ON THE REAL AND IMAGINARY CURVES OF THE RIEMANN ZETA-FUNCTION

by

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Real curves of the zeta-function

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We describe the curves Re $\zeta(s) = 0$ and Im $\zeta(s) = 0$. Subject to certain hypotheses we establish a connection between zeros of $\zeta(s)$ and "Gram-points" which involves these curves.

§1. Statement of Results

In this paper we present some theorems and a conjecture about the curves $\mathrm{Re}\,\zeta(s) = 0$ and $\mathrm{Im}\,\zeta(s) = 0$ where $\zeta(s)$ is Riemann's zeta-function. In particular, we show (subject to the conjecture) that the zeros of $\zeta(s)$ and the "Gram-points", well-known from the literature on calculations related to the zeros of $\zeta(s)$, are connected via these curves. There seems to be a good deal of regularity to these curves which provides some insight into the nature of the zeta-function.

Following van de Lune [3] we shall refer to maximally connected subsets of $\{s : \mathrm{Re}\,\zeta(s) = 0\}$ as R-lines and to maximally connected subsets of $\{s : \mathrm{Im}\,\zeta(s) = 0\}$ as I-lines. Since an R-line or an I-line is a curve on which a harmonic function has constant value it follows by the maximum-minimum principle for harmonic functions that each such curve must have a singularity of $\zeta(s)$ (that is, either $s = 1$ or $s = \infty$) as a limit point. Figure 1 from Utzinger [8,p.27] (see also Jahnke and Emde [1,p.270]) shows the R-lines and I-lines near $s = 1$.

It is easily seen from the expression

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} \qquad (s = \sigma + it,\ \sigma > 1) \tag{1}$$

that the R-lines are bounded above in the σ-direction. van de Lune [3] showed that the values of s on any R-line satisfy $\sigma < \sigma_R$ where σ_R is defined by

$$\sum_{p} \arcsin p^{-\sigma_R} = \frac{\pi}{2}; \tag{2}$$

the sum is over positive primes p. He did not calculate σ_R explicitly, but showed, using the primes < 100000, that $\sigma_R > 1.18283$. We show, without calculating the primes,

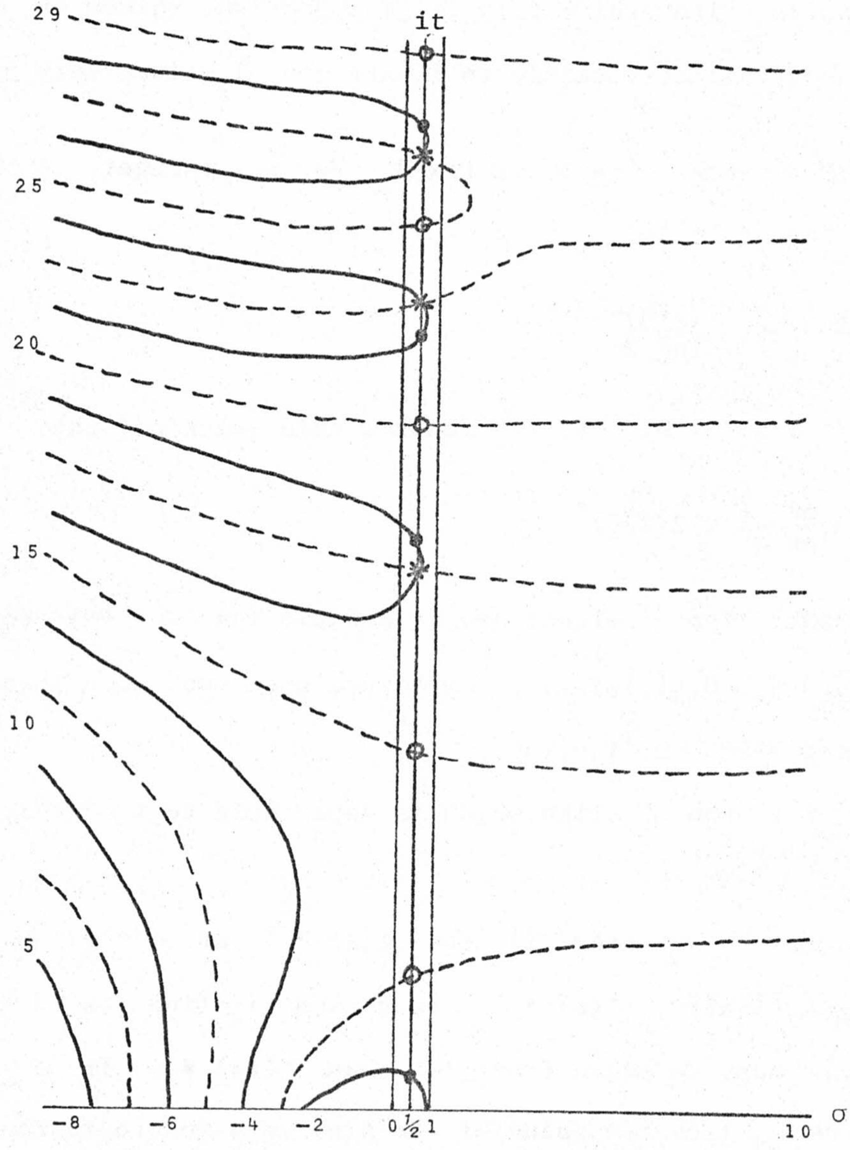

Figure 1

THEOREM 1. Let σ_R be defined by (2). Then

$$\sigma_R = 1.192347 + 2\theta 10^{-6}$$

where $|\theta| \leq 1$.

It is apparent from Figure 1 that for some of the I-lines the values of σ are bounded above; we call these the I_1-lines. The I-lines which have points s with arbitrarily large (positive) values of σ will be called I_2-lines. It is possible to locate the I_2-lines very precisely.

THEOREM 2. Let $\sigma \geq 5$ be fixed. For any integer n there is precisely one value of t with

$$\frac{(n-\frac{1}{2})\pi}{\log 2} < t \leq \frac{(n+\frac{1}{2})\pi}{\log 2}$$

such that $\mathrm{Im}\,\zeta(\sigma + it) = 0$. If t_n denotes this solution, then

$$\left| t_n - \frac{n\pi}{\log 2} \right| < 2(2/3)^{\sigma}.$$

COROLLARY. The I_2-lines are asymptotic (as $\sigma \to +\infty$) to the lines $t = \frac{n\pi}{\log 2}$, $n = 0, \pm 1, \pm 2, \ldots$. Moreover, each such line is an asymptote for precisely one I_2-line.

We refer to the I_2-line which is asymptotic to $t = n\pi/\log 2$ as the $I_{2,n}$-line.

It is easy to see from (1) that $\zeta(s) \to 1$ as $\sigma \to +\infty$ uniformly for all t. In particular, $\zeta(s) \to 1$ along each I_2-line as $\sigma \to +\infty$. It is well-known (and easy to prove from (1)) that $\zeta'(s) \neq 0$ if $\sigma \geq 4$. Therefore, on any I_2-line the value of $\mathrm{Re}\,\zeta(s)$ is a monotonic function of σ for $\sigma \geq 4$. We refer to the I_2-lines for which $\mathrm{Re}\,\zeta(s) \downarrow 1$ as $\sigma \to +\infty$ as G-curves and those for which $\mathrm{Re}\,\zeta(s) \uparrow 1$ as $\sigma \to \infty$ will be called Z-curves. The reason for this notation will be apparent later.

THEOREM 3. If n is even, then the $I_{2,n}$-line is a G-curve. If n is odd, then the $I_{2,n}$-line is a Z-curve.

Now we describe the R- and I-lines for a left half-plane.

THEOREM 4. Let $\sigma \leq -3$ be fixed. There are

$$T \log \frac{T}{2\pi e} + O_\sigma(1)$$

solutions t of $\operatorname{Re} \zeta(\sigma + it) = 0$ with $0 < t < T$. The same estimate hold for the number of solutions of $\operatorname{Im} \zeta(\sigma + it) = 0$. Moreover, the solutions of $\operatorname{Re} \zeta(\sigma + it) = 0$ are interlaced with the solutions of $\operatorname{Im} \zeta(\sigma + it) = 0$ for $t \geq 10$.

COROLLARY. In the region $\sigma \leq -3$, $t \geq 10$ the R-lines and the I-lines alternate. Moreover, $\sigma \to -\infty$, $t \to \infty$, and $|\zeta(s)| \to \infty$ as $|s| \to \infty$ on any of the R-lines or I-lines in this region.

Most of our results about the R-lines and I-lines in the strip $-3 \leq \sigma \leq 5$ are conditional. We can say something about these lines when $\sigma = \frac{1}{2}$, however. Let $\chi(s)$ have its usual meaning in the theory of the zeta-function:

$$\chi(s) = \zeta(s)/\zeta(1-s) = 2^s \pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s). \tag{3}$$

It is well-known that $|\chi(s)| = 1$ if and only if $\sigma = \frac{1}{2}$.

THEOREM 5. Let $s = \frac{1}{2} + it$. Then $\operatorname{Re} \zeta(s) = 0$ if and only if $\zeta(s) = 0$ or $\chi(s) = -1$ and $\operatorname{Im} \zeta(s) = 0$ if and only if $\zeta(s) = 0$ or $\chi(s) = +1$.

The points s for which $\chi(s) = +1$ are the Gram-points mentioned earlier. We will refer to these as the Gram$^+$-points and to the points s

for which $\chi(s) = -1$ as the Gram$^-$-points. These points are very well spaced on the $\sigma = \frac{1}{2}$ line. The distance between two consecutive Gram$^+$-points (or Gram$^-$-points) at height T is

$$\frac{2\pi}{\log \frac{T}{2\pi}} + O\left(\frac{1}{T \log^2 T}\right). \tag{4}$$

This is also the average distance between consecutive ordinates of zeros of the zeta-function, which explains how the Gram points might be of use in locating zeros of $\zeta(s)$ on the critical line: Gram's "law" is that there is usually a zero of $\zeta(s)$ between consecutive Gram points. However, it is likely that there exist consecutive Gram points $\frac{1}{2} + it$ and $\frac{1}{2} + it'$ with any number of ordinates of zeros of $\zeta(s)$ between t and t' and it is also likely that there are stretches of any number of consecutive Gram points between $\frac{1}{2} + it_1$ and $\frac{1}{2} + it_1'$, say, without any ordinates of zeros of $\zeta(s)$ in (t_1, t_2). Nevertheless, there is a definite one-to-one correspondence between Gram-points and zeros of $\zeta(s)$, subject to some conjectures.

CONJECTURE. If $s = \sigma + it$, $t \neq 0$, and $\zeta'(s) = 0$, then $\operatorname{Re} \zeta(s) \neq 0$ and $\operatorname{Im} \zeta(s) \neq 0$.

In other words, we conjecture that zeros of ζ' do not lie on the R-lines or the I-lines except for the real axis (which is an I-line). Since ζ' has only countably many zeros, while there are uncountably many curves $\operatorname{Re} \zeta(s) = a$ or $\operatorname{Im} \zeta(s) = a$, and since there is no contrary evidence, the Conjecture seems plausible. We also have some numerical evidence for the Conjecture which we will cite later. The importance of this Conjecture for our purposes is contained in the following

PROPOSITION. Suppose that R is an R-line which has no zeros of $\zeta'(s)$ on it. Then there is a homeomorphism f from $\mathbb{R}$ (the real-line) to R such that the function $g : \mathbb{R} \to \mathbb{R}$ given by $g(x) = \operatorname{Im} \zeta(f(x))$ is monotone. A similar statement holds for I-lines with $\operatorname{Im} \zeta$ replaced by $\operatorname{Re} \zeta$.

In less technical language, the Proposition asserts that the R-lines do not intersect each other and that if you trace along an R-line without reversing directions, then $\operatorname{Im} \zeta(s)$ always increases or always decreases. This Proposition is a consequence of the well-known fact that an analytic function F is univalent in every sufficiently small neighborhood of a point z_0 if and only if $F'(z_0) \neq 0$ (see Titchmarch [6,§§6.4 and 6.43]).

We can now state our main result.

THEOREM 6. Suppose that the Conjecture holds. Then the following three statements are equivalent:

(i) all the zeros of $\zeta(s)$ with $t \neq 0$ are on the line $\sigma = \frac{1}{2}$ (i.e. the Riemann-Hypothesis);

(ii) any R-line which has no real points on it intersects the line $\sigma = \frac{1}{2}$ in precisely two points, one of which is a zero of $\zeta(s)$ and the other is a Gram^--point;

(iii) any I_1-line which has no real points on it intersects the line $\sigma = \frac{1}{2}$ in precisely two points, one of which is a zero of $\zeta(s)$ and the other is a Gram^+-point. Any I_2-line without real points intersects $\sigma = \frac{1}{2}$ in precisely one point; for a G-curve the point is a Gram^+-point and for a Z-curve the point is a zero of $\zeta(s)$.

It will follow from our proofs that assuming the Conjecture and the Riemann-Hypotheses the region between consecutive G-curves is in some sense a unit as far as the R-lines and I-lines are concerned. For the G-curves do not intersect any other R-lines or I-lines so that between consecutive G-curves the zeros of $\zeta(s)$ and the Gram^--points are in one-to-one correspondence and are paired up by the R-lines. The Gram^+-points in this region are paired with all but one of the zeros of $\zeta(s)$ here via the I_1-lines, and the remaining zero of $\zeta(s)$ is the intersection of $\sigma = \frac{1}{2}$ and a Z-curve.

Regarding the violations of Gram's law mentioned earlier, it is likely that a Gram^--point and its corresponding zero of $\zeta(s)$ will occasionally be separated by "many" other Gram-points. In situations such as these, it might be expected that the R-lines cross each other, in contradiction to the Conjecture. We have tested some points which are mentioned in Lehman's paper [2, p.531] as being points where $\zeta(s)$ behaves erratically as far as Gram's law is concerned, and found the results to be in agreement with our Conjecture. At these places we found that the R-lines do not intersect each other; rather they are looped as in Figure 2.

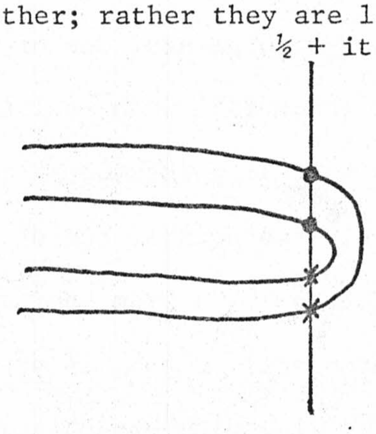

Figure 2

The intervals where we found this phenomenon are of the form (g_n, g_{n+2}) where g_n is the solution t of $-\arg \chi(\frac{1}{2} + it) = 2n\pi$ (the nth Gram-point) and we found this phenomenon of Figure 2 for $n = 171382, 206715, 209783, 233173$, and 234500. (We must stress, however, that the computations were not carried out to an extent that we claim to have rigorously proved that the R-lines behave as in Figure 2 for these values of n. We would have had to evaluate the ζ-function at a large number of points to completely justify our statements, but this investigation is intended to be theoretical rather than computational.)

If this looping effect of Figure 2 is accurate then it seems to question the winding effect that van de Lune [3, p.337] found. He asserted that there are R-lines of the shape in Figure 3.

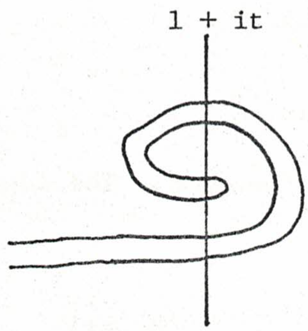

Figure 3

It appears, though, that he has inferred this shape from considerations of $\zeta(s)$ for $\sigma > 1$. Since it is unlikely (by Theorem 6) that such an R-line could wind back to $\sigma = \frac{1}{2}$ we wonder if perhaps Figure 3 should be three R-lines looped around each other in the manner of Figure 2. Such a configuration would match that of Figure 3 for $\sigma > 1$.

The author would like to thank D. R. Heath-Brown for an interesting conversation on this material. In particular he suggested the possibility of computing σ_R of Theorem 1 without using primes and he pointed out a simple proof for Theorem 4.

§2. Proof of Theorem 1

We will let θ denote a (possibly complex) number with absolute value ≤ 1 which is, in general, different at each occurence.

We first establish some lemmas.

LEMMA 1. For $\sigma > 1$,

$$\sum_p \arcsin p^{-\sigma} = \sum_{N=1}^{\infty} \frac{A_N}{N} \log \zeta(N\sigma)$$

where

$$A_N = \sum_{\substack{n(2m+1)=N \\ m\geq 0, n\geq 1}} \mu(n) \binom{2m}{m} 2^{-2m}.$$

Here μ is the Mobius-function. The identity may be deduced from the expansions

$$\arcsin p^{-\sigma} = \sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{(2n+1)} 2^{-2n} (p^{-\sigma})^{2n+1}$$

and

$$\log \zeta(\sigma) = \sum_{p,m}\sum m^{-1} p^{-m\sigma}$$

by equating the coefficients of powers of $p^{-\sigma}$.

LEMMA 2. With A_N defined as above, $|A_N| \leq 1.8$ for all N.

PROOF. By Stirlings formula, $k! = (k/e)^k (2\pi k)^{\frac{1}{2}} e^{\theta/(12k)}$.

Hence

$$\binom{2m}{m} 2^{-2m} = (m\pi)^{-\frac{1}{2}} e^{5\theta/(24m)} = (m\pi)^{-\frac{1}{2}} (1+.25\theta m^{-1})$$

$$= (m\pi)^{-\frac{1}{2}} + .15\theta m^{-3/2}$$

for $m \geq 1$. By the mean-value-theorem,

$$m^{-\frac{1}{2}} = (m+\tfrac{1}{2})^{-\frac{1}{2}} = .25\theta m^{-\frac{1}{2}}.$$

Therefore,

$$\binom{2m}{m} 2^{-2m} = (m+\tfrac{1}{2})\pi^{-\frac{1}{2}} + .3\theta m^{-3/2}$$

so that

$$A_N = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \sum_{n(2m+1)=N} \mu(n)(2m+1)^{-\frac{1}{2}} + .9\theta$$

by Lemma 1 and the fact that $\Sigma m^{-3/2} < 3$. Suppose that N is odd. Then the above sum is a convolution of multiplicative functions and is therefore multiplicative. Let $f(n) = \sum_{d|n} \mu(n/d) d^{-\frac{1}{2}}$. Then f is multiplicative and

$$f(p^r) = p^{-r/2} - p^{-(r-1)/2}$$

so that $|f(p^r)| \leq 1$. Hence $|f(n)| \leq 1$ for all n. Thus, for odd N,

$$A_N \leq (2/\pi)^{\frac{1}{2}} + .9 \leq 1.8.$$

But if N is odd then $A_{2N} = -A_N$, and $A_{2^r N} = 0$ if $r \geq 2$. This proves the Lemma.

LEMMA 3. If $\sigma > 1$ and $M \geq 1$ then

$$E_{2M+1} = \left| \sum_{N=2M+1}^{\infty} \frac{A_N}{N} \log \zeta(N\sigma) \right| \leq \frac{6}{2M+1} 2^{-(2M+1)\sigma},$$

PROOF. If $\sigma > 1$, then

$$\zeta(\sigma) = \sum_{n=1}^{\infty} n^{-\sigma} \leq 1 + 2^{-\sigma} + \int_2^{\infty} x^{-\sigma} d\sigma = 1 + \left(\frac{\sigma+1}{\sigma-1}\right) 2^{-\sigma}.$$

Hence

$$\log \zeta(\sigma) \leq 2^{1-\sigma}$$

for $\sigma \geq 3$. Thus, if N is odd, then by Lemma 2,

$$\left|\frac{A_N}{N} \log \zeta(N\sigma) - \frac{A_{2N}}{2N} \log \zeta(2N\sigma)\right| = \frac{|A_N|}{N} (\log \zeta(N\sigma) - \tfrac{1}{2} \log \zeta(2N\sigma))$$

$$\leq \frac{|A_N|}{N} \log \zeta(N\sigma) \leq \frac{2^{2-N\sigma}}{N}.$$

Hence we may ignore the even terms. Thus,

$$E_{2M+1} \leq \sum_{n=M}^{\infty} \frac{2^{2-(2n+1)\sigma}}{2n+1} \leq \frac{2^{2-(2M+1)\sigma}}{(2M+1)(1-4^{-\sigma})} \leq \frac{6}{2M+1} 2^{-(2M+1)\sigma}$$

as stated.

We list the first few A_N with N odd:

N	1	3	5	7	9	11	13	15
A_N	1	$-\frac{1}{2}$	$-\frac{5}{8}$	$-\frac{11}{16}$	$-\frac{29}{128}$	$-\frac{193}{256}$	$-\frac{793}{1024}$	$\frac{3079}{6144}$

By the last lemma with $M = 8$ and $\sigma > 1.1$,

$$\sum_p \arcsin p^{-\sigma} = \sum_{N=1}^{15} \frac{A_N}{N} \log \zeta(N\sigma) + 2\theta 10^{-6}.$$

We use the Euler-Maclaurin summation formula to evaluate $\zeta(\sigma)$ (see Rademacher [4,§38].). Thus,

$$\zeta(\sigma) = \sum_{n=1}^{N-1} n^{-\sigma} + \frac{N^{1-\sigma}}{\sigma-1} + \tfrac{1}{2}N^{-\sigma} + \sum_{k=1}^{v} S_k + R_{v+1}$$

where

$$S_k = \frac{B_{2k}}{(2k)!} \sigma(\sigma+1)\ldots(\sigma+2k-2)N^{-\sigma-2k+1}$$

and $|R_{v+1}| \leq |S_{v+1}|$. Here B_{2k} is the $2k^{th}$ Bernoulli number: $B_2 = 1/6$, $B_4 = 1/30$, and $B_6 = 1/42$. In our calculations we used $N = 10$, $v = 2$ for $1 < \sigma \leq 3$; $N = 10$, $v = 1$ for $3 < \sigma \leq 6$; $N = 10$,

$v = 0$ for $6 < \sigma \leq 13$; and $N = 7$, $v = 0$ for $13 < \sigma \leq 20$. It is easily checked that the error $|R_{v+1}|$ in each calculation of $\zeta(\sigma)$, $1.1 < \sigma \leq 20$, is $\leq \frac{1}{6} 10^{-6}$. This leads to an error in $\log \zeta(\sigma)$, $1.1 < \sigma \leq 20$, which is $\leq \frac{1}{3} 10^{-6}$. Hence the error in calculating

$$\sum_{N=1}^{15} \frac{A_N}{N} \log \zeta(N\sigma)$$

by replacing $\zeta(N\sigma)$ by the appropriate Euler-Maclaurin sum is

$$\leq \frac{2}{3} 10^{-6} \sum_{\substack{N=1 \\ 4+N}}^{15} \frac{1}{N} \leq 2\ 10^{-6}$$

for $1.1 < \sigma \leq 1.3$.

The round off errors are negligible in a small calculation such as this, but are certainly $< 10^{-6}$. Hence by Lemmas 1, 2, and 3 we are able to calculate $\sum_p \arcsin p^{-\sigma}$ with an error $< 5 \times 10^{-6}$. But our calculations give $\sum_p \arcsin p^{-\sigma} < \frac{\pi}{2} - 5\cdot 10^{-6}$ for $\sigma = 1.192349$ and $\sum_p \arcsin p^{-\sigma} > \frac{\pi}{2} + 5\cdot 10^{-6}$ for $\sigma = 1.192345$. Hence the result.

§3. Proofs of Theorems 2, 3, 4 and 5

Theorem 2 is an easy consequence of

LEMMA. If $\sigma \geq 5$, then $\arg(\zeta(\sigma+it)-1)$ is a decreasing function of t.

PROOF. We observe that if f is regular and not zero on the vertical line $\sigma = \sigma_1$, then

$$\frac{d}{dt} \arg f(\sigma_1+it) = \operatorname{Re} \frac{f'}{f}(s)\Big|_{s=\sigma_1+it}. \tag{5}$$

Therefore

$$\frac{d}{dt}\arg(\zeta(\sigma+it)-1) = \text{Re}\,\frac{\zeta'(s)}{\zeta(s)-1} = \text{Re}\,\frac{\zeta'(s)\overline{(\zeta(s)-1)}}{|\zeta(s)-1|^2}\,.$$

If $\sigma \geq 3$, then

$$\sum_{n=3}^{\infty} n^{-\sigma} \leq 3^{-\sigma} + \int_3^{\infty} x^{-\sigma}dx = 3^{-\sigma}(1+\frac{3}{\sigma-1}) \leq 2^{-\sigma} \qquad (6)$$

so that $\zeta(s) \neq 1$ for $\sigma \geq 3$. Thus it suffices to show that

$$\text{Re}\ \zeta'(s)\overline{(\zeta(s)-1)} < 0$$

for $\sigma \geq 5$. If $\sigma \geq 5$ then

$$\sum_{n=3}^{\infty} \frac{\log n}{n^{\sigma}} \leq 3^{-\sigma}\log 3 + \int_3^{\infty} x^{-\sigma}\log x\,dx = 3^{-\sigma}(\log 3 + \frac{3\log 3}{\sigma-1} + \frac{3}{(\sigma-1)^2})$$

$$\leq 2.2\cdot 3^{-\sigma}$$

so that

$$\zeta'(s) = -\sum_{n=2}^{\infty}\frac{\log n}{n^s} = \frac{-\log 2}{2^s} + 2.2\theta 3^{-\sigma}.$$

By (6),

$$\overline{\zeta(s)} - 1 = \sum_{n=2}^{\infty} n^{-\bar{s}} = 2^{-\bar{s}} + 1.75\cdot 3^{-\sigma}$$

for $\sigma \geq 5$. Hence

$$\text{Re}\ \zeta'(s)\ \overline{(\zeta(s)-1)} = \frac{-\log 2}{2^{2\sigma}} + 4\theta(6^{-\sigma}+9^{-\sigma})$$

$$= \frac{-\log 2}{2^{2\sigma}} + 5\theta 6^{-\sigma} < 0$$

for $\sigma \geq 5$ which proves the Lemma.

To prove Theorem 2, observe that $\text{Im}\ \zeta(s) = 0$ precisely when $\arg(\zeta(s)-1) \equiv 0 \bmod \pi$. But by (6),

$$\arg(\zeta(s)-1) = \arg 2^{-s} + \arg(1+2^s\sum_{n=3}^{\infty} n^{-s})$$

$$= -t\log 2 + \arg(1+\theta 2^{\sigma}3^{-\sigma}\cdot 1.75) \qquad (7)$$

$$= -t\log 2 + 1.1\theta(2/3)^{\sigma}$$

for $\sigma \geq 5$. The first part of Theorem 2 now follows from (7) and Lemma 4. Moreover, by (7) we see that t_n satisfies

$$-t_n \log 2 + 1.1\theta(2/3)^{\sigma} = -n\pi$$

or

$$\left| t_n - \frac{n\pi}{\log 2} \right| \leq \frac{1.1}{\log 2} (2/3)^{\sigma} < 2(2/3)^{\sigma}$$

which proves the theorem.

To prove Theorem 3 we observe that

$$2^{in\pi/\log 2} = \begin{cases} +1 & \text{if } n \text{ is even} \\ -1 & \text{if } n \text{ is odd} \end{cases}$$

so that if s is on an $I_{2,n}$ - line with n even then

$$\zeta(s) = 1 + \frac{1}{2^{\sigma+\varepsilon(\sigma)}} + O(3^{-\sigma})$$

where $\varepsilon(\sigma) \to 0$ as $\sigma \to \infty$ while if s is on an $I_{2,n}$ line with n odd, then

$$\zeta(s) = 1 - \frac{1}{2^{\sigma+\varepsilon(\sigma)}} + O(3^{-\sigma}).$$

In the case that n is even, $\mathrm{Re}\, \zeta(s) > 1$ for all sufficiently large σ while if n is odd, then $\mathrm{Re}\, \zeta(s) < 1$ for all sufficiently large σ. Taken with the remarks immediately preceding Theorem 3, this establishes the theorem.

To prove Theorem 4 we need an estimate for the logarithmic derivative of $\zeta(s)$.

LEMMA 5. If $\sigma < 0$ and $t > 0$ then

$$\frac{\chi'}{\chi}(s) = -\log \frac{s}{2\pi i} + \theta\left(\frac{1}{|\sigma|} + \frac{\pi e^{-\pi t}}{1-e^{-\pi t}}\right).$$

PROOF. We have $\chi(s) = 2^s\pi^{s-1} \sin \frac{\pi s}{2} \Gamma(1-s)$ so that

$$\frac{\chi'}{\chi}(s) = \log 2\pi + \frac{\pi}{2} \cot \frac{\pi s}{2} - \frac{\Gamma'}{\Gamma}(1-s).$$

It is easy to see that

$$\cot z = -i + \frac{2\theta e^{-2y}}{1-e^{-2y}}$$

for $z = x + iy$, $y > 0$. Also (see Whittaker and Watson [9, 12.31]),

$$\frac{\Gamma'}{\Gamma}(z+1) = \log z + \frac{\theta}{2x}$$

for $x > 0$. The Lemma follows easily from the formulae.

The assertion in Theorem 4 that the solutions of $\operatorname{Re} \zeta(\sigma+it) = 0$ are interlaced with the solutions of $\operatorname{Im} \zeta(\sigma+it) = 0$ follows immediately from

LEMMA 6. Let $\sigma \leq -3$ be fixed. Then $\arg \zeta(\sigma+it)$ is a decreasing function of t for $t \geq 10$.

PROOF. By (3) and (5),

$$\frac{d}{dt} \arg \zeta(\sigma+it) = \operatorname{Re} \frac{\zeta'}{\zeta}(s) = \operatorname{Re} \frac{\chi'}{\chi}(s) + \operatorname{Re} \frac{\zeta'}{\zeta}(1-s).$$

But

$$\frac{\zeta'}{\zeta}(1-s) = \theta \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^{1-\sigma}} = \sigma \sum_{n=2}^{\infty} \frac{\log n}{n^{1-\sigma}} = \theta\left(\frac{\log 2}{2^{1-\sigma}} + \int_2^{\infty} x^{\sigma-1} \log x \, dx\right)$$

$$= \theta 2^{\sigma}$$

if $\sigma \leq -3$. Hence, by Lemma 5,

$$\frac{d}{dt} \arg \zeta(\sigma+it) = -\tfrac{1}{2} \log\left(\frac{\sigma^2+t^2}{4\pi^2}\right) + \theta\left(2^{\sigma} + \frac{1}{|\sigma|} + \frac{\pi e^{-\pi t}}{1-e^{-\pi t}}\right) < 0$$

if $t \geq 10$ and $\sigma \leq -3$. This proves the Lemma.

The rest of Theorem 4 follows from

LEMMA 7. If $\sigma \leq -2$ and $t > 1$, then

$$\arg \zeta(\sigma+it) = -t \log \frac{t}{2\pi e} + \frac{\pi}{4} + (\tfrac{1}{2}-\sigma)\arctan \frac{1-\sigma}{t} - \frac{t}{2} \log(1 + \frac{(\sigma-1)^2}{t^2})$$

$$+ O(e^{-\pi t} + \frac{1}{|s|} + 2^{\sigma}). \tag{8}$$

$$= -t \log \frac{t}{2\pi e} + O_{\sigma}(1)$$

$$= -\sigma \frac{\pi}{2} - t \log |\sigma| + O_t(1).$$

PROOF. By the functional equation (1),

$$\arg \zeta(s) = \arg \chi(s) + \arg \zeta(1-s)$$

$$= \mathrm{Im}(s \log 2\pi + \log \sin \frac{\pi s}{2} + \log \Gamma(1-s) + \log \zeta(1-s))$$

$$= t \log 2\pi + \frac{\pi}{2} - \frac{\pi\sigma}{2} + O(e^{-\pi t}) + \mathrm{Im} \log \Gamma(1-s) + O(2^{\sigma}).$$

Also,

$$\log \Gamma(z) = (z-\tfrac{1}{2})\log z - z + \tfrac{1}{2} \log 2\pi + O(\frac{1}{|z|}) \tag{9}$$

for $-\pi + \delta < \arg z < \pi - \delta$ (see Whittaker and Watson [9,§12.31]). The Lemma follows easily.

From equation (8) we see that on the curve

$$\arg \zeta(s) = \alpha$$

$\alpha >> 1$, we have $\sigma \to -\infty$ and $t \to +\infty$ as $|s| \to \infty$. By (1) and (9) it is easy to see that $|\zeta(s)| \to \infty$ as $|s| \to \infty$ on this curve. This proves the Corollary to Theorem 4.

Theorem 5 is easy to prove. By the functional equation (1) we have for $s = \frac{1}{2} + it$,

$$\mathrm{Re}\ \zeta(s) = \tfrac{1}{2}(\zeta(s) + \zeta(1-s)) = \tfrac{1}{2}\zeta(1-s)(1+\chi(s))$$

and

$$\mathrm{Im}\ \zeta(s) = -\tfrac{1}{2}i(\zeta(s) + \zeta(1-s)) = -\tfrac{1}{2}i\zeta(1-s)(1+\chi(s))$$

from which the Theorem follows.

§4. Proof of Theorem 6

In this section we assume the validity of the Conjecture. We need to show that all of the R-lines and I-lines, except those which intersect the real axis, are described by Theorems 2 and 4. That is, we need to show that there are no R- or I-lines located entirely in the half strip $-3 \leq \sigma \leq 5$, $t > 0$. We can do this with the help of the Conjecture.

LEMMA 8. Suppose that $s_1 = \sigma_1 + it_1$ with $\sigma_1 > \frac{1}{2}$, $t_1 > 0$, and that s_1 is on an R-line (or I-line) which lies entirely in the upper half-plane. Then that R-line (or I-line) intersects the line $\sigma = \sigma^*$ for any $\sigma^* \leq \frac{1}{2}$. Moreover, any R-line or I-line is bounded above in the t-direction for $\sigma \geq \sigma^*$.

REMARK. Speiser [5,p.517] proves something similar to this.

PROOF. We first prove the Lemma for I_2-lines. We can identify the I_2-lines, as before, according to the value of n for which the I_2-line is asymptotic to $t = \frac{n\pi}{\log 2}$; we call this the $I_{2,n}$-line. Note that if $n \neq n'$ then $I_{2,n}$ and $I_{2,n'}$ really are distinct I_2-lines.

For as $\sigma \to \infty$ on any I_2-line we have $\zeta(s) \to 1$, so that $I_{2,n}$ and $I_{2,n'}$ are distinct by the Proposition. Now if $I_{2,N}$, $N > 0$, did not intersect $\sigma = \sigma^*$, then it must contain points s with arbitrarily large t. However, this would imply, by the Proposition, that if $n > N$ then $I_{2,n}$ also does not intersect $\sigma = \sigma^*$. But then the line segment $\sigma + iT$, $\sigma^* \leq \sigma \leq 3$, would contain (by Theorem 2) $\gg T$ points at which $\mathrm{Im}\, \zeta(s) = 0$. However, it is well-known how to use Jensen's Theorem to show that the number of such points is $\ll_{\sigma^*} \log T$. (See Titchmarsh [7,§9.4], for example.) Hence all the I_2-lines intersect $\sigma = \sigma^*$.

Now it is clear that an I_1-line which contains a point s_1 with $\sigma_1 > \frac{1}{2}$ must intersect $\sigma = \sigma^*$. For if not it would intersect the I_2-lines, contrary to the Proposition.

Finally, an R-line which contains a point s_1 with $\sigma > \frac{1}{2}$ must intersect $\sigma = \sigma^*$. For if not it would intersect infinitely many I_2-lines. But each such intersection is at a zero of $\zeta(s)$ and any R-line (or I-line) can have at most one zero of $\zeta(s)$ on it, by the Proposition. This proves the Lemma.

It follows from this Lemma that on any R-line or I_1-line which does not intersect the real axis, $\sigma \to -\infty$ as $|s| \to \infty$, so that by the Corollary to Theorem 4, $|\zeta(s)| \to \infty$ as $|s| \to \infty$ on such a curve. For an I_2-line, $|\zeta(s)| \to \infty$ as $|s| \to \infty$ in one direction while $\zeta(s) \to 1$ as $|s| \to \infty$ in the other direction. Hence, by the monotonicity asserted in the Proposition, we have proved

LEMMA 9. Any R-line, I_1-line, or Z-curve which does not intersect the real-axis has precisely one zero of $\zeta(s)$ on it. A G-curve does not have any zeros of $\zeta(s)$ on it.

We now prove Theorem 6. Suppose, first of all, that the Riemann-Hypothesis is false. We will show that (ii) and (iii) of Theorem 6 are also false. Let $s_1 = \sigma_1 + it_1$, $\sigma_1 > \frac{1}{2}$, be a zero of $\zeta(s)$. Then there is an R-line which passes through s_1. This R-line could not intersect $\sigma = \frac{1}{2}$ at a zero of $\zeta(s)$ by Lemma 9. Hence (ii) is false. Similarly there is an I-line, which is either an I_1-line or a Z-curve, which passes through s_1. But this I-line could not intersect $\sigma = \frac{1}{2}$ in a zero of $\zeta(s)$, again by Lemma 9.

To complete the proof of Theorem 6 it suffices to show that the Riemann-Hypothesis implies (ii) and (iii). Assume the Riemann-Hypothesis. As usual we let $S(T) = \frac{1}{\pi} \arg \zeta(\frac{1}{2} + iT)$ where the value of $\arg \zeta(s)$ is determined by continuous variation along line segments from $s = 3$ to $s = 3 + iT$ to $s = \frac{1}{2} + iT$ (and $\arg \zeta(3) = 0$). Since we are assuming the Riemann-Hypothesis it follows from the argument principle that we may determine $S(T)$ by continuous variation along any path from 3 to $\frac{1}{2} + iT$ which does not cross $\sigma = \frac{1}{2}$.

LEMMA 10. Suppose that a G-curve traced leftward (i.e. for decreasing σ) from $\sigma = 5$ first intersects $\sigma = \frac{1}{2}$ at the point $\frac{1}{2} + iT$. Then $S(T) = 0$.

This Lemma is obvious since $\operatorname{Re} \zeta(s) > 0$ for any s on a G-curve and since $\zeta(\frac{1}{2} + iT)$ is real and positive.

The significance of $S(T)$ is apparent in the following

LEMMA 11. Let $N(T)$ be the number of zeros of $\zeta(\frac{1}{2} + it)$ with $0 < t \leq T$ and let $L(T) = 1 - \frac{1}{2\pi} \arg \chi(\frac{1}{2} + it)$ where $\arg \chi(s)$ is determined by continuous variation from $s = \frac{1}{2}$ to $s = \frac{1}{2} + iT$ along the line segment and $\arg \chi(\frac{1}{2}) = 0$. Then

$$N(T) = L(T) + S(T).$$

The proof is well-known (see Titchmarsh [7,§9.3] for example).

We now proceed to deduce (ii) and (iii). Every R-line or I-line which does not intersect the real axis must intersect $\sigma = \frac{1}{2}$. This is true for I_2-lines by Lemma 8, and it is true for R-lines and I_1-lines by Lemma 9 and the Riemann-Hypothesis. Moreover, an R-line or an I_1-line must intersect $\sigma = \frac{1}{2}$ an even number of times (counting multiplicities) while an I_2-line must intersect $\sigma = \frac{1}{2}$ an odd number of times. Hence any R-line has at least one Gram^--point and an I_1-line has at least one Gram^+-point. We will also use the fact that the R-lines may be identified by the unique zero of $\zeta(s)$ which they contain. We prove that (ii) and (iii) hold on each segment $\frac{1}{2} + it$, $T_1 < t < T_2$, of the line $\sigma = \frac{1}{2}$, where T_1 and T_2 are the ordinates of the first intersection of consecutive G-curves with $\sigma = \frac{1}{2}$ (as in Lemma 10). By Lemmas 10 and 11, $\zeta(s)$ has precisely

$$\frac{1}{2\pi} (\arg \chi(\tfrac{1}{2} + iT_1) - \arg \chi(\tfrac{1}{2} + iT_2))$$

zeros in this interval. Clearly, this is the same as the number of Gram^--points in the interval. Thus the Gram^--points in this interval are in one-to-one correspondence with the zeros of $\zeta(s)$, which are in one-to-one correspondence with the R-lines they are on. Since each R-line has at least one Gram^--point, it follows that each has precisely one Gram^--point. This proves (ii). Similarly, in this same interval the Gram^--points outnumber the Gram^+-points by one. Thus, the number of zeros of $\zeta(s)$ exceeds the number of Gram^+-points by one. But there is a unique Z-curve which intersects this interval in a zero of $\zeta(s)$. The rest of the zeros of $\zeta(s)$ are in one-to-one correspondence with the Gram^+-points of the interval and with the I_1-lines which intersect the interval in a zero of $\zeta(s)$.

But each I_1-line has at least one Gram$^+$-point on it. This accounts for all the Gram$^+$-points in the interval, and it follows that an I_1-line or a G-curve has precisely one Gram$^+$-point, while a Z-curve has no Gram$^+$-points. This proves (iii) and the Theorem.

§5. Open Questions

Since the Gram-points can easily be determined to any degree of accuracy and the same is true for the G-curves to the right of $\sigma = 3$, it might be possible to determine which Gram-points are on G-curves. Moreover, the midpoint between successive Gram-points on G-curves might be very close to a zero of $\zeta(s)$ on a Z-curve. Thus it may be possible to predict the location of certain zeros of $\zeta(s)$ with greater accuracy than usual.

Let σ_I denote the infimum of numbers σ^* such that if s is on an I_1-line then $\sigma < \sigma^*$. It would be of interest to give an alternate characterization of σ_I, such as (2) for σ_R, and to determine it numerically.

From Lemmas 1 and 2 we see that $A_1 = 1$ and

$$A_p = -1 + O(p^{-\frac{1}{2}}).$$

We wonder if $|A_N| \leq 1$ for all N.

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Legends for Figures

Figure 1 Real and Imaginary Curves for $\zeta(s)$ with s near 1

(—— R-line, --- I-line, ● Gram^--point, o Gram^+-point,

× zero of $\zeta(s)$)

Figure 2 Looping Effect for R-lines near $\sigma = \frac{1}{2}$

(—— R-line, ● Gram^--point, × zero of $\zeta(s)$)

Figure 3 Possible Winding Effect for R-lines near $\sigma = 1$

(—— R-line)