MEAN VALUES OF THE RIEMANN ZETA-FUNCTION, III

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Introduction. Let

$$I_k = I_k(T) = \int_1^T |\zeta(1/2+it)|^{2k}\,dt. \tag{1}$$

Then $I_0 \sim T$, $I_1 \sim T\log T$, and $I_2 \sim \frac{1}{2\pi^2}T\log^4 T$ as $T \to \infty$. However, the asymptotic behavior of I_k is not known for any other value of k. It is known that I_k has order of magnitude $T(\log T)^{k^2}$ when $k = 1/n$ for positive integer n and, if the Riemann Hypothesis is true, when $0 < k < 2$ (see Heath-Brown [6]). It is expected that the order of magnitude of I_k is $T(\log T)^{k^2}$ for all $k \ge 0$; in fact, one may conjecture that

$$I_k \sim c_k T(\log T)^{k^2} \tag{2}$$

as $T \to \infty$ for some numbers c_k. No conjecture has been given for the c_k. In [2] we prove that the Riemann Hypothesis implies that

$$I_k \ge (1+o(1)) f_k a_k T(\log T)^{k^2} \tag{3}$$

as $T \to \infty$ for all fixed $k \ge 0$ where

$$f_k = \Gamma(1+k^2)^{-1}$$

and

$$\begin{aligned} a_k &= \prod_p \left((1-1/p)^{k^2} \sum_{m=0}^{\infty} \left(\frac{\Gamma(k+m)}{\Gamma(k)m!} \right)^2 p^{-m} \right) \\ &= \prod_p \left((1-1/p)^{k^2} {}_2F_1(k,k,1,1/p) \right) \end{aligned} \tag{4}$$

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where ${}_2F_1$ is the hypergeometric function. Gonek, in a paper to appear in *Mathematika* has extended this lower bound to k with $-1/2 < k < 0$ under the additional assumption that all zeros of ζ are simple. We note in [2] that $c_k = a_k f_k$ for $k = 0$, and $k = 1$ and suggest that it might be the case that $c_k = a_k f_k$ for $0 \le k \le 1$. Indeed, the numerical evidence of Odlyzko [9] indicates that c_k and $a_k f_k$ are very close in this range of k. Recently, Heath-Brown [7] showed that the Riemann Hypothesis implies that

$$I_k \le (1 + o(1)) a_k f_k \frac{2}{(k^2+1)(2-k)} T(\log T)^{k^2} \tag{5}$$

as $T \to \infty$ for $0 \le k < 2$. Note that the maximum of $2/[(k^2+1)(2-k)]$ on $[0,1]$ is 27/25. Thus, on RH, c_k and $a_k f_k$ are in fact very close. (See the appendix for more on a_k.)

Let

$$A_k(s) = \sum_{n=1}^{N} d_k(n) n^{-s} \tag{6}$$

be a partial sum of

$$\zeta(s)^k = \sum_{n=1}^{\infty} d_k(n) n^{-s} \qquad (\Re s > 1) \tag{7}$$

If $N \gg |t|$, then it is well known that $A_1(1/2+it)$ is a good approximation (pointwise) for $\zeta(1/2+it)$. Our lower bound for c_k is obtained via an argument which is similar to the proof of Bessel's inequality using $A_k(s)$ as our approximation to $\zeta(s)^k$. Thus, we considered the integral

$$\int_0^T |\zeta^k(1/2+it) - A_k(1/2+it)|^2 \, dt.$$

On one hand the integral is non-negative. On the other hand we square out the integrand and evaluate asymptotically the terms involving A_k. In this way we obtain, essentially,

$$\int_0^T |\zeta(1/2+it)|^{2k} \, dt \succeq \int_0^T |A_k(1/2+it)|^2 \, dt.$$

Our argument shows that for any N and k the best Dirichlet polynomial approximation to ζ^k in the mean square sense is given by A_k.

In this paper we wish to consider

$$\int_0^T |\zeta(1/2+it)|^2 |\zeta^r(1/2+it) - A_r(1/2+it, P)|^2 \, dt$$

in order to obtain a better lower bound than (3) for I_k when $k > 1$. (We use $k = r+1$ throughout the paper.) The definition of $A_r(s, P)$ is given by

$$A_r(s, P) = \sum_{n \le N} d_k(n) P(\log n / \log N) n^{-s}.$$

The reason for the P factor is that it transpires that there are choices for P which lead to better lower bounds for I_k than $P = 1$ does. In fact, only when $k = 2$ and $\theta = 1$ is it the case that $P = 1$ is the best choice where θ is defined below.

Of major importance in this work is the value of N. In order to compare it to T we set

$$N = T^{\theta}.$$

If $\theta < 1/2$ we can evaluate the terms of (8) which involve A_k asymptotically. However, we believe that our formulae hold for any fixed $\theta < 1$. Indeed, for the "square term" this is a consequence of the conjecture in [1]. The "cross term" is of a similar nature and admitting any $\theta < 1$ for its evaluation is as plausible as the conjecture of 1. Thus, we will state some results subject both to RH (or the Lindelöf Hypothesis) and the hypothesis that our formulae are valid for all $\theta < 1$.

In addition to the new lower bounds for I_k, we also consider the first and second derivatives of $C(k) = I_k/TL^{k^2}$ with respect to k at $k = 0$ and $k = 1$. As $T \to \infty$, we expect that $C(k) \to c_k$, as has already been mentioned. (The dependance of $C(k)$ on T has been supressed.) We show that $C'(0)$ approaches a limit as $T \to \infty$ and if RH is true so does $C'(1)$. If RH and the pair correlation conjecture are true, then $C''(0)$ has a limit. We cannot show that $C''(1)$ has a limit, though we suspect that it does. In addition, the first derivatives of C are equal to the first derivatives of our asymptotic lower bound, $a_k f_k$, at both $k = 0$ and $k = 1$. This indicates that Heath-Brown's upper bound is probably too large in the neighborhood of $k = 0$.

As far as our conclusions about the precise value of c_k for $0 < k < 1$, we would have to speculate, in light of this work, that Dirichlet polynomials of length $<< T$ are inadequate approximations of $\zeta^k(1/2 + it)$ for $t \approx T$ if k is not an integer. Another manifestation of this phenomenon is seen when one considers sums over zeros ρ of ζ of $A_k(\rho)$: if $k = 1$ then the mean value is 0 (if $N \approx T$) whereas if $k \neq 1$ then the mean value is non-zero.

Now we state our results. For the mean value theorems we state separately the evaluations of the "square" term and the "cross" term. Let

$$J_{r,N}(T) = \int_1^T |\zeta(1/2 + it)|^2 |A_r(1/2 + it, P)|^2 dt \qquad (8)$$

with $r = k - 1$.

Theorem 1. *If $N = T^{\theta}$ for some θ with $0 < \theta < 1/2$, and if P is a polynomial, then* $J_{r,N}(T)$

$$\sim T(\log N)^{k^2} \frac{a_k}{\Gamma(r+1)^2\Gamma(r^2)} \int_0^1 \alpha^{r^2-1} \left(\frac{1}{\theta} h'(\alpha)^2 + 2rh(\alpha)h'(\alpha)\right) d\alpha$$

as $T \to \infty$ where

$$h(\alpha) = \int_{\alpha}^1 (\beta - \alpha)^r P(\beta)\, d\beta.$$

Now let

$$K_{r,N}(T) = \int_1^T |\zeta(1/2+it)|^2 \zeta(1/2+it)^r \overline{A_r(1/2+it,P)}\, dt.$$

Then

Theorem 2. *If P is a polynomial and $0 < \theta < 1/2$, then*

$$K_{r,N}(T) \sim T(\log N)^{k^2} \frac{a_k \theta^{-1-r}}{\Gamma(r+2)\Gamma(r^2+r)}$$
$$\int_0^1 P(\alpha)\alpha^{k^2-r-2} {}_2F_1(-r, -r-1, k^2-r-1, -\alpha\theta)\, d\alpha$$

as $T \to \infty$ where ${}_2F_1$ is the usual hypergeometric function.

By making appropriate choices of P we are led to

Corollary 1. *We have*

$$I_3 \succeq 10.13 a_3 f_3 T L^9$$

as $T \to \infty$. Moreover, if the Lindelöf Hypothesis is true, then we have the following asymptotic lower bounds for $F_k = I_k / a_k f_k T L^{k^2}$ $(4 \le k \le 6)$: 205, 3242, 28130.

Because of the θ^{k^2} factor in our lower bound, the result for $k \ge 7$ is actually worse than the bound given in (3). However, if we assume that any $\theta < 1$ is admissible in Theorems 1 and 2, then we can do considerably better; in fact, we can get $F_k \to \infty$:

Corollary 2. *Assuming that Theorems 1 and 2 hold for any $\theta < 1$, we have $F_3 \succeq 38.76$, and if the Lindelöf Hypothesis is true in addition then $F_4 \succeq 21528$, $F_5 \succeq 48438800$, and as $k \to \infty$*

$$F_k \succeq c(ek/2)^{2k-3/2}$$

where $c = 1/(e\sqrt{2\pi e})$.

We mention that if any $\theta < 1$ is admissible in Theorem 1 then we can also obtain an upper asymptotic bound of 56 for F_3. Thus, we expect that I_3 is between 38.76 and 56 times $a_3 f_3 T L^9$.

We also have calculated some lower bounds for F_k for fractional k between 1 and 2. As we do not expect that these are the correct values of c_k we have not worked too hard to obtain the optimal bounds that are deducible from Theorems 1 and 2. However, we do not expect that our bounds are off by much.

Corollary 3. *We present the results in tabular form. The first row is the value of k. The second row is the upper bound for $I_k/(a_k f_k T L^{k^2})$ obtained by Heath-Brown under the assumption of RH. The third row is our lower bound obtained under the assumption of RH and that Theorems 1 and 2 hold for any $\theta < 1$. The fourth row is our lower bound obtained under assumption of RH. The numbers are truncations of their actual values.*

k	1.1	1.2	1.3	1.4	1.5	1.6	1.7	1.8	1.9	2.0
H-B	1.005	1.02	1.06	1.12	1.23	1.40	1.71	2.35	4.33	∞
$\theta = 1$	1.003	1.01	1.04	1.08	1.14	1.22	1.34	1.49	1.71	2
$\theta = \frac{1}{2}$	1.001	1.01	1.03	1.06	1.10	1.16	1.24	1.34	1.46	1.6

We remark that the correct value at $k = 1$ is 1 and the correct value at $k = 2$ is 2. Now we state our results about derivatives of $C(k)$.

Theorem 3. *If $\kappa = 0$ or if the Riemann Hypothesis holds and $\kappa = 1$, then*

$$\lim_{T\to\infty} \frac{d}{dk}\left(\frac{1}{T(\log T)^{k^2}}\int_1^T |\zeta(1/2+it)|^{2k}\,dt\right)\Bigg|_{k=\kappa} = \frac{d}{dk}(a_k f_k)\Bigg|_{k=\kappa}$$

We remark that the factor $\frac{2}{[(k^2+1)(2-k)]}$ from Heath-Brown's work [7] is 1 at $k = 0$ and at $k = 1$ while its derivative is $1/2$ at $k = 0$ and is 0 at $k = 1$. This would seem to indicate that Heath-Brown's estimate is too large near $k = 0$.

Theorem 4. *Assuming RH and the pair correlation conjecture*

$$\lim_{T\to\infty} \frac{d^2}{dk^2}\left(\frac{1}{T(\log T)^{k^2}}\int_1^T |\zeta(1/2+it)|^{2k}\,dt\right)\Bigg|_{k=\kappa} = \frac{d^2}{dk^2}(a_k f_k)\Bigg|_{k=0} + 2.$$

Theorem 4 is deduced from the following considerations. Let

$$M(k) = M(k,T) = \frac{1}{T(\log T)^{k^2}}\int_1^T |\zeta(1/2+it)|^{2k}\,dt \tag{11}$$

and

$$\begin{aligned} D(k,n) &= D(k,n,T) \\ &= \frac{1}{T(\log T)^{k^2}}\int_1^T |\zeta(1/2+it)|^{2k}(\log|\zeta(1/2+it)|)^n\,dt \end{aligned} \tag{12}$$

we see that

$$M'(k) = 2D(k,1) - 2kD(k,0)\log\log T \tag{13}$$

and

$$\begin{aligned} M''(k) = 4D(k,2) &- 8kD(k,1)\log\log T \\ &+ (4k^2(\log\log T)^2 - 2\log\log T)D(k,0). \end{aligned} \tag{14}$$

Regarding the $D(k,n)$, it is trivial that $D(0,0) = 1 + o(1)$ and it is not difficult to show that $D(0,1) = o(1)$. Assuming the Riemann Hypothesis, it follows from work of Goldston [4] that

$$\begin{aligned} D(0,2) = \frac{1}{2}\Bigg[\log\log T &+ \int_1^\infty \frac{F(\alpha,T)}{\alpha^2}\,d\alpha + \gamma + \\ &\sum_p \sum_{m=2}^\infty \left(-\frac{1}{m} + \frac{1}{m^2}\right)p^{-m}\Bigg] + o(1) \end{aligned} \tag{15}$$

where $F(\alpha, T)$ is the Fourier transform of the pair correlation function for the zeros of the zeta-function introduced by Montgomery [8]. (We have corrected the sign in front of the sum over m which is mistakenly a "minus" in Goldston's paper.) If Montgomery's pair correlation conjecture is true, then the integral here is $1 + o(1)$. It is well known that $D(1,0) = 1 + O(1/\log T)$ (see Titchmarsh [12,§7.4]). To these, we add

Theorem 5 (Conrey, Ghosh, Goldston). *Assuming the Riemann Hypothesis,*

$$D(1,1) = (\log\log T + \gamma - 1)(1 + O(1/\log T))$$

and

$$D(1,2) = (\log\log T)^2 + (2\gamma - 3/2)\log\log T + O(1).$$

Thus, we see that if the Riemann Hypothesis and the pair correlation conjecture are true, then the limit in (10) does exist when $k = 0$, and is equal to

$$2 + 2\gamma + 2\sum_p \sum_{m=1}^{\infty}\left(-\frac{1}{m} + \frac{1}{m^2}\right).$$

Also, if the O(1) in the formula for $D(1,2)$ above could be replaced by $C + o(1)$, then the limit in (10) would exist for $k = 1$ as well and would be equal to $4C$.

We would like to thank D. Goldston for kindly permitting us to include Theorem 5 in this paper.

Lemmas on d_r. In this section we give some preliminary material on the multiplicative arithmetic function $d_r(n)$. We assume throughout that $r > 0$. Our estimates are not in general uniform in r.

By (7) and the Euler product for $\zeta(s)$ it follows that

$$(1-x)^{-r} = \sum_{m=0}^{\infty} d_r(p^m)x^m \tag{20}$$

for $|x| < 1$ and any prime p. In particular,

$$d_r(p^m) = \frac{\Gamma(r+m)}{\Gamma(r)m!}. \tag{21}$$

Using this and some simple calculations we have

Lemma 1. *For n fixed, $d_r(n)$ is increasing in r for $r > 0$. If $0 < r \le 1$, then*

$$0 < d_r(n) \le 1$$

for any n and

$$d_r(mn) \le d_r(n)$$

for all m and n. If $r \ge 1$, then

$$0 < d_r(mn) \le d_r(m)d_r(n)$$

for all m and n.

As might be imagined from Lemma 1 we will have to distinguish the cases $r > 1$ and $0 < r < 1$ throughout our considerations. Next, regarding the average size of $d_r(n)$, the following is well known:

Lemma 2. *For fixed $r \geq 0$,*

$$\sum_{n \leq x} d_r(n) \ll x \log^{r-1} x.$$

This is also a consequence of Lemma 4.

We will need estimates for $d_r(hn)$ averaged over n which are uniform in h. To obtain these, we first consider the generating function. Using the fact that $F(n) = f(hn)/f(h)$ is a multiplicative function whenever f is (provided that $f(h) \neq 0$) it is easy to prove

Lemma3. *Let $h = \prod_p p^{h_p}$. Then,*

$$\sum_{n=1}^{\infty} \frac{d_r(hn)}{n^s} = \zeta(s)^r \prod_{p|h} \left((1 - p^{-s})^r \sum_{m=0}^{\infty} d_r(p^{m+h_p}) p^{-ms} \right)$$
$$=: \zeta(s)^r D_r(h, s)$$

for $\Re(s) > 1$.

One may find similar formulae in Chapter 1 of Titchmarsh [13]. We need to bound the Dirichlet series $D_r(h,s)$ defined in the above lemma. Let

$$D_r(h,s) = \sum_{n=1}^{\infty} b(h,n) n^{-s}.$$

For two Dirichlet series $A(s) = \sum a_n n^{-s}$ and $B(s) = \sum b_n n^{-s}$ we say that A is majorized by B and write $A(s) \prec\prec B(s)$ if $|a_n| \leq b_n$ for all n. It follows easily from Lemma 1 that

$$D_r(h,s) \prec\prec \begin{cases} d_r(h) F_{-2r}(h,s) & \text{if } r > 1 \\ F_{-2r}(h,s) & \text{if } 0 < r \leq 1 \end{cases}$$

where

$$F_j(h,s) = \prod_{p|h} (1 - p^{-s})^j.$$

We wish to apply Theorem 2 of Selberg [11]. Thus, we observe that if R is the least integer greater than or equal to r, we have

$$\sum_{n=1}^{\infty} |b_r(h,n)| n^{-1} (\log n)^{r+3} \leq \delta_r(h) \left(\frac{d^{R+3}}{ds^{R+3}} (F_{-2r}(h,s)) \right) \bigg|_{s=1}$$

where

$$\delta_r(h) = \begin{cases} 1 & \text{if } 0 \leq r \leq 1 \\ d_r(h) & \text{if } r > 1 \end{cases}$$

We now describe a class $\mathcal{C}$ of arithmetic functions of integer variable h and complex parameter s whose generic member is denoted by $\mathcal{F}(h)$. First of all, $F_r(h,s)$ is in $\mathcal{C}$ for any real r and any complex s with positive real part. In addition, any finite linear combination of functions in $\mathcal{C}$ is again in $\mathcal{C}$. Also, $\mathcal{C}$ is closed under differentiation with respect to s and is closed under multiplication. We complete the definition by requiring that $\mathcal{C}$ be the smallest collection of functions satisfying these conditions. Then, applying Theorem 2 of Selberg [11] and summation by parts, we easily obtain

Lemma 4. *For each $r \geq 0$ there is a $\mathcal{F} \in C$ such that*

$$\sum_{n \leq x} d_r(hn)n^{-1} = \frac{D_r(h,1)(\log x)^r}{\Gamma(r+1)} + O(\delta_r(h)\mathcal{F}(h)).$$

uniformly for all h.

The important thing to know about the class C is that its members are bounded on average:

Lemma 5. *For any fixed $r \geq 0$ and $\mathcal{F} \in C$,*

$$\sum_{h \leq H} \mathcal{F}(h)\delta_r(h) \ll H \log^{r-1} H.$$

Proof. It suffices to prove this for $r \geq 1$. It is not difficult to check that for any $\mathcal{F}$ in C there exists $K > 0$ and $\sigma > 0$ such that

$$\mathcal{F}(h) \leq \prod_{p|h}(1 + Kp^{-\sigma}) = \sum_{n|h} n^{-\sigma} K^{\omega(n)}$$

where $\omega(n)$ is the number of prime factors of n. Thus, the sum in question is

$$\begin{aligned} &\leq \sum_{n \leq H} n^{-\sigma} K^{\omega(n)} \sum_{h \leq H/n} d_r(hn) \\ &\ll H \log^{r-1} H \sum_{n \leq H} n^{-\sigma} K^{\omega(n)} d_r(n) n^{-1} \\ &\ll H \log^{r-1} H \end{aligned}$$

since

$$K^{\omega(n)} d_r(n) \ll n^{\sigma/2}$$

for sufficiently large n.

Proof of Theorem 1. We need Theorem 1 of Balasubramanian, Conrey, and Heath-Brown [1] which we state for convenience:

Lemma 6. *Let $A(s) = \sum_{m \leq M} a(m)m^{-s}$ and let*

$$I = \int_1^T |\zeta A(1/2 + it)|^2 \, dt.$$

Then

$$I = T \sum_{h,k \leq M} \frac{a(h)\overline{a(k)}}{hk}(h,k)\left(\log \frac{T(h,k)^2}{2\pi hk} + 2\gamma - 1\right) + o(T)$$

provided that $(\log M)/(\log T) \le \theta_0 < 1/2$ *and* $a(m) \ll_\epsilon m^\epsilon$ *for any* $\epsilon > 0$.

For our application,

$$a(h) = d_r(h) P\left(\frac{\log h}{\log N}\right).$$

In order to evalute the sum, we first consider

$$S_r(H,K) = \sum_{\substack{h \le H \\ k \le K}} \frac{d_r(h) d_r(k)}{hk} (h,k) \left(\log \frac{T(h,k)^2}{2\pi hk} + 2\gamma - 1\right).$$

We assume without loss of generality that $H \le K$. Then we separate the variables h and k using the Möbius inversion formula:

$$(h,k)(A + \log(h,k)) = \sum_{\substack{\alpha | h \\ \alpha | k}} \sum_{\beta | \alpha} \mu(\beta) \frac{\alpha}{\beta} \left(A + \log \frac{\alpha}{\beta}\right). \tag{26}$$

Inserting this equation into (20), rearranging the order of summation, and replacing h by $h\alpha$ and k by $k\alpha$ we obtain

$$S = \sum_{\alpha \le H} \frac{1}{\alpha} \sum_{\beta | \alpha} \frac{\mu(\beta)}{\beta} \sum_{\substack{h \le H/\alpha \\ k \le K/\alpha}} \frac{d_r(h\alpha) d_r(k\alpha)}{hk} \left(\log \frac{T\beta^2}{2\pi hk} + 2\gamma - 1\right). \tag{27}$$

Now

$$\sum_{\beta | \alpha} \frac{\mu(\beta)}{\beta} = \frac{\phi(\alpha)}{\alpha} \tag{28}$$

and

$$\begin{aligned} \sum_{\beta | \alpha} \frac{\mu(\beta)}{\beta} \log \beta &= -\frac{d}{ds} \sum_{\beta | \alpha} \frac{\mu(\beta)}{b^s} \Bigg|_{s=1} \\ &= -\frac{d}{ds} F(\alpha, s) \Bigg|_{s=1} = \mathcal{F}(\alpha). \end{aligned} \tag{29}$$

Therefore,

$$S = \sum_{\alpha \le H} \frac{\phi(\alpha)}{\alpha^2} (S_0(H) S_0(K) \log T - S_0(H) S_1(K) - S_0(K) S_1(H)) + E \tag{30}$$

where

$$S_0(\alpha, H) = S_0(H) = \sum_{h \le H/\alpha} \frac{d_r(h\alpha)}{h}, \tag{31}$$

(32) $$S_1(\alpha, H) = S_1(H) = \sum_{h \le H/\alpha} \frac{d_r(h\alpha)}{h} \log h,$$

and

$$E = \sum_{\alpha \le H} \frac{\mathcal{F}(\alpha)}{\alpha} S_0(H) S_0(K).$$

By Lemma 4,

$$S_0(H) \ll \begin{cases} d_r(h)\mathcal{F}(\alpha)(\log H)^r, & \text{if } r > 1 \\ \mathcal{F}(\alpha)(\log H)^r, & \text{if } 0 \le r \le 1 \end{cases}$$

Thus, by Lemmas 1, 2, and 5,

(33) $$E \ll \sum_{\alpha \le H} \frac{\mathcal{F}(\alpha)}{\alpha} \log^{2r} H \ll \log^{2r+1} H.$$

if $0 < r \le 1$, while if $r > 1$, then

(33) $$E \ll \sum_{\alpha \le H} \frac{\mathcal{F}(\alpha) d_r(\alpha)^2}{\alpha} \log^{2r} H \ll \log^{k^2-1} H.$$

Then, by Lemma 4 and summation by parts,

(34) $$S = \sum_{\alpha \le H} \frac{\phi(\alpha) D_r(\alpha, 1)^2}{\alpha^2 \Gamma(r+1)^2} \Big((\log H/\alpha)^r (\log K/\alpha)^r \log \frac{T}{2\pi}$$
$$- \frac{r}{r+1} (\log H/\alpha)^r (\log K/\alpha)^{r+1} - \frac{r}{r+1} (\log K/\alpha)^r (\log H/\alpha)^{r+1} \Big)$$
$$+ E_1$$

where, by (24) and Lemma 5,

(35) $$E_1 \ll (\log T)^{2r+1} + \sum_{\alpha \le H} \frac{\mathcal{F}(\alpha)}{\alpha} (\log T)^{2r} \ll (\log T)^{2r+1}.$$

(We have also used the fact that $\frac{\phi(\alpha)}{\alpha}$ is in $\mathcal{C}$.)

To further evaluate the sum it is necessary to get an expression for the residue of

$$\sum_{\alpha=1}^{\infty} \frac{D_r(\alpha, 1)^2 \phi(\alpha)/\alpha}{\alpha^s}$$

at $s = 1$. Then we apply Theorem 2 of Selberg's paper [11]. Thus, we obtain

(36) $$\sum_{\alpha \le N} \frac{D_r(\alpha, 1)^2 \phi(\alpha)}{\alpha} \sim \frac{H(1)}{\Gamma(r^2)} N (\log N)^{r^2-1}$$

where

$$(37)\qquad H(1)=\prod_p(1-1/p)^{r^2}\sum_{m=0}^{\infty}\phi(p^m)D_r(p^m,1)^2p^{-2m}.$$

Now we claim that

$$(38)\qquad H(1)=a_k$$

where a_k is defined in the introduction. Note that

$$(39)\qquad D_r(p^m,1)=(1-1/p)^r\sum_{n=0}^{\infty}d_r(p^{m+n})p^{-n}$$

and $\phi(p^m)/p^m=1-1/p$ if $m>0$ and $=1$ if $m=0$. Since $r=k-1$, to prove (33) it suffices to show that

$$(40)\qquad (1-x)^{-2r-1}+\sum_{m=1}^{\infty}x^m\left(\sum_{n=0}^{\infty}d_r(p^{m+n})x^n\right)^2=\sum_{m=0}^{\infty}d_{r+1}(p^m)^2x^m.$$

Now $d_{r+1}(n)=\sum_{\alpha|n}d_r(\alpha)$ so that the right hand side of (35) is

$$(41)\qquad =\sum_{m=0}^{\infty}\left(\sum_{l=0}^{m}g_l\right)^2x^m$$

where we have used $g_l=d_r(p^l)$. Writing $(1-x)^{-2r-1}$ as

$$(1-x)^{-r}(1-x)^{-r}(1-x),$$

we see that (35) is equivalent to

$$(42)\qquad \sum_{m=0}^{\infty}x^m\left(\sum_{l=0}^{\infty}g_lx^l\right)^2+\sum_{m=1}^{\infty}x^m\left(\sum_{l=0}^{\infty}g_{l+m}x^l\right)^2=\sum_{m=0}^{\infty}x^m\left(\sum_{l=0}^{m}g_l\right)^2.$$

But it is easily checked by comparing coefficients of $g_{l_1}g_{l_2}$ on both sides that this is a purely formal identity which holds for indeterminates x and g_l.

Now we derive our required estimate from the above and a two variable summation by parts:

Lemma 7. *Let $a(h,k)$ be an arithmetical function and let $f(x,y)$ have continuous first partial derivatives in $[1,N]$. Let*

$$S(H,K)=\sum_{\substack{1\le h\le H\\1\le k\le K}}a(h,k).$$

Then

$$\sum_{\substack{1\le h\le H\\ 1\le k\le K}} a(h,k)f(h,k) = f(N,N)S(N,N) - \int_1^N f_y(N,y)S(N,y)\,dy$$
$$- \int_1^N f_x((x,N)S(x,N)\,dx$$
$$+ \int_1^N \int_1^N f_{xy}(x,y)S(x,y)\,dx\,dy$$

The proof follows easily from the one variable summation by parts. Now by partial summation we get

Lemma 8.

$$S(H,K) \sim \frac{a_k}{\Gamma(r^2-1)}(\log H)^{k^2} \int_0^1 \eta^{r^2-2}(1-\eta)^r(\lambda-\eta)^r$$
$$\left(\frac{\log T/2\pi}{\log H} - \frac{r}{r+1}(\lambda-\eta) - \frac{r}{r+1}\eta\right)\,d\eta$$

Theorem 1 is a straightforward application of Lemmas 7 and 8.

Proof of Theorem 2. We consider the integral in question as a complex integral by letting $s = 1/2 + it$. We then use Cauchy's Theorem to move the path of integration to the line segment with real part $1 + (\log T)^{-1}$. If $k = 3$ then by standard estimates we may do this with an error term which is $\ll NT^{1/2}$. If k is an integer greater than 3, then we assume the Lindelöf Hypothesis, while if k is non-integral, then we assume the Riemann Hypothesis in order to move the path with an acceptable error term. Thus, the integral is

$$(44) \qquad = \frac{1}{i}\int_{c+i}^{c+iT} \chi(1-s)\zeta(s)^{k+1}A_r(1-s,P)\,ds + O_\epsilon(T^{c-1/2+\epsilon})$$

for any $\epsilon > 0$ and $c > 1$. We let $\chi(s)$ denote the factor from the asymmetric functional equation

$$(45) \qquad \zeta(s) = \chi(s)\zeta(1-s)$$

for $\zeta(s)$ and use this in (40) to replace $\zeta(1-s)$. Next, we apply the following lemma which amounts to integrating term-by-term. (See [3], Lemma 1 and also the remarks after Lemma 10 in this paper.)

Lemma 9. *Suppose that* $A(s) = \sum a(n)n^{-s}$ *for* $\sigma > 1$ *where*

$$a(n) \ll d_\kappa(n)(\log n)^\ell$$

for some non-negative integers κ and ℓ. Suppose also that $B(s) = \sum b(n)n^{-s}$ where $b_n \ll d_\lambda(n)$. Let $c = 1 + (\log T)^{-1}$.Then

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s)A(s)B(1-s)\, ds = \sum_{n\le N} \frac{b(n)}{n} \sum_{m\le \frac{nT}{2\pi}} a(n)e(-m/n)$$
$$+ O(T^{1/2}N(\log T)^{\kappa+\lambda+\ell})$$

We apply this Lemma and summation by parts and see that it suffices to evaluate

$$\sum_{n\le N} \frac{d_r(n)}{n} \sum_{m\le} \frac{nT}{2\pi} d_{r+2}(m)e(-m/n).$$

The thing to do now is work out the generating function for the inner sum. To do this, we replace the exponential by a sum over Dirichlet characters. The formula is

$$e(-n/m) = \sum_{\substack{d|m \\ d|n}} \phi\left(\frac{m}{d}\right)^{-1} \sum_{\chi \bmod m/d} \tau(\overline{\chi})\chi\left(\frac{-n}{d}\right)$$

Now letting $D_r(s, a/m) = \sum_{n=1}^{\infty} \frac{d_r(n)e(na/m)}{n^s}$ we have

$$D_r(s, a/m) = \sum_{g|m} \frac{g^{-s}}{\phi(m/g)} \sum_{\chi \bmod m/g} \tau(\overline{\chi}) \sum_{n} \frac{d_r(ng)\chi(n)}{n^s}$$

If r is an integer, then the only pole of this is at $s = 1$ and occurs in the term involving the principal character χ_0. If r is not an integer then by the Riemann Hypothesis the only singularity in $\sigma > 1/2$ is at $s = 1$ again arising from the term with the principal character. In either event, if we subtract off the term involving χ_0 the remainder is regular in $\sigma > 1/2$. The term with χ_0 can be evaluated as $\zeta(s)^r\mathcal{E}_r(m,s)$ where

$$\mathcal{E}_r(m,s) = \frac{1}{m^s}\sum_{g|m} \frac{\mu(g)g^s}{\phi(g)} \sum_{\ell|g} \frac{\mu(\ell)}{\ell^s} H_r(\ell m/g, s)$$

with

$$H_r(\lambda, s) = \prod_{p|\lambda}(1-p^{-s})^r \sum_{m=0}^{\infty} \frac{d_r(p^{m+e_p(\lambda)})}{p^{ms}}$$

Dealing with the main term, we find that if $c > 1$, then

$$K_{r,N} \sim \sum_{m\le N} \frac{d_r(m)}{m}\frac{1}{i}\int_{(c)} \zeta(s)^{r+2}\mathcal{E}_{r+2}(m,s)\left(\frac{mT}{2\pi}\right)^s \frac{ds}{s}$$
$$\sim T\sum_{m\le N} d_r(m)\frac{1}{2\pi i}\int_{L_{/epsilon}} (mT/2\pi)^w \mathcal{E}_{r+2}(m, w+1)\frac{dw}{w^{r+1}}$$

where L_ϵ ($\epsilon > 0$) denotes the path which starts at $-1/4 - i\epsilon$ proceeds to $-i\epsilon$ then around a positive oriented semicircle to $i\epsilon$ then to $-1/4 + i\epsilon$. Now we consider the generating function

$$\sum_m \frac{d_r(m)}{m^{s-w}} \mathcal{E}_{r+2}(m, w+1)$$

By considering Euler products it turns out that the above is

$$= \zeta(s+1)^{r(r+2)} \zeta(s+1-w)^{-r} \mathcal{H}_r(s, w)$$

where $\mathcal{H}$ can be expressed by an Euler product which is absolutely convergent if w and s both have real part greater than -1/4 . Moreover, it can be shown, as in the proof of Theorem 1, that $\mathcal{H}_r(1) = a_{r+1}$. This leads to

$$\begin{aligned} K_{r,N} &\sim a_k T \frac{1}{(2\pi i)^2} \int_{L_\epsilon} \int_{L_{2\epsilon}} \frac{(T/2\pi)^w N^s (s-w)^r}{w^{r+2} s^{k^2}} \, ds \, dw \\ &= a_k T L^{k^2} \frac{1}{(2\pi i)^2} \int_{L_\epsilon} \int_{L_{2\epsilon}} \frac{e^{w+\theta s}(s-w)^r}{w^{r+2} s^{k^2}} \, ds \, dw \end{aligned}$$

where $N = T^\theta$. This is easily seen to be

$$\sim a_k T L^{k^2} \frac{\theta^{r^2+r}}{\Gamma(k+1)\Gamma(k^2-r)} {}_2F_1(-r, -k, k^2-r, -\theta)$$

in the usual notation of hypergeometric functions. After partial summation, we have the main term of the Theorem.

It remains to discuss the error terms. In case $k \neq 3$ this is easy in light of the assumption of either LH or RH. If $k = 3$, the argument is much the same as in Conrey-Ghosh [3]. The error term is roughly

$$\sum_{n \le N} \frac{a(n)}{n\phi(n)} \int_{-U}^{U} \sum_{\chi \bmod n} \tau(\chi) L(s,\chi)^4 \left(\frac{nT}{2\pi}\right)^s \frac{ds}{s}$$

where $s = 1/2 + it$. This expression may be estimated by the large sieve inequalities, just as in [3] so we don't give the precise details. We ultimately end up with an error term that is

$$\ll_\epsilon NT^{1/2+\epsilon}$$

for any k.

<u>**Corollaries**</u> In this section we discuss the choices of the function P from $A(s, P)$ which lead to the numerical results stated in the Corollaries. Here we have used the calculus of variations to make our choices optimally.

Since $\int |\zeta|^2 |\zeta^r - A_r|^2 \ge 0$ we have

$$I_k \ge 2\Re\{K_{r,N}(T)\} - J_{r,N}(T)$$

Then, using the notation

$$F_k = I_k/(TL^{k^2} a_k f_k)$$

of Corollary 1, we have by Theorems 1 and 2, $F_k \succeq \theta^{k^2} M$ where $M =$

$$\frac{2\theta^{-r}}{\Gamma(r+2)\Gamma(r^2+r)} \int_0^1 P(\alpha)\alpha^{k^2-r-2} {}_2F_1(-r, -r-1, k^2-r-1, -\alpha\theta)\, d\alpha$$
$$- \frac{1}{\Gamma(r^2)\Gamma(r+1)^2} \int_0^1 \alpha^{r^2-1}(h'(\alpha)^2 + 2\theta r h(\alpha) h'(\alpha))\, d\alpha$$

Here

$$h(\alpha) = \int_\alpha^1 (\gamma - \alpha)^r P(\gamma) d\gamma$$

Now let

$$c_1 = \frac{\theta^{-r}}{\Gamma(r+2)\Gamma(r^2-1)}$$

and

$$c_2 = \frac{1}{\Gamma(r^2)\Gamma(r+1)^2}$$

Then the integral we want to maximize is (in an abbreviated notation)

$$\frac{-2\theta^{k^2-1}}{\Gamma(r+1)} \int_0^1 \alpha^{r^2-1}(-c_1 h F \alpha^{-1} + c_2(h'^2 + 2r\theta h h'))\, d\alpha$$

The Euler-Lagrange equation is

$$\alpha h'' + (r^2-1)h' + r(r^2-1)\theta h = \frac{-\alpha^{r^2-2} c_1 F}{2c_2 \alpha^{r^2-2}}$$

Because of the definition of h in terms of P it is also required that

$$h(1) = h'(1) = \cdots = h^{(q)}(1) = 0$$

where q is the integer part of r. In practice we will only satisfy the first of these initial conditions and show that our solution can be approximated by functions which do satisfy all of the initial conditions. Thus we take

$$h(\alpha) = h_p(\alpha) + C \frac{J_{r^2-2}(\sqrt{4r(r^2-1)\theta\alpha})}{(4r(r^2-1)\theta\alpha)^{(r^2-2)/2}}$$

where J is the usual Bessel function, C is chosen so as to make $h(1) = 0$ and h_p is a particular solution of the differential equation whose power series expansion in $|\alpha| < 1$

may be easily found. In case k is an integer, h_p is a polynomial since in this case the hypergeometric function reduces to a polynomial. In this case, the right hand side is

$$\sum_{n=0}^{r} b_n \alpha^n$$

where

$$b_0 = \frac{-(r-1)\Gamma(r+1)}{2\theta^r}$$

and

$$b_n = b_{n-1} \frac{-\theta(n-1-r)(n-2-r)}{n(r^2+n-2)}$$

from which we deduce that a particular solution is given by

$$h_p(\alpha) = \sum_{n=0}^{r} h_n \alpha^n$$

where

$$h_n = \frac{b_n - (n+1)(n+r^2-1)h_{n+1}}{r(r^2-1)\theta}$$

for $n = r, r-1, \ldots, 0$. One simplification in the computation results from the following: if for some function $\mathcal{F}$,

$$I(h) = \int_0^1 (h\mathcal{F} + c\alpha^{r^2-1}(h'^2 + 2r\theta hh'))\, d\alpha$$

then the Euler-Lagrange equation is

$$\alpha h'' + (r^2-1)h' + r(r^2-1)\theta h = \frac{\mathcal{F}}{2c\alpha^{r^2-2}}$$

If h satisfies this differential equation and $h(1) = 0$, then

$$I(h) = \frac{1}{2}\int_0^1 h\mathcal{F}$$

We illustrate what happens for the case $k = 3$. Our lower bound is

$$\int |\zeta|^6 \succeq a_3 T L^9 \theta^8 2 \int_0^1 \left(P(\alpha)\left(\frac{\alpha^5}{\theta^2 6!} - \frac{\alpha^6}{\theta 6!} + \frac{\alpha^7}{7!}\right) - \frac{1}{24}\alpha^3 (h'(\alpha)^2 + 4\theta h(\alpha)h'(\alpha)) \right) d\alpha$$

where

$$h(\alpha) = \int_\alpha^1 P(\gamma)(\gamma - \alpha)^2\, d\gamma$$

After integration by parts several times, we have

$$\int |\zeta|^6 \succeq -\frac{a_3}{24} T L^9 \theta^8 \left(\int \mathcal{F} h + \alpha^3 (h'^2 + 4\theta h h') \right)$$

where

$$\mathcal{F} = -\frac{2\alpha^2}{\theta^2} + \frac{4\alpha^3}{\theta} - \alpha^4$$

Then

$$h(\alpha) = \frac{C J_2(\sqrt{24\theta\alpha})}{24\theta\alpha} - \frac{\alpha^2}{12\theta} + \frac{4\alpha}{9\theta^2} - \frac{7}{18\theta^3}$$

where

$$C = \left(2 - \frac{32}{3\theta} + \frac{28}{3\theta^2}\right) / J_2(\sqrt{24\theta})$$

The integral simplifies to give

$$\int |\zeta|^6 \succeq -\frac{a_3 T L^9 \theta^8}{48} \int \mathcal{F} h$$

For this choice of h it is true that $h'(1) \neq 0$ and $h''(1) \neq 0$. However, it is easy to see that for any $\epsilon > 0$ there exist functions H which satisfy the right conditions for which the difference between $I(h)$ and $I(H)$ is $< \epsilon$ in absolute value.

For k between 1 and 2 we have not tried to optimize our lower bounds as carefully since it seems that in any event we don't have the correct answers. Here we take P to be a constant and choose the constant optimally. The lower bound so obtained is given by

$$\int |\zeta|^{2k} \succeq \frac{a_k T L^{k^2} {}_2F_1(-r, -r-1, k^2 - r, -\theta)^2}{k\Gamma(k^2 - r)^2 \theta^{2r+2} \Gamma(2k-1)(k^3/\theta - 4k^2 + 6k - 2)}$$

from which Corollary 3 follows. Assuming that $\theta = 1$ is admissible, and using Stirling's formula and

$${}_2F_1(-r, -r-1, k^2 - r, -1) \to 1/e$$

as $k \to \infty$ we obtain the asymptotic result claimed in Corollary 2.

Proof of Theorem 5. It will be convenient to initially consider

$$I(n) = \int_1^T |\zeta(1/2 + it)|^2 \log^n \zeta(1/2 + it) \, dt. \tag{43}$$

We consider this as a complex integral by letting $s = 1/2 + it$. We then use Cauchy's Theorem to move the path of integration to the right of $\sigma = 1$. Since we are assuming the Riemann Hypothesis we have

$$I(n) = \frac{1}{i} \int_{c+i}^{c+iT} \zeta(s)\zeta(1-s) \log^n \zeta(s) \, ds + O_\epsilon(T^{c-1/2+\epsilon}) \tag{44}$$

for any $\epsilon > 0$ and $c > 1$. We let $\chi(s)$ denote the factor from the asymmetric functional equation

$$\zeta(s) = \chi(s)\zeta(1-s) \tag{45}$$

for $\zeta(s)$ and use this in (40) to replace $\zeta(1-s)$. Next, we apply a lemma from Gonek [5] (which is actually just Lemma 9 with $N = 1$).

Lemma 10 (Gonek). *If $b_n \ll_\epsilon n^\epsilon$ for any $\epsilon > 0$ and if $c > 1$, then*

$$\frac{1}{2\pi i}\int_{c+i}^{c+iT} \chi(1-s)\sum_{n=1}^{\infty}\frac{b_n}{n^s}\,ds = \sum_{n\le \frac{T}{2\pi}} b_n + O(T^{c-1/2}).$$

We remark that the proof of Lemma 9 has not appeared in print but may be proven along the lines of Gonek's proof of this Lemma 10.

Thus, it suffices to evaluate

$$\sum_{lm\le\frac{T}{2\pi}} \Lambda_1(l)^{*n} d(m) \tag{46}$$

where Λ_1 is the generating function of $\log\zeta(s)$ which is supported on prime powers : $\Lambda_1(p^k) = 1/k$. (Alternatively, $\Lambda_1(n) = \frac{\Lambda(n)}{\log n}$ for $n > 1$.) Also, $*$ denotes Dirichlet convolution. By Perron's formula, the sum in (46) is

$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s)^2\log^n\zeta(s)\left(\frac{T}{2\pi}\right)^s\frac{ds}{s}$$

where $c > 1$. This integral is easily evaluated using standard estimates for $\zeta(s)$ and $\log\zeta(s)$ near $s = 1$. In fact, letting $\tau = \frac{T}{2\pi}$ for the moment, we find that

$$I(n) = T\int_L \frac{\tau^z}{z^2}\log^n\left(\frac{1}{z}\right)\,dz + O(\tau(\log\log\tau)^n)$$

where L denotes the path which is a line segment from $-1/2 - i/\log\tau$ to $-i/\log\tau$ followed by a semicircle from $-i\tau/\log\tau$ to $i/\log\tau$ on which $\Re z \ge 0$ followed by a line segment from $i/\log\tau$ to $-1/2 + i/\log\tau$. The integral here is

$$= \left(\frac{d}{ds}\right)^n \int_L \frac{\tau^z}{z^s}\,dz\bigg|_{s=2}$$

from which it follows that

$$I(n) = T\left(\frac{d}{ds}\right)^n\left(\frac{(\log\tau)^{s-1}}{\Gamma(s)}\right)\bigg|_{s=2} + O(T(\log\log T)^n) \tag{47}$$

Now, we have

$$D(1,1) = \Re I(1)/(T\log T) = \log\log T - \Gamma'(2) + O\left(\frac{\log\log T}{\log T}\right). \tag{48}$$

Letting $\psi = \Gamma'/\Gamma$, we have $\psi(z+1) = 1/z + \psi(z)$ so that $\Gamma'(2) = \gamma - 1$ since $\Gamma'(1) = -\gamma$. Now the first estimate of Theorem 5 follows. We note also that $I(2)/(T\log T)$

$$= ((\log\log T)^2 - 2\Gamma'(2)\log\log T + 2\Gamma'(2)^2 - \Gamma''(2))(1 + O(1/\log T)) \tag{49}$$

To evaluate $D(1,2)$, we use the fact that

$$\log^2|\zeta(1/2+it)| = \Re\log^2\zeta(1/2+it) + (\pi S(t))^2 \tag{50}$$

where, $S(t) = \frac{1}{\pi}\arg\zeta(1/2+it)$. Thus, we need to estimate

$$I = \int_1^T |\zeta(1/2+it)|^2 S(t)^2\,dt. \tag{51}$$

For this purpose we use Selberg's [10] approximate formula for S(t):

Lemma 11 (Selberg). *Assuming RH, if $t > 2$ and $4 \le x \le t^2$, then*

$$S(t) = -\frac{1}{\pi} \sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma_1}} \frac{\sin(t \log n)}{\log n} + O\left(\frac{1}{\log x} \left| \sum_{n<x^2} \frac{\Lambda_x(n)}{n^{\sigma_1+it}} \right| \right) + O\left(\frac{\log t}{\log x} \right)$$

where

$$\Lambda_x(n) = \begin{cases} \Lambda(n) & \text{if } 1 \le n \le x \\ \frac{\Lambda(n)\log(x^2/n)}{\log x} & \text{if } x \le n \le x^2 \end{cases}$$

We will take $x = T^{1/8}$. We substitute Selberg's formula for $S(t)$ into I and estimate the contributions from the O-terms. Now $(\log T)/(\log x) \ll 1$, so that the second O-term contributes an amount which is $\ll T \log T$ to I since this is precisely the order of magnitude of

$$\int_1^T |\zeta(1/2+it)|^2 \, dt.$$

To estimate the contribution from the first O-term we use Cauchy's inequality and Lemma 6. Thus,

$$\int_1^T |\zeta(1/2+it)|^2 \Big| \sum_{m \le x} \frac{\Lambda_x(m)}{m^{\sigma_1+it}} \Big|^2 \, dt \ll T \log T \sum_{h,k \le x^2} \frac{\Lambda(h)\Lambda(k)}{hk}(h,k) \tag{52}$$

since $\Lambda_x(h) \ll \Lambda(h)$, $h^{-\sigma_1} \ll h^{-1/2}$, and $\log h \ll \log T$. For the terms with $(h,k) = 1$ the sum is

$$\ll \left(\sum_{h \le T} \frac{\Lambda(h)}{h} \right)^2 \ll \log^2 T$$

while for the terms with $(h,k) > 1$ the sum is

$$\ll \sum_{p \le T} \frac{\log^2 p}{p} \ll \log^2 T.$$

Thus, the total contribution from the first O-term to I is $\ll T \log T$. Now we consider

$$\int_1^T |\zeta(1/2+it)|^2 \left(\sum_{h \le x^2} \frac{\Lambda_x(h) \sin(t \log h)}{h_1^{\sigma} \log h} \right)^2 dt = \frac{1}{2}(I_1 - \Re I_2) \tag{53}$$

say, where

$$I_1 = \int_1^T |\zeta(1/2+it)|^2 \left| \sum_{h \le x^2} \frac{\Lambda_x(h)}{h^{\sigma_1+it} \log h} \right|^2 dt \tag{54}$$

and

$$(55) \qquad I_2 = \int_1^T |\zeta(1/2+it)|^2 \left(\sum_{h \le x^2} \frac{\Lambda_x(h)}{h^{\sigma_1+it} \log h} \right)^2 dt.$$

We can evaluate I_1 by Lemma 6 and I_2 by Lemma 9. Let $a(h) = \Lambda_x(h) h^{1/2-\sigma_1} (\log h)^{-1}$ for now. Then,

$$(56) \qquad I_1 = T \sum_{h,k \le x^2} \frac{a(h)a(k)}{hk}(h,k) \left(\log \frac{T(h,k)^2}{2\pi hk} + 2\gamma - 1 \right) + o(T)$$

and

$$(57) \qquad \begin{aligned} I_2 &= 2\pi T \sum_{h,k \le x^2} a(h)a(k) \sum_{n \le \frac{T}{2\pi hk}} d(n) + O(T^{1/2+\epsilon}) \\ &= T \sum_{h,k \le x^2} \frac{a(h)a(k)}{hk} \left(\log \left(\frac{T}{2\pi hk} \right) + O(1) \right). \end{aligned}$$

Thus, (using some of the estimates from above)

$$(58) \qquad I_1 - I_2 = T \sum_{\substack{h,k \le x^2 \\ (h,k)>1}} \frac{a(h)a(k)}{hk}(h,k) \left(\log \frac{T(h,k)^2}{2\pi hk} \right) + O(T \log T)$$

Now $a(h) \ll 1$, so it is clear that the terms of $I_1 - I_2$ for which h or k is not prime will give a total contribution which is $\ll T \log T$. Hence, we have

$$(59) \qquad I_1 - I_2 = T \log T \sum_{p \le x^2} \frac{a(p)^2}{p} + O(T \log T).$$

Finally, this last sum over primes is easily evaluated as in Selberg [10] and we obtain:

Lemma 12. *Assuming the Riemann Hypothesis,*

$$\int_1^T |\zeta(1/2+it)|^2 S(t)^2 \, dt = \frac{1}{2\pi^2} T \log T \log\log T + O(T \log T)$$

Now combining (49) and (50) with this lemma, the second assertion of Theorem 5 is proved.

We note that by (14) and Theorem 5 we have $M'(0) = o(1)$ and assuming the Riemann Hypothesis, $M'(1) = -2 + 2\gamma$. Also, if the Riemann Hypothesis and the pair correlation conjecture are true, then

$$M''(0) = 2 \sum_p \sum_{m=1}^{\infty} \left(\frac{1}{m^2} - \frac{1}{m} \right) p^{-m} + 2\gamma + 2 + o(1).$$

Proofs of Theorems 3 and 4. In this section we calculate the derivatives of a_k and use (13), (14), and Theorem 5 to deduce Theorems 3 and 4.

We first observe that

$$a_k = \prod_p (1-1/p)^{k^2} l_0(k,p) \tag{60}$$

where $\mathrm{e}(\theta) = e^{2\pi i\theta}$ and

$$l_j = l_j(k) = l_j(k,p) = \int_0^1 |1-\mathrm{e}(\theta)/p^{1/2}|^{-2k} \log^j |1-\mathrm{e}(\theta)/p^{1/2}|\, d\theta. \tag{61}$$

We observe that $l_j' = -2l_{j+1}$ where the differentiation is with respect to k. Thus,

$$\frac{a_k'}{a_k} = \sum_p 2k\log(1-1/p) - 2l_1/l_0 \tag{62}$$

and

$$\left(\frac{a_k'}{a_k}\right)' = 2\sum_p \left(\log(1-1/p) + 2\frac{l_0 l_2 - l_1^2}{l_0^2}\right). \tag{63}$$

Now it is easy to calculate that $l_0(0) = 1$, $l_1(0) = 0$, and

$$\begin{aligned} l_2(0) &= \int_0^1 \sum_{m,n=1}^{\infty} \frac{\cos(m\theta)\cos(n\theta)}{mnp^{(m+n)/2}}\, d\theta \\ &= \frac{1}{2}\sum_{m=1}^{\infty} \frac{1}{m^2 p^m}. \end{aligned} \tag{64}$$

Also, by Parseval's formula,

$$l_0(1) = \int_0^1 \left|1 - \frac{\mathrm{e}(\theta)}{p^{1/2}}\right|^{-2} d\theta = \sum_{n=0}^{\infty} p^{-n} = (1-1/p)^{-1}, \tag{65}$$

and

$$\begin{aligned} l_1(1) &= \Re \int_0^1 \sum_{l=1}^{\infty} \sum_{m,n=0}^{\infty} \frac{\mathrm{e}((l+m-n)\theta)}{lp^{(l+m+n)/2}}\, d\theta \\ &= -\sum_{l=1}^{\infty}\sum_{m=0}^{\infty} \frac{1}{lp^{l+m}} = (1-1/p)^{-1}\log(1-1/p). \end{aligned} \tag{66}$$

Thus, we easily find by (62) and (63) that $a_0 = 1$, $a_0' = 0$, and

(67)
$$a_0'' = \sum_p (2\log(1-1/p) + 4l_2(0))$$
$$= 2\sum_p \sum_{m=1}^{\infty} \left(\frac{1}{m^2} - \frac{1}{m}\right)\frac{1}{p^m}.$$

Now, we observe that

(68)
$$a_k = \prod_p (1-1/p)^{k^2-k} P_{k-1}\left(\frac{p+1}{p-1}\right)$$

where P_k is the Legendre function of order k. (See Titchmarsh [12,§8.9] for a proof of this.) Therefore, since $P_{k-1} = P_{-k}$, it follows that

(69)
$$a_k = a_{1-k}.$$

Hence, $a_1 = a_0 = 1$, $a_1' = -a_0' = 0$, and

(70)
$$a_1'' = a_0'' = 2\sum_p \sum_{m=1}^{\infty} \left(\frac{1}{m^2} - \frac{1}{m}\right)\frac{1}{p^m}.$$

Now we consider derivatives of f_k. We have

(71)
$$\frac{f_k'}{f_k} = -2k\frac{\Gamma'}{\Gamma}(1+k^2)$$

and

(72)
$$\frac{d}{dk}\left(\frac{f_k'}{f_k}\right) = \frac{f_k f_k'' - f_k'^2}{f_k^2}$$
$$= -2\frac{\Gamma'}{\Gamma}(1+k^2) - 4k^2\frac{\Gamma\Gamma'' - \Gamma'^2}{\Gamma^2}(1+k^2)$$

so that $f_0 = 1$, $f_0' = 0$, and $f_0'' = -2\Gamma'(1) = 2\gamma$. Also, $f_1 = 1$, and in the notation of the last section, $f_1' = -2\psi(2) = -2\Gamma'(2)$, and $f_1'' = f_1'^2 - 2\psi(2) - 4\psi'(2)$. It follows that $f_1' = -2 + 2\gamma$ and

$$f_1'' = 8\Gamma'(2)^2 - 2\Gamma'(2) - 4\Gamma''(2).$$

Thus, we find that

$$(a_k f_k)'|_{k=0} = 0,$$
$$(a_k f_k)'|_{k=1} = -2 + 2\gamma,$$
$$(a_k f_k)''|_{k=0} = 2\sum_p \sum_{m=1}^{\infty} \left(\frac{1}{m^2} - \frac{1}{m}\right) p^{-m} + 2\gamma,$$

and

(73)
$$(a_k f_k)''|_{k=1} = 2\sum_p \sum_{m=1}^{\infty} \left(\frac{1}{m^2} - \frac{1}{m}\right) + 8\Gamma'(2)^2 - 2\Gamma'(2) - 4\Gamma''(2).$$

Upon comparing these equations with the results of the last section, we see that Theorems 3 and 4 are proved.

REFERENCES

1. R. Balasubramanian, J. B. Conrey, and D.R. Heath-Brown, *Asymptotic mean square of the product of the Riemann zeta-function and a Dirichlet polynomial*, J. Reine Agnew. Math. **357** (1985), 161–181.
2. J. B. Conrey and A. Ghosh, *On mean values of the zeta-function*, Mathematika **31** (1984), 159–161.
3. J. B. Conrey, A. Ghosh, and S.M. Gonek, *Simple zeros of the Riemann zetafunction*, (preprint).
4. D. A. Goldston, *On the function $S(T)$ in the theory of the Riemann zeta-function*, J. Number Theory **27** (1987), 149–177.
5. S. M. Gonek, *Mean values of the Riemann zeta-function and its derivatives*, Inventiones Math. **75** (1984), 123–141.
6. D. R. Heath-Brown, *Fractional moments of the Riemann zeta-function*, J. London Math. Soc.(2) **24** (1981), 65–78.
7. D. R. Heath-Brown, *Fractional moments of the Riemann zeta-function. II*, (to appear).
8. H. L. Montgomery, *The pair correlation of zeros of the Riemann zeta function*, Proc. Symp. Pure Math., vol. 24, A.M.S., Providence, 1973.
9. A. M. Odlyzko, *The 10^{20}-th zero of the Riemann zeta function and 70 million of its neighbors*, (preprint).
10. A. Selberg, *On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip $0 < t < T$*, Avhandlinger Norske Vid. Akad. Oslo no. No. 1 (1944).
11. A. Selberg, *Note on a paper by L. G. Sathe*, J. Indian Math. Soc. **18** (1954), 83–87.
12. E. C. Titchmarsh, *The Theory of the Riemann Zeta-function*, Second Edition, Clarendon Press, Oxford, 1986.

Appendix

Further properties of a_k.

In this appendix we prove that a_k has a local maximum at $k = 1/2$ and local minima at $k = 0$ and $k = 1$ and no other extrema for real k.

To do this we consider a_s as a function of a complex variable s. We show that a_s is an entire function of order 2 which satisfies the functional equation

$$a_s = a_{1-s}$$

Also, all the zeros of a_s are on the line $\sigma = 1/2$.

From these properties of a_s we easily deduce that a_s' has all its zeros on $\sigma = 1/2$ with two possible exceptions; if the exceptions exist, then they are real zeros. In the last section we showed that $a_0' = a_1' = 0$, so that the exceptions do exist.

To prove most of these assertions we use the representation

$$a_s = \prod_p \left(1 - \frac{1}{p}\right)^{s^2-s} P_{s-1}\left(\frac{p+1}{p-1}\right)$$

where P is the Legendre function for which we have the formulae

$$P_s(x) = \frac{1}{\pi}\int_0^\pi \left(x + \sqrt{x^2-1}\cos\theta\right)^s \, d\theta$$

for $x > 1$ and all s and

$$P_{s-1/2}(\cosh x) = \frac{\sqrt{2}}{\pi}\int_0^x \frac{\cosh su}{\sqrt{\cosh x - \cosh u}}\, du$$

valid for real x and all s.

From the first formula for P we easily deduce that

$$|P_{s-1}(x)| \leq (2x)^{|\sigma-1|}$$

Also,

$$\left|(1-1/p)^{s^2-s}\right| = (1-1/p)^{\sigma^2-t^2-\sigma}$$

Let $C(s)$ be a real function of s to be determined later. In the product formula for a_s we estimate the factors for $p \leq C(s)$ by the above, while the factors for $p > C(s)$ are

$$= 1 + O(|s|^4/p^2)$$

provided that $C(s)$ is sufficiently large as is easily seen by considering the power series expansion of each factor. Thus, we have

$$|a_s| \leq \prod_{p \leq C(s)} (1-1/p)^{\sigma^2-t^2-\sigma} \left(2\frac{p+1}{p-1}\right)^{|\sigma-1|} \prod_{p>C(s)} (1+O(|s|^4/p^2))$$
$$\ll (A \log C(s))^{|s|^2} \exp(A|s|^4/C(s))$$

for some A. (We have used the estimate $\prod_{p\leq x}(1-1/p) \approx (\log x)^{-1}$.) Now choosing $C(s) = |s|^4$, we see that

$$a_s \ll_\epsilon \exp\left(|s|^{2+\epsilon}\right)$$

Thus, a_s is entire of order at most 2. To see that a_s has order exactly 2 it suffices to show that the number $N(T)$ of zeros of $a_{1/2+it}$ in $0 < t < T$ satisfies

$$\liminf_{T\to\infty} \frac{\log N(T)}{\log T} \geq 2$$

We will prove this estimate later.

To see that all the zeros of a_s have real part 1/2 we use the second representation of P given above and the theorem in Polya-Szego, Problems and Theorems in Analysis (Vol. I, Part III, No. 205) to conclude that the entire function of t given by

$$P_{-1/2+it}(x)$$

has all real zeros provided that $x > 1$.

Next we show that a'_s has all but two of its zeros on the 1/2-line. First of all, by the functional equation,

$$b_z = a_{1/2+i\sqrt{z}}$$

is an entire function of order 1 which is real on the real line. By the theory of entire functions of finite order we may write

$$b_z = e^{A+Bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}$$

where the z_n are all real and positive and A and B are real. Then

$$\frac{b'_z}{b_z} = B + \sum_n \left(\frac{1}{z - z_n} + \frac{1}{z_n} \right)$$

Now

$$\Im \frac{b'_z}{b_z} = -y \sum_n \frac{1}{|z - z_n|^2}$$

so that all zeros of b'_z are real. For real $z = x$ each term

$$\frac{1}{z - z_n} + \frac{1}{z_n}$$

is decreasing. Thus, b'_z/b_z can have at most one negative real zero. A real negative zero of b'_z corresponds to a pair of real zeros of a'_z. Thus, with two possible exceptions the zeros of a'_s are on the half-line. We conclude that the real zeros of a'_s are at $s = 0$, $s = 1/2$, and $s = 1$.

It remains to prove that a_s is of order at least two (though this assertion was not required in the above deductions). By applying Laplace's method to obtain the asymptotic evaluation of $P_{s-1}(x)$ on the half-line it is easily deduced that the first zero of this function is at a height

$$t \sim \frac{1}{2} h(x)$$

and the spacing between consecutive zeros is

$$\sim h(x)$$

where

$$h(x) = \frac{2\pi}{\log \frac{x+\sqrt{x^2-1}}{x-\sqrt{x^2-1}}}$$

Taking $x = (p+1)/(p-1)$ we have

$$h(x) \sim \frac{\pi}{2}\sqrt{p}$$

Thus the number of zeros to height T is

$$\gg T \sum_{p<T^2} p^{-1/2} \gg \frac{T^2}{\log T}$$

Hence,

$$\liminf \frac{\log N(T)}{\log T} \geq 2$$

as desired.

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