s)B(s)$ on the rectangle with vertices $a+i$, σ_k+i, σ_k+iT, $a+iT$ where $\sigma_k \ll_k 1$ is a number for which $\zeta^{(k)}(s)$ has no zeros in $\sigma > \sigma_k$. Now $B(s)$ is a Dirichlet polynomial with leading coefficient 1, bounded coefficients and length $\ll T^{1/2}$. Thus, in a completely standard way (see Levinson and Montgomery [7] Section 3 and Levinson [6] Section 1 for exact details) we obtain

$$2\pi \sum_{\substack{\beta_k>a \\ 0<\gamma_k<T}} (\beta_k - a) \le \int_2^T \log|\zeta^{(k)}B(a+it)|\,dt + T(a\log 2 - k\log\log 2) + \mathrm{O}(\log T). \tag{11}$$

Now with $Q(x) = x^k$ we have $\zeta^{(k)}(s) = L^k G(s)$ with G as in Theorem 1. Then by the arithmetic mean-geometric mean inequality, the integral in (11) is

$$\le \frac{T}{2}\log\left(\frac{1}{T}\int_2^T |GB(a+iT)|^2\,dt\right)$$

so that

$$2\pi \sum_{\substack{\beta_k > a \\ 0<\gamma_k<T}} (\beta_k - a) \le kT \log\log T + \frac{T}{2}\log I(a, x^k)$$
$$+ T(a\log 2 - k\log\log 2) + O(\log T).$$

Then by (10), for $|a - 1/2| = o(1)$, we have that

$$2\pi \sum_{\substack{\beta_k > a \\ 0<\gamma_k<T}} (\beta_k - a)$$

is

$$\le \begin{cases} kT\log\log T + T(1/2 - a)\log T \\ \quad + T(a\log 2 - k\log\log 2 + O(T)) & \text{if } (1-2a)L \to \infty \\ kT\log\frac{1}{(2a-1)} + O(T) & \text{if } (1-2a)L \to -\infty \\ kT\log\log T + O(T) & \text{if } |1-2a|L \ll 1 \end{cases} \tag{12}$$

Next we note that using (1) and (6) we obtain

$$2\pi \sum_{\substack{\beta_k < a \\ 0<\gamma_k<T}} (a - \beta_k) = -kT\log\log T + 2\pi \sum_{\substack{\beta_k > a \\ 0<\gamma_k<T}} (\beta_k - a)$$
$$+ (a - 1/2)T\log\frac{T}{2\pi e} - T(a\log 2 - k\log\log 2)$$
$$+ 2\pi k\,\mathrm{li}(\frac{T}{2\pi}) + O(\log T).$$

Combining this with (12) we get that

$$2\pi \sum_{\substack{\beta_k < a \\ 0<\gamma_k<T}} (a - \beta_k)$$

is

$$\le \begin{cases} kT\log\frac{1}{(2a-1)L} + (a-1/2)TL + O(T) & \text{if } (2a-1)L \to \infty \\ O(T) & \text{if } (1-2a)L \le C \end{cases} \tag{13}$$

for any fixed $C > 0$. Then, (a) follows in a straightforward way. Next we prove (b). Let $c > 0$ and suppose that almost all of the zeros of $\zeta^{(k)}(s)$ are in the region $\sigma < 1/2 + c/\log|t|, |t| \geq 2$. Then for $c' > c$ we have

$$\begin{aligned}\sum_{\substack{\gamma_k \leq T\\ \beta_k < 1/2 + \frac{c'}{L}}} (1/2 + \frac{c'}{L} - \beta_k) &= \sum_{\substack{\gamma_k \leq T\\ \beta_k < 1/2 + \frac{c}{L}}} (1/2 + \frac{c'}{L} - \beta_k) \\ &\quad + O(L^{-1} \sum_{\substack{\beta_k < 1/2 + \frac{c'}{L}\\ \gamma_k \leq T}} 1) \\ &= \sum_{\substack{\gamma_k \leq T\\ \beta_k < 1/2 + c/L}} (1/2 + \frac{c'}{L} - \beta_k) + O(T) \\ &\geq \frac{c' - c}{L} \sum_{\substack{\gamma_k \leq T\\ \beta_k < 1/2 + c/L}} 1 + O(T) \\ &\geq (c' - c)\frac{T}{2\pi} - AT \end{aligned} \tag{14}$$

for some fixed $A \geq 0$. On the other hand, using (13) we see that the left hand side of (14) is

$$\leq \frac{kT}{2\pi} \log \frac{1}{c'} + \frac{c'T}{2\pi} + BT.$$

for some number B which is independent of T. This is a contradiction if c' is sufficiently large ($c' > e^{(c+A+B)/k}$); thus, (b) follows.

3. Proof of Theorem 2

In this section we assume the Riemann Hypothesis. The proof of Theorem 2 follows the lines of the proof in Conrey-Ghosh [3], so in some places we refer to that paper rather than give all the details. To begin with, we note that the complex poles of $\zeta^{(k+1)}(s)/\zeta^{(k)}(s)$ are in $\sigma \geq 1/2$, by Speiser's theorem if $k = 1$ and by (i) if $k > 1$. Thus, with T large and $U = TL^{-10}$,

$$S := \sum_{T < \gamma_k \leq T+U} \chi(\rho_k) = \frac{1}{2\pi i} \int_C \chi(s) \frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)}\, ds$$

where C is the positively oriented rectangle with vertices $\sigma_k + iT$, $\sigma_k + i(T + U)$, $1/2 - \delta + iT$, $1/2 - \delta + i(T + U)$ where $\sigma_k \geq \min\{3, 1 + \sup_{\rho_k} \beta_k\}$, δ is fixed with $0 < \delta < 1/8$ and where we assume that the horizontal sides of this rectangle are a distance $\gg L^{-1}$ from any zero of $\zeta^{(k)}(s)$. This last assumption entails no loss of generality since by (1) there are $\ll \log T$ zeros of $\zeta^{(k)}(1/2 + it)$ in an interval $(T, T + 1)$ so we only have to adjust T and U by an amount $\ll 1$ to justify the assumption and by (8) this involves an addition or deletion of $\ll \log T$ terms of size $\ll 1$. By (8) and the definition of σ_k the integrand is

$$\ll T^{-5/2}$$

for $s = \sigma_k + it$, $T \le t \le T+U$, while on the horizontal parts of the segment the integrand is

$$\ll L^2 T^\delta$$

by (8) and since $\zeta^{(k+1)}/\zeta^{(k)}(s) \ll L^2$ on the horizontal sides. (This can be proved in the case $k \ge 1$ exactly as for the case $k = 0$; see also equation (6.1) of Levinson and Montgomery [7].) Thus,

$$S = \frac{-1}{2\pi i} \int_{1/2-\delta+iT}^{1/2-\delta+i(T+U)} \chi(s) \frac{\zeta^{(k+1)}(s)}{\zeta^{(k)}(s)} \, ds + O(T^{1/2}L^2).$$

We make a change of variable $s \to 1-s$ here and have

$$\overline{S} = \frac{1}{2\pi i} \int_{1/2+\delta+iT}^{1/2+\delta+i(T+U)} \chi(1-s) \frac{\zeta^{(k+1)}(1-s)}{\zeta^{(k)}(1-s)} \, ds + O(T^{1/2}L^2). \qquad (15)$$

Now we derive another expression for $\zeta^{(k+1)}/\zeta^{(k)}$. First of all,

$$\frac{\chi'}{\chi}(s) = -\log \frac{|t|}{2\pi} + O\left(\frac{1}{|t|}\right),$$

and

$$\left(\frac{d}{ds}\right)^n \frac{\chi'}{\chi}(s) \ll |t|^{-n}.$$

From these and the functional equation

$$\zeta(s) = \chi(s)\zeta(1-s)$$

it easily follows that for $\sigma \le 1/2$

$$(-1)^m \zeta^{(m)}(s) = \chi(s)(1 + O(1/|t|)) \left(\ell - \left(\frac{d}{ds}\right)\right)^m \zeta(1-s) \qquad (16)$$

where $\ell = \log \frac{|t|}{2\pi}$ (see Conrey [2], Lemma 2). Now let

$$G_k(s,z) = \left(z + \frac{d}{ds}\right)^k \zeta(s) = z^k \zeta(s) + k z^{k-1} \zeta'(s) + \cdots + \zeta^{(k)}(s).$$

Then using (16) in the numerator and denominator it is not hard to see that

$$\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s) = -(\ell + \frac{G_k'}{G_k}(s,\ell))(1+\mathrm{O}(\frac{1}{|t|})) \tag{17}$$

where differentiation is with respect to s. (Use the relation $\binom{k+1}{j} = \binom{k}{j} + \binom{k}{j-1}$.) Next we observe that

$$\frac{G_k'}{G_k}(s,z) = \frac{\frac{\zeta'}{\zeta}(s) + \frac{k}{z}\frac{\zeta''}{\zeta}(s) + \cdots + \frac{1}{z^k}\frac{\zeta^{(k+1)}}{\zeta}(s)}{1 + \frac{k}{z}\frac{\zeta'}{\zeta}(s) + \cdots + \frac{1}{z^k}\frac{\zeta^{(k)}}{\zeta}(s)}.$$

Now assuming the Riemann Hypothesis it is not hard to show that

$$\frac{\zeta^{(j)}}{\zeta}(s) \ll (\log t)^{j+1-2\sigma}$$

uniformly for $1/2 < \sigma_0 \le \sigma \le \sigma_1 < 1, t \ge 2$. To prove this estimate one may proceed by Cauchy's theorem and induction starting from the case $j = 1$ which is well-known (see Titchmarsh [11], Theorem 14.55) For example, we see by Cauchy's theorem that

$$\frac{d}{ds}\frac{\zeta'}{\zeta}(s) = \frac{1}{2\pi i}\int_{|w-s|=\ell^{-1}} \frac{\zeta'/\zeta(w)}{(w-s)^2}\, ds \ll \ell^{3-2\sigma}$$

so that

$$\frac{\zeta''}{\zeta}(s) = \frac{d}{ds}\frac{\zeta'}{\zeta}(s) + (\frac{\zeta'}{\zeta}(s))^2 \ll \ell^{3-2\sigma} + \ell^{4-4\sigma} \ll \ell^{3-2\sigma}$$

for $1/2 < \sigma_0 \le \sigma \le \sigma_1 < 1$. To establish the case $j = 3$ we differentiate ζ''/ζ, and so on. We conclude that in the region $\sigma \ge 1/2+\delta$, $T \le t \le T+U$, $T \le \Re z \le T+U$, $\Im z \ll 1$, $|s-1| \gg 1$ there are no poles of G_k'/G_k and that

$$G_k'/G_k(s,z) = o(L) \tag{18}$$

uniformly. Then by Cauchy's Theorem

$$\frac{d}{dz}\frac{G_k'}{G_k}(s,z) \ll L^2 \tag{19}$$

there. Now it follows from (19) and the ordinary mean-value theorem of differential calculus that

$$\frac{G_k'}{G_k}(s,\ell) = \frac{G_k'}{G_k}(s,L) + \mathrm{O}(L^{-8})$$

for $T \le t \le T+U, \sigma \ge 1/2+\delta$. Thus, by (17) and (18)

$$\frac{\zeta^{(k+1)}}{\zeta^{(k)}}(1-s) = -L - \frac{G_k'}{G_k}(s,L) + O(L^{-8})$$

for $T \le t \le T+U, \sigma \ge 1/2+\delta$. We insert this in (15) and obtain

$$\overline{S} = \frac{-1}{2\pi i}\int_{1/2+\delta+iT}^{1/2+\delta+i(T+U)} \chi(1-s)(L+\frac{G_k'}{G_k}(s,L))\,ds + O(T^{\delta}L^{-8}).$$

Then by Cauchy's theorem and the estimates (8) and (18) we have

$$\overline{S} = \frac{-1}{2\pi i}\int_{1+\delta+iT}^{1+\delta+i(T+U)} \chi(1-s)(L+\frac{G_k'}{G_k}(s,L))\,ds + O(T^{1/2+\delta}) \qquad (20)$$

where $\delta > 0$ is still fixed. Next we expand $G_k'/G_k(s,L)$ into a Dirichlet series. Let

$$\alpha(s) = \frac{1}{L}\frac{\zeta'}{\zeta}(s) + \frac{k}{L^2}\frac{\zeta''}{\zeta}(s) + \cdots + \frac{1}{L^k}\frac{\zeta^{(k)}}{\zeta}(s).$$

Then

$$|\alpha(s)| \le C(\delta,k)L^{-1}$$

for $\sigma \ge 1+\delta$ and a positive constant $C = C(\delta,k)$. Thus, for T sufficiently large and $\sigma \le 1+\delta$,

$$(1+\alpha(s))^{-1} = 1 + \sum_{j=1}^{\infty}(-1)^j\alpha(s)^j = 1 + \sum_{j=1}^{J}(-1)^j\alpha(s)^j + O(T^{-1})$$

where $J = [2L/\log L]$. Now

$$\alpha(s) = \sum_{n=1}^{\infty}\frac{a(n,L)}{n^s}$$

where

$$|a(n,L)| \le C_1(\epsilon,k)n^{\epsilon}/L$$

for any $\epsilon > 0$ and some positive constant $C_1 = C_1(\epsilon,k)$. Thus,

$$1/G_k(s,L) = \sum_{n=1}^{\infty}\frac{b(n,L)}{n^s} + O(T^{-1}) \qquad (\sigma \ge 1+\delta) \qquad (21)$$

where

$$|b(n,L)| \le n^{\epsilon} \sum_{j=1}^{J} \frac{C_1^j}{L^j} d_j(n).$$

Then by (8), (20), and (21),

$$\overline{S} = \frac{-1}{2\pi i} \int_{1+\delta+iT}^{1+\delta+i(T+U)} \chi(1-s)(L + \sum_{n=1}^{\infty} \frac{\beta(n,L)}{n^s})\, ds + O(T^{1/2+\delta} L)$$

where

$$G_k'(s,L) \sum_{n=1}^{\infty} \frac{b(n,L)}{n^s} = \sum_{n=1}^{\infty} \frac{\beta(n,L)}{n^s} \qquad (\sigma \ge 1+\delta). \tag{22}$$

Now

$$\sum_{n=1}^{\infty} \frac{|\beta(n,L)|}{n^{1+\delta}} \ll 1 \tag{23}$$

and according to some work of Karl Norton (unpublished),

$$d_j(n) \le n^{(\log j/(\log\log n)(1+o(1)))}$$

uniformly for $j \ll (\log n)/(\log\log n)$ so that for $T/2 < n < 3T/2$,

$$\sum_{j=1}^{J} (C_1/L)^j d_j(n) \le \sum_{j=1}^{J} (C_1/L)^j (3T/2)^{\frac{\log j}{\log\log T}(1+O(\frac{1}{L}))} \ll_{\epsilon} T^{\epsilon} \tag{24}$$

for any $\epsilon > 0$. Thus $|\beta(n,L)| \ll_{\epsilon} n^{\epsilon}$ for $n \approx T$. Then by (23), (24), and Lemmas 2 and 5 of Gonek [5],

$$\overline{S} = - \sum_{\frac{T}{2\pi} \le n \le \frac{T+U}{2\pi}} \beta(n,L) + O(T^{1/2+\delta} L). \tag{25}$$

Then by Perron's formula, (21), and (22),

$$\sum_{n \le x} \beta(n,L) = \frac{1}{2\pi i} \int_{1/2+\delta-iT}^{1/2+\delta+iT} \frac{G_k'}{G_k}(s,L) \frac{x^s}{s}\, ds + O((1+\frac{x}{T})T^{\delta})$$

for $x \ll T$. By Cauchy's theorem and (18),

$$\begin{aligned} \sum_{n \le x} \beta(n,L) &= \frac{1}{2\pi i} \int_{1/2+\delta-iT}^{1/2+\delta+iT} \frac{G_{k'}}{G_k}(s,L) \frac{x^s}{s}\, ds + O(\frac{xT^{\delta}}{T}) + \sum_R \\ &= \sum_R + O(\frac{xT^{\delta}}{T} + x^{1/2+\delta} L^2) \end{aligned} \tag{26}$$

where $\sum_R$ is the sum of the residues of the integrand at its poles in $|s-1| \leq 1$. We now account for the poles of G'/G. Using the definition of G below (16), we see that G has a pole of order $k+1$ at $s=1$. Therefore, G_k'/G_k has a simple pole at $s=1$ with residue $-k-1$. Next, we apply the argument principle to $G_k(s,z)/(z^k\zeta(s))$ on the circle $|s-1|=1$. The estimate for $\zeta^{(k)}/\zeta$ given earlier shows that the total change in argument is 0. But $G_k(s,z)/(z^k\zeta(s))$ has a pole of order k at $s=1$ and no other poles whence $G_k(s,z)$ has k zeros (counting multiplicities) in $|s-1| \leq 1$. Thus, G_k'/G_k has a simple pole at $s=1$ with residue $-k-1$ and simple poles at the zeros of

$$G_k(s,L) = L^k\zeta(s) + kL^{k-1}\zeta'(s) + \cdots + \zeta^{(k)}(s)$$

with residue equal to the multiplicity of the zero. In the neighborhood of $s=1$ we have

$$\zeta^{(j)}(s) = \frac{(-1)^j j!}{(s-1)^{j+1}} + O(1).$$

Thus

$$\begin{aligned}
G_k(s,L) &= \sum_{j=0}^{k} \binom{k}{j} \zeta^{(j)}(s) L^{k-j} \\
&= \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} \left(\frac{j!(-1)^j}{(s-1)^{j+1}} + O(1) \right) L^{k-j} \\
&= \frac{(-1)^k k!}{(s-1)^{k+1}} \sum_{j=0}^{k} (-1)^j ((s-1)L)^j + O(\frac{1}{|s-1|^k}) \\
&= \frac{(-1)^k k!}{(s-1)^{k+1}} f_k((1-s)L) + O(|s-1|^{-k})
\end{aligned}$$

where f_k is as defined in the statement of Theorem 2. Denoting the zeros of $f_k(z)$ by z_ν, $1 \leq \nu \leq k$, we see that the poles of $G_k'/G_k(s,L)$ are at

$$s_\nu = 1 - \frac{z_\nu}{L} + O_k\left(\frac{1}{L^2}\right).$$

Thus by (26),

$$\sum_{n \leq x} \beta(n,L) = x(-k-1+\sum_{\nu=1}^{k} x^{s_\nu - 1}) + O(x^{1/2+\delta}L^2)$$

for $x \approx T$. Now it follows in a straightforward way that

$$S = (k+1-\sum_{\nu=1}^{k} e^{-z_\nu})\frac{U}{2\pi} + O(U/L)$$

which implies Theorem 2.

4. Conclusion

We remark that the techniques used in the proof of Theorem 2 can also be used to derive asymptotic formulae (on RH) for

$$\sum_{0<\gamma_k<T} \zeta^{(j)}(\rho_k)$$

for any positive integers j and k.

In the absence of precise knowledge of the horizontal distribution of zeros of derivatives of ζ we ask two questions which may be approachable: Let us use the notation

$$\begin{aligned} N_k^-(\sigma,T) &:= \#\{\rho_k : 0<\gamma_k \le T, \beta_k < \sigma\}, \\ N_k^+(\sigma,T) &:= \#\{\rho_k : 0<\gamma_k \le T, \beta_k \ge \sigma\}. \end{aligned}$$

Then

(α) does there exist a $c>0$ for which

$$N_k^+\left(1/2+\frac{c\log\log T}{\log T}, T\right) \gg N_k(T)?$$

(β) is there a $c>0$ for which

$$N_k^-\left(1/2+\frac{c}{\log T}, T\right) \gg N_k(T)?$$

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