The matrix completion problem regarding various classes of $P_{0,1}$- matrices

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CHAPTER 1. Introduction

Applications of matrix completion problems occur in situations where some information is known but other information not available, but it is known that the complete matrix must have certain properties. Examples include computer engineering problems including data transmission, coding, decompression, and image enhancement. Matrix completion problems also arise in optimization and in the study of Euclidean distance matrices.

All matrices discussed here are real $n \times n$ matrices, and α is a subset of $N = \{1, 2, \ldots, n\}$. We will denote the matrix $\begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$ by $\operatorname{diag}(d_1, d_2, \ldots, d_n)$. The two entries a_{ij} and a_{ji} of a matrix are *symmetrically placed* entries.

An element a_{ij} of a matrix A is the element in row i and column j of the matrix A. A *principal submatrix* $A(\alpha)$ of a matrix A is a square array lying in the rows and columns of A indexed by α. A *principal minor* is the determinant of a principal submatrix. A *permutation matrix* is a matrix in which each entry is either 0 or 1 and there is exactly one 1 in each row and each column of the matrix. A *permutation similarity* of A is a product PAP^{-1} where P is a permutation matrix. A *diagonal similarity* of A is a product DAD^{-1} where D is a diagonal matrix.

A *partial matrix* is a rectangular array in which some entries are specified and some are left unspecified. We usually label the specified entries as a_{ij} and the unspecified entries as x_{ij} where the entry is in the ith row and jth column of the partial matrix. A *fully specified principal submatrix* is a principal submatrix of a partial matrix which contains only specified entries. A *completion* of a partial matrix is a specific choice of values for the unspecified entries. A *pattern* for $n \times n$ matrices is a list of positions of an $n \times n$ matrix, that is, a subset of $N \times N$ where $N = \{1, \ldots, n\}$. A pattern is assumed to contain all diagonal positions. A partial matrix specifies a pattern if its specified entries lie exactly in those positions listed in the pattern. Many of the definitions in this chapter are taken from [2], [12], and [5].

A *P-matrix* is a matrix in which every principal minor of the matrix is positive. A *P_0-matrix* is a matrix in which every principal minor of the matrix is nonnegative. A *$P_{0,1}$-matrix* is a matrix in which every principal minor of the matrix is nonnegative, and the diagonal of the matrix is positive. A *weakly sign symmetric matrix* is a matrix in which $a_{ij}a_{ji} \geq 0$, $1 \leq i, j \leq n$. A *sign symmetric matrix* is a matrix in which either $a_{ij} = a_{ji} = 0$ or $a_{ij}a_{ji} > 0$, $1 \leq i, j \leq n$. We will consider the classes of sign symmetric $P_{0,1}$-matrices, weakly sign symmetric $P_{0,1}$-matrices, and $P_{0,1}$-matrices.

If a partial matrix that specifies a pattern contains a fully specified principal submatrix, the determinant of that submatrix is called an *original minor.* A *partial $P_{0,1}$-matrix* is a partial matrix where every original minor is nonnegative and the diagonal entries are all strictly positive. A *partial sign symmetric $P_{0,1}$-matrix* is a partial $P_{0,1}$-matrix with the additional condition that the symmetrically placed specified entries a_{ij} and a_{ji} fulfill the property that either $a_{ij} = a_{ji} = 0$ or $a_{ij}a_{ji} > 0$. Likewise, a *partial weakly sign symmetric $P_{0,1}$-matrix* is a partial $P_{0,1}$ matrix in which $a_{ij}a_{ji} \geq 0$ when both a_{ij} and a_{ji} are specified.

Let Π be in the set of classes $\{P_{0,1}$, weakly sign symmetric $P_{0,1}$, sign symmetric $P_{0,1}\}$ of matrices. We consider the question: Given a pattern, does every partial Π-matrix which specifies the pattern have completion to a Π-matrix? We say a *partial Π-matrix A has Π-completion* if we can choose the unspecified entries so that the new matrix $\widehat{A}$ is a Π-matrix. We say a *pattern has Π-completion* if every partial Π-matrix specifying that pattern can be completed to a Π-matrix. Since any permutation similarity of a Π-matrix is still a Π-matrix, if we answer this question for one matrix A, we answer the question for any matrix B where $B = PAP^{-1}$ for some permutation matrix P. Since multiplying a Π-matrix by a positive diagonal matrix produces a Π-matrix, we can multiply a Π-matrix A on the left by the diagonal matrix $D = \mathrm{diag}(1/a_{11}, 1/a_{22}, \ldots, 1/a_{nn})$. This changes the matrix A into a new matrix A'

in which all its diagonal entries are equal to 1. If the matrix A' has Π-completion, then so does the matrix A, because we can multiply the completed matrix $\widehat{A}'$ on the left by the matrix D^{-1} to obtain a completion of the matrix A. Since all the classes we consider have positive diagonal elements, we assume that the diagonal elements of every matrix are equal to 1.

A graph $G(V,E)$ of order $n > 0$ is a set V, with $|V| = n$, and a set E composed of a subset of the set $\{\{i,j\}|i,j \in V \text{ and } i \neq j\}$. The elements of E are called edges. A digraph $G(V,E)$ of order $n > 0$ is a set V with $|V| = n$, and a set E composed of a subset of the set $\{(i,j)|i,j \in V \text{ and } i \neq j\}$. The elements of E in a digraph are called arcs or directed edges.

An $n \times n$ partial matrix A (and the pattern describing the specified entries of A) is associated with a digraph $G(V,E)$ where the vertices V of G are $\{1,2,3,\ldots,n\}$, and an arc $(i,j) \in E$ if and only if a_{ij} is specified in A. All patterns we consider have specified diagonal entries. A digraph G is said to have Π-completion if every partial Π-matrix associated with G has completion to a Π-matrix. Notice that a permutation similarity of the matrix is equivalent to a renumbering of the vertices of the digraph. We will often describe the pattern by its digraph. We consider the digraphs of order up to 4 and graphs up to order 6 as listed in [9], and we number the digraphs (graphs) where q is the number of arcs (edges) and n is the diagram number.

A *path of length* k in a digraph is a digraph $P(V,E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$, $E = \{(x_0x_1), (x_1x_2), \ldots, (x_{k-1}x_k)\}$, where the x_i are all distinct. A *path of length* k in a graph is a graph $P(V,E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$, $E = \{\{x_0x_1\}, \{x_1x_2\}, \ldots, \{x_{k-1}x_k\}\}$, where the x_i are all distinct. A *cycle* can be created in a graph from a path $P(V,E) = \{x_0, x_1, \ldots, x_k\}$, with $k \geq 2$, by including the edge $\{x_k, x_0\}$ in the edge set E. A *cycle* can be created in a digraph from a path $P(V,E) = \{x_0, x_1, \ldots, x_k\}$, with $k \geq 1$, by including the (arc (x_k, x_0)) in the arc set E.

A *symmetric* k*-cycle* is a digraph on k vertices with arc set $E = \{(v_i, v_{i+1}), (v_{i+1}, v_i), (v_k, v_1), (v_1, v_k)|i = 1, 2, \ldots, k-1\}$. A *semipath* is a digraph $P(V,E)$ of the form $V = \{x_0, x_1, \ldots, x_k\}$, $E = \{e_{x_1}, e_{x_2}, \ldots, e_{x_k}\}$, where e_{x_i} is either the arc (x_{i-1}, x_i) or the arc (x_i, x_{i-1}), and the x_i are all distinct. A *semicycle* can be created from a semipath $P(V,E) =$

$\{x_0, x_1, \ldots, x_k\}$ by including the arc e in the arc set E where e is either (x_k, x_0) or (x_0, x_k). The *distance* between two vertices u and v in a digraph is the number of arcs in the shortest semipath between them and is denoted by $\text{dist}(u, v)$. If there is no semipath between the vertices u and v, then $\text{dist}(u, v)$ is defined to be infinite.

A *chord* of a cycle is an edge not in the cycle whose endpoints are on the cycle. A digraph $G(V, E)$ is *symmetric* if $(i, j) \in E$ if and only if $(j, i) \in E$. For a graph G, we define a digraph $D(G)$ with the same vertex set as G as follows: the arcs (i, j) and (j, i) are in the digraph if and only if the edge $\{i, j\}$ is in the graph. Likewise, for a symmetric digraph D, we define a graph $G(D)$ with the same vertex set as D where $\{i, j\}$ is an edge in G if and only if (i, j) and (j, i) are arcs in D. An *asymmetric digraph* $G(V, E)$ is a digraph where for each vertex i and j, no more than one of the arcs (i, j) and (j, i) is in E. An *asymmetric cycle* is an asymmetric digraph consisting only of one cycle. An *asymmetric pattern* is a pattern associated with an asymmetric digraph.

A digraph is *strongly connected* if for all $i, j \in V$, there is a path from i to j. A digraph is *connected* if for all $i, j \in V$, there is a semipath from i to j. A *cut-vertex* of a connected digraph is a vertex whose deletion from G disconnects the digraph. A connected digraph is *nonseparable* if it has no cut-vertices. A *block* is a maximal nonseparable subdigraph. A *complete digraph* or *clique* is a digraph which contains all possible arcs. A *block-clique digraph* is a digraph whose blocks are all cliques. A connected component of a digraph (graph) is a maximal connected subdigraph (subgraph). A strongly connected component of a digraph is a maximal strongly connected subdigraph.

A *tree* is a connected digraph that contains no semicycle. A forest is a digraph whose connected components are trees. A vertex u is a *neighbor* of another vertex v in the digraph $G(V, E)$ if either $(u, v) \in E$ or $(v, u) \in E$.

We use the following theorem in order to simplify the proofs of the completion of a graph (digraph). However, the method used in the proof of the theorem can also be used in other contexts to change the values of certain entries of a matrix to whatever is desired through the use of a diagonal similarity.

Theorem 1.1 *Let $S \subset \{a_{ij} | 1 \leq i, j \leq n$ and $i \neq j\}$, $|S| \leq n-1$ be a set of nonzero entries from an $n \times n$ matrix A. Let $T(V, E)$ be the digraph on n vertices where the arc $(i, j) \in E$ if and only if $a_{ij} \in S$. If the digraph of S contains no semicycle, then the elements of S can be made equal to 1 through the use of a diagonal similarity.*

Proof: If T is a forest, we can partition the matrix A into blocks A_{ij}, where all of the entries in S lie in the disjoint principal submatrices $A_{11}, A_{22}, \ldots, A_{tt}$, corresponding to the connected components of T. Then the diagonal entries which affect one of these principal submatrices do not affect any of the other principal submatrices A_{qq}, for $q \in \{1, 2, \ldots, t\}$. We can work with each connected component separately, so that if $D_p A_{pp} D_p^{-1}$ has 1's in the appropriate positions for each $p \in \{1, 2, \ldots, t\}$, then the matrix

$$B = \begin{bmatrix} D_1 & 0 & 0 & \cdots & 0 \\ 0 & D_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & D_t \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{tt} \end{bmatrix} \begin{bmatrix} D_1 & 0 & 0 & \cdots & 0 \\ 0 & D_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & & \vdots \\ 0 & 0 & 0 & \cdots & D_t \end{bmatrix}^{-1}$$

has 1's in the appropriate positions.

Therefore, we may assume T is a tree. Then there exists exactly one semipath from any vertex to any other vertex of T. Let $D = \text{diag}(d_1, d_2, \ldots, d_n)$. Set $d_1 = 1$. For any vertex v at distance k from 1, let $P = (v_0 = 1, v_1, \ldots, v_{k-1}, v_k = v)$ be the semipath from 1 to v with vertex set $V = \{v_0, v_2, \ldots, v_k\}$ and arc set $E = \{e_{v_1}, e_{v_2}, \ldots, e_{v_k}\}$ where $e_{v_l} = (v_{l-1}, v_l)$ or (v_l, v_{l-1}). Set $d_v = g_{v_1} g_{v_2} \ldots g_{v_k}$ where

$$g_{v_l} = \begin{cases} a_{v_{l-1}v_l} & \text{if } e_{v_l} = (v_{l-1}, v_l), \\ \dfrac{1}{a_{v_l v_{l-1}}} & \text{if } e_{v_l} = (v_l, v_{l-1}), \end{cases} \qquad \text{for } l = 1, 2, \ldots, k. \tag{1.1}$$

We know there is a vertex w on the semipath between vertex v and vertex 1 such that $\text{dist}(1, w) = k - 1$. Then $d_w = g_{v_1} g_{v_2} \ldots g_{v_{k-1}}$ and $d_v = g_{v_1} g_{v_2} \ldots g_{v_{k-1}} g_{v_k}$ where

$$g_{v_k} = g_v = \begin{cases} a_{wv} & \text{if } e_v = (w, v), \\ \dfrac{1}{a_{vw}} & \text{if } e_v = (v, w). \end{cases}$$

Then for $B = DAD^{-1}$, $\begin{cases} \text{if } e_v = (w,v), & b_{wv} = (g_{v_1} g_{v_2} \dots g_{v_{k-1}})(a_{wv}) \left(\dfrac{1}{g_{v_1} g_{v_2} \dots g_{v_{k-1}} a_{wv}} \right) = 1 \\ \text{if } e_v = (v,w), & b_{vw} = \left(\dfrac{g_{v_1} g_{v_2} \dots g_{v_{k-1}}}{a_{vw}} \right) (a_{vw}) \left(\dfrac{1}{g_{v_1} g_{v_2} \dots g_{v_{k-1}}} \right) = 1, \end{cases}$

so the entry corresponding to e_v is made equal to 1. Considering the induced subdigraph $H(V', E')$ of T with $V' = \{u | \text{dist}(1,u) \leq k\}$, we see that v is a leaf of H. Since the entry d_v changes only the entries of A corresponding to arcs incident with v, the choice of d_v does not change any other entry corresponding to an arc of H. The same is true for any other vertex u at distance k from 1. □

Example 1.2 *Here is an example of this procedure applied to a* 5×5 *matrix.*

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$. Assume the entries a_{12}, a_{31}, a_{24}, and a_{53} are nonzero, and we want to make the entries $a_{12} = a_{31} = a_{24} = a_{53} = 1$ through the use of a diagonal similarity. Then the distance tree of the arcs corresponding to these four entries with root vertex 1 is shown in Figure 1.1.

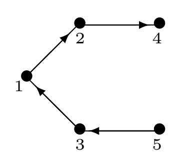

Figure 1.1 Tree T associated with the entries of A chosen in Example 1.2

In order to make these 4 entries equal to one, we set $d_1 = 1$, $d_2 = a_{12}$, $d_3 = 1/a_{31}$, $d_4 = a_{12}a_{24}$, and $d_5 = \dfrac{1}{a_{31}a_{53}}$.

Then $B = DAD^{-1} =$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & a_{12} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{a_{31}} & 0 & 0 \\ 0 & 0 & 0 & a_{12}a_{24} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{a_{31}a_{53}} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{a_{12}} & 0 & 0 & 0 \\ 0 & 0 & a_{31} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{a_{12}a_{24}} & 0 \\ 0 & 0 & 0 & 0 & a_{31}a_{53} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11} & 1 & a_{13}a_{31} & \frac{a_{14}}{a_{12}a_{24}} & a_{15}a_{31}a_{53} \\ a_{12}a_{21} & a_{22} & a_{12}a_{23}a_{31} & 1 & a_{12}a_{25}a_{31}a_{53} \\ 1 & \frac{a_{32}}{a_{12}a_{31}} & a_{33} & \frac{a_{34}}{a_{12}a_{24}a_{31}} & a_{35}a_{53} \\ a_{12}a_{24}a_{41} & a_{24}a_{42} & a_{12}a_{24}a_{31}a_{43} & a_{44} & a_{12}a_{24}a_{31}a_{45}a_{53} \\ \frac{a_{51}}{a_{31}a_{53}} & \frac{a_{52}}{a_{12}a_{31}a_{53}} & 1 & \frac{a_{54}}{a_{12}a_{24}a_{31}a_{53}} & a_{55} \end{bmatrix}$$

This has 1's in the appropriate 4 positions. □

It is not necessarily the case that $n-1$ nonzero entries can be made equal to one when the digraph of these entries contains a semicycle. For example, consider a 4×4 matrix A, and attempt to make entries a_{21}, a_{23}, and a_{13} equal to one. Then $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}$ and the digraph of the $n-1$ entries is shown in Figure 1.2.

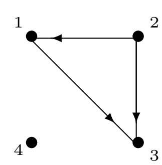

Figure 1.2 Digraph corresponding to the entries of A in S

We would like to choose d_1, d_2, d_3, and d_4 so that if $D = \mathrm{diag}(d_1, d_2, d_3, d_4)$ and $DAD^{-1} = B = [b_{ij}]$, then b_{21}, b_{23}, and b_{13} are equal to one. This occurs if and only if Equations (1.2), (1.3), and (1.4) hold.

$$d_2 a_{21} d_1^{-1} = b_{21} = 1 \tag{1.2}$$

$$d_2 a_{23} d_3^{-1} = b_{23} = 1 \tag{1.3}$$

$$d_1 a_{13} d_3^{-1} = b_{13} = 1 \tag{1.4}$$

Equation 1.2 holds if and only if

$$d_2 = \frac{d_1}{a_{21}} \tag{1.5}$$

Substituting Equation 1.5 into Equation 1.3, we get

$$\frac{d_1 a_{23}}{a_{21} d_3} = 1 \tag{1.6}$$

Solving Equation 1.6 for d_1, we get

$$d_1 = \frac{a_{21} d_3}{a_{23}} \tag{1.7}$$

Substituting Equation 1.7 into Equation 1.4, we get

$$\frac{a_{21} d_3}{a_{23}} a_{13} \frac{1}{d_3} = 1 \tag{1.8}$$

But this is true if and only if

$$\frac{a_{21} a_{13}}{a_{23}} = 1 \tag{1.9}$$

Therefore, we can only find such a set of elements, d_1, d_2, d_3, and d_4 if $\frac{a_{21}a_{13}}{a_{23}} = 1$. Since this is not necessarily the case, we cannot necessarily make these three elements equal to 1 through the use of a diagonal similarity.

Since the entry a_{ij} becomes $d_i a_{ij} \frac{1}{d_j}$ through the diagonal similarity DAD^{-1}, the cycle product $a_{ij}a_{ji}$ becomes $d_i a_{ij} \frac{1}{d_j} d_j a_{ji} \frac{1}{d_i} = a_{ij}a_{ji}$, which is the same cycle product as before. Similarly, a diagonal similarity does not change the cycle products of a matrix A of any size. Therefore, if the product of two symmetrically placed entries of a matrix are equal 1, making the entry $a_{ij} = 1$ through a diagonal similarity automatically makes the entry $a_{ji} = 1$.

We will use the term Π to refer to any one of the classes $P_{0,1}$, sign symmetric $P_{0,1}$, and weakly sign symmetric $P_{0,1}$. The following theorem and lemma will be used extensively in the subsequent chapters.

Theorem 1.3 *[10] If the pattern Q has Π-completion, then any principal subpattern R has Π-completion. Equivalently, if a digraph has Π-completion, then so does any induced subdigraph.*

Lemma 1.4 *Let G be a digraph that has Π-completion. Let H be a digraph obtained from G by deleting one arc (u, v) such that u and v are not both contained in any clique of order 3. Then H has Π-completion.*

Proof: Let G and H satisfy the hypothesis, and let A be a partial Π-matrix specifying H. We choose a value for the unspecified (u, v)-entry of A to obtain a partial Π-matrix B specifying G as follows. If $\langle u, v\rangle$ is a clique in G, choose a value c for the u, v-entry that completes $A(\{u, v\})$ to a Π-matrix. Such a c is guaranteed to exist because the order 2 digraph with one arc has Π-completion by [10]. Otherwise, set the (u, v)-entry equal to zero. Then since G has Π-completion, we can complete B to a Π-matrix C, which also completes A. $\square$

The Appendix consists of the computations that were used to check the results in a proposition, theorem and a number of lemmas in the chapters that follow. These computations were done in the software package Mathematica, and will aid the reader in working through the proofs of the following: the classification of the digraph $q = 7$, $n = 2$ regarding weakly sign symmetric $P_{0,1}$-completion in Proposition 2.5; the classification of the digraphs $q = 5$, $n = 7$ in Lemma 4.7; $q = 5$, $n = 8$ in Lemma 4.8; $q = 5$, $n = 9$ in Lemma 4.9; $q = 6$, $n = 4$ in Lemma 4.10; $q = 6$, $n = 5$ in Lemma 4.11; $q = 6$, $n = 6$ in Lemma 4.12; and $q = 6$, $n = 7$ in Lemma 4.13; regarding sign symmetric $P_{0,1}$-completion, and the classification of the graph $q = 5$ in Theorem 5.10 (the double triangle) regarding $P_{0,1}$-completion.

Matrix completion problems have been studied for many classes of matrices, but the most important class is the class of positive definite matrices. A positive definite matrix is a symmetric P-matrix. The positive definite matrix completion problem was first studied by Burg in his PhD thesis [1], and was then applied to geophysical problems. The question for positive definite tridiagonal matrices was considered by Dym and Gohberg [6]. Since positive definite matrices are symmetric, only graphs, not digraphs, need to be considered. The entire question was answered by Grone, et al in 1984: A graph has positive definite completion if and only if it is chordal [8].

This led to interest in completion problems for additional classes of matrices which generalize the positive definite matrix class. The matrix completion problem regarding the classes of P- and P_0-matrices were first studied by Johnson and Kroshel [13]. They showed that any graph and any order 3 digraph has P-completion, and that there exists a digraph of order 4 which does not have P-completion. Specifically, they showed that the digraph which is complete except for one missing arc does not have P-completion. Because any graph has P-completion, the double triangle ($q = 10$, $n = 1$) and the symmetric 4-cycle ($q = 8$, $n = 2$) have P-completion. Johnson and Kroshel also showed that the double triangle does not have P_0 completion. The double triangle is of importance because for many classes Π, it does not have Π-completion. Choi et al classified the order 4 digraphs as to P_0-completion and showed that every asymmetric digraph has P_0-completion and the symmetric n-cycle has P_0-completion for $n \geq 5$ [2]. DeAlba and Hogben [3] made further progress in classifying the digraphs as to P-completion. Not all order 4 digraphs have been classified as to P-completion.

Fallet, et al considered the completion problem of graphs regarding P-, P_0-, (weakly) sign-symmetric P-, and (weakly) sign symmetric $P_{0,1}$-matrices [7]; Π is used to represent any one of the classes they studied. In this paper, it is proved that any block-clique graph has Π-completion. It is also shown that the double triangle ($q = 10$, $n = 1$) does not have X-completion for X any of the classes weakly sign symmetric P_0-, weakly sign symmetric $P_{0,1}$-, sign symmetric P_0-, sign symmetric $P_{0,1}$-matrices. The symmetric 4-cycle does not have sign symmetric P- or weakly sign symmetric P-completion.

Hogben considered a number of classes of matrices in her paper on graph theoretic methods [10]. This paper surveys what had been done so far on matrix completions, and also provides further results. In the paper, Hogben shows that for Π any of the classes P_0-, $P_{0,1}$-, P-, weakly sign symmetric P_0-, weakly sign symmetric $P_{0,1}$-, weakly sign symmetric P-, and sign symmetric P-matrices, any digraph for which every strongly connected induced subdigraph has Π-completion has completion to a Π-matrix. She also shows that a number of specific digraphs do not have Π-completion for certain Π.

DeAlba et al proved the symmetric 6-cycle has (weakly) sign symmetric P-completion, as does the symmetric n-cycle for $n > 6$ [4]. They also proved that the asymmetric n-cycle has (weakly) sign symmetric P-completion for all $n \geq 4$ and classified all order 4 digraphs as to completion for these classes. Approximately 30 papers on matrix completion problems have been published primarily in the last 10 years.

CHAPTER 2. Classification of digraphs of order less than or equal to 4 regarding weakly sign symmetric $P_{0,1}$-completion

Theorem 2.1 *[11] Any pattern that has weakly sign symmetric P_0-completion also has weakly sign symmetric $P_{0,1}$-completion.*

Theorem 2.2 *[11] Any pattern that has weakly sign symmetric $P_{0,1}$-completion also has weakly sign symmetric P-completion.*

Corollary 2.3 *Any pattern that does not have weakly sign symmetric P-completion does not have weakly sign symmetric $P_{0,1}$-completion.*

Lemma 2.4 *All patterns for 2×2 matrices have weakly sign symmetric $P_{0,1}$-completion. A pattern for 3×3 matrices has weakly sign symmetric $P_{0,1}$-completion if and only if its digraph does not contain a 3-cycle or is complete.*

Proof: Any partial weakly sign symmetric P_0-matrix specifying any one of the order two digraphs or any of the order three digraphs except $q = 3$, $n = 2$; $q = 4$, $n = 2$; $q = 5$ may be completed to a weakly sign symmetric P_0-matrix by Lemma 4.1 of [4]. Therefore, any partial weakly sign symmetric $P_{0,1}$-matrix specifying any one of the order two digraphs or any of the order three digraphs except $q = 3$, $n = 2$; $q = 4$, $n = 2$; $q = 5$ may be completed to a weakly sign symmetric $P_{0,1}$-matrix by Theorem 2.1. These are exactly those digraphs which do not contain a 3-cycle or are complete.

A pattern whose digraph is one of $q = 3$, $n = 2$; $q = 4$, $n = 2$; $q = 5$ does not have weakly sign symmetric P-completion by Lemma 4.1 of [4]. Therefore, it also does not have weakly sign symmetric $P_{0,1}$-completion by Corollary 2.3. □

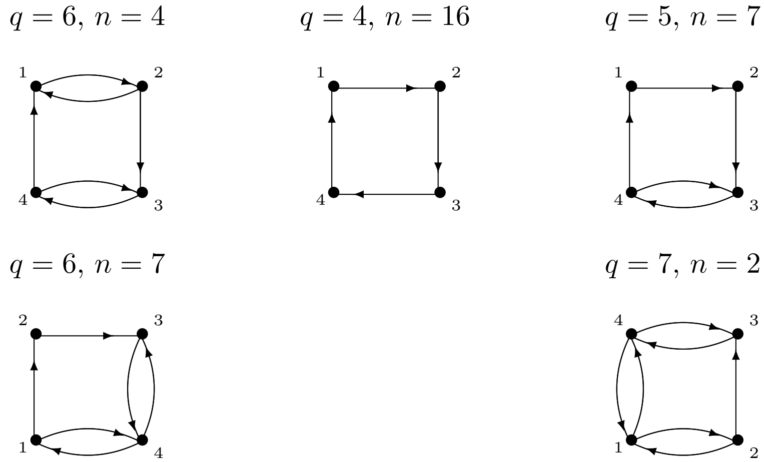

Figure 2.1 The digraphs $q = 6$, $n =$ 4-8 and $q = 7$, $n = 2$

Proposition 2.5 *A 4×4 matrix satisfying the pattern with digraph $q = 7$, $n = 2$; $q = 4$, $n = 16$; $q = 5$, $n = 7$; $q = 6$, $n = 4, 7$ has weakly sign symmetric $P_{0,1}$-completion.*

Proof: We may complete these matrices in the same manner as in Lemma 4.2 in [4].

Consider the digraph $q = 7$, $n = 2$ shown in Figure 2.1. Let $A = \begin{bmatrix} 1 & a_{12} & x_{13} & a_{14} \\ a_{21} & 1 & a_{23} & x_{24} \\ x_{31} & x_{32} & 1 & a_{34} \\ a_{41} & x_{42} & a_{43} & 1 \end{bmatrix}$ be a partial weakly sign symmetric $P_{0,1}$-matrix specifying the digraph $q = 7$, $n = 2$. Clearly, the original 2×2 principal minors, $1 - a_{12}a_{21}$, $1 - a_{34}a_{43}$, and $1 - a_{41}a_{14}$ are nonnegative. We consider two cases: (1) $a_{12}a_{23}a_{34}a_{41} \leq 0$ and (2) $a_{12}a_{23}a_{34}a_{41} > 0$.

Case 1: $a_{12}a_{23}a_{34}a_{41} \leq 0$. Set $x_{13} = a_{14}a_{43}$, $x_{24} = a_{21}a_{14}$, $x_{31} = 0$, $x_{32} = 0$, $x_{42} = 0$. The determinant of A and the 3×3 principal minors of A are shown in Table 2.1. The determinant of A can also be written as $a_{12}a_{21}a_{14}a_{41}a_{43}a_{34} - a_{12}a_{23}a_{34}a_{41} + (1 - a_{12}a_{21})(1 - a_{14}a_{41})(1 - a_{34}a_{43})$, the $A(1, 2, 4)$ minor of A can also be written as $(1 - a_{12}a_{21})(1 - a_{14}a_{41})$, and the $A(1, 3, 4)$ minor of A also be written as $(1 - a_{14}a_{41})(1 - a_{34}a_{43})$. Therefore, the determinant of A can be written as a product of original minors of A plus some nonnegative terms, and all of the 3×3 principal minors of A are either an original minor, or can be written as a product of original minors. Therefore, the determinant of A and all the 3×3 principal minors of A are nonnegative. Each of the 2×2 principal minors of A is either equal to 1, or is an original minor of A.

Case 2: $a_{12}a_{23}a_{34}a_{41} > 0$. Set $x_{13} = a_{14}a_{43}$, $x_{24} = a_{21}a_{14} + a_{23}a_{34}$, $x_{31} = 0$, $x_{32} = 0$, and $x_{42} = 0$. The determinant of A and the 3×3 principal minors of A are shown in Table 2.1. The determinant of A can also be written as $a_{12}a_{21}a_{14}a_{41}a_{43}a_{34} + (1 - a_{12}a_{21})(1 - a_{14}a_{41})(1 - a_{34}a_{43})$, the $A(1,2,4)$ minor can also be written as $a_{12}a_{23}a_{34}a_{41} + (1 - a_{12}a_{21})(1 - a_{14}a_{41})$, and the $A(1,3,4)$ minors can also be written as $(1 - a_{14}a_{41})(1 - a_{34}a_{43})$, so all of the 3×3 principal minors of A and the determinant of A can be written as a product of the original minors plus (possibly) some nonnegative terms.

Therefore, the digraph $q = 7$, $n = 2$ has weakly sign symmetric $P_{0,1}$-matrix completion.

Table 2.1 Det$A$ and 3×3 principal minors of A in Case 1

$A(\alpha)$	Det$A(\alpha)$ for Case 1	Det$A(\alpha)$ for Case 2
$A(1,2,3,4)$	$1 - a_{12}a_{21} - a_{14}a_{41} + a_{12}a_{14}a_{21}a_{41} - a_{12}a_{23}a_{34}a_{41} - a_{34}a_{43} + a_{12}a_{21}a_{34}a_{43} + a_{14}a_{34}a_{41}a_{43}$	$1 - a_{12}a_{21} - a_{14}a_{41} + a_{12}a_{14}a_{21}a_{41} - a_{34}a_{43} + a_{12}a_{21}a_{34}a_{43} + a_{14}a_{34}a_{41}a_{43}$
$A(1,2,3)$	$1 - a_{12}a_{21}$	$1 - a_{12}a_{21}$
$A(1,2,4)$	$1 - a_{12}a_{21} - a_{14}a_{41} + a_{12}a_{14}a_{21}a_{41}$	$1 - a_{12}a_{21} - a_{14}a_{41} + a_{12}a_{14}a_{21}a_{41} + a_{12}a_{23}a_{34}a_{41}$
$A(1,3,4)$	$1 - a_{14}a_{41} - a_{34}a_{43} + a_{14}a_{34}a_{41}a_{43}$	$1 - a_{14}a_{41} - a_{34}a_{43} + a_{14}a_{34}a_{41}a_{43}$
$A(2,3,4)$	$1 - a_{34}a_{43}$	$1 - a_{34}a_{43}$

Consider the digraphs $q = 4$, $n = 16$; $q = 6$, $n =$ 4,7; and $q = 5$, $n = 7$ shown in Figure 2.1. Let A_1 be a partial weakly sign symmetric matrix specifying the digraph $q = 4$, $n = 16$. Set $x_{21} = 0$, $x_{43} = 0$, and $x_{14} = 0$. Call this new partial matrix B_1. Let A_2 be a partial weakly sign symmetric matrix specifying the digraph $q = 5$, $n = 7$. Set $x_{21} = 0$, and $x_{14} = 0$. Call this new partial matrix B_2. Let A_3 be a partial weakly sign symmetric matrix specifying the digraph $q = 6$, $n = 4$. Set $x_{14} = 0$. Call this new partial matrix B_3. Let A_4 be a partial weakly sign symmetric matrix specifying the digraph $q = 6$, $n = 7$. Set $x_{21} = 0$. Call this new partial matrix B_4.

Each of the partial matrices B_1, B_2, B_3, and B_4 are weakly sign symmetric $P_{0,1}$-matrices specifying the digraph $q = 7$, $n = 2$. Then each of the matrices B_1, B_2, B_3, and B_4 can be completed to a weakly sign symmetric $P_{0,1}$-matrix, $\widehat{B}_1$, $\widehat{B}_2$, $\widehat{B}_3$, and $\widehat{B}_4$. Therefore, the digraphs $q = 4$, $n = 16$; $q = 6$, $n = 4$,7 and $q = 5$, $n = 7$ have weakly sign symmetric $P_{0,1}$-matrix completion. □

Theorem 2.6 *(Classification of Patterns of* 4×4 *matrices regarding weakly sign symmetric* $P_{0,1}$*-matrix completion). Let* Q *be a pattern for* 4×4 *matrices that includes all diagonal positions. The pattern* Q *has weakly sign symmetric* $P_{0,1}$*-completion if and only if its digraph is one of the following (numbered as in [9],* q *is the number of arcs,* n *is the diagram number).*

q =0

q =1;

q =2;   n =1-5;

q =3;   n =1-11, 13;

q =4;   n =1-12, 14-19, 21-23, 25-27;

q =5;   n =1-5, 7-10, 14-17, 21-24, 26-29, 31, 33-34, 36-37;

q =6;   n =1-8, 13, 15, 17, 19, 23, 26-27, 32, 35, 38-40, 43, 46;

q =7;   n =2,4-5, 9, 14, 24, 29, 34, 36;

q =8;   n =1, 10, 12, 18;

q =9;   n =8, 11;

q =12.

Proof:

Part 1. Digraphs that have weakly sign symmetric $P_{0,1}$-completion.

The following digraphs have weakly sign symmetric P_0-completion by Theorem 4.4 of [4], and therefore also have weakly sign symmetric $P_{0,1}$-completion by Theorem 2.1:

$q = 0$

$q = 1$;

$q = 2$; $n = 1$-5;

$q = 3$; $n = 1$-11, 13;

$q = 4$; $n = 1$-12, 14-15, 17-19, 21-23, 25-27;

$q = 5$; $n = 1$-5, 8-10, 14-17, 21-24, 26-29, 31, 33-34, 36-37;

$q = 6$; $n = 1$-3, 5-6, 8, 13, 15, 17, 19, 23, 26-27, 32, 35, 38-40, 43, 46;

$q = 7$; $n = 4$-5, 9, 14, 24, 29, 34, 36;

$q = 8$; $n = 1$, 10, 12, 18;

$q = 9$; $n = 8$, 11;

$q = 12$.

Equivalently, the previous list of digraphs are those digraphs listed in the theorem, excluding the following: $q = 4$, $n = 16$; $q = 5$, $n = 7$; $q = 6$, $n = 4, 7$; $q = 7$, $n = 2$. These excluded digraphs have weakly sign symmetric $P_{0,1}$-completion by Proposition 2.5.

Part 2. Digraphs that do not have weakly sign symmetric $P_{0,1}$-completion.

The remaining digraphs do not have weakly sign symmetric P-completion. Therefore, by Corollary 2.3, they also do not have weakly sign symmetric $P_{0,1}$-completion.

This finishes the classification of the 4×4 patterns regarding completion of the class of weakly sign symmetric $P_{0,1}$-matrices.$\square$

CHAPTER 3. Classification of graphs of orders 5 and 6 and regarding weakly sign symmetric $P_{0,1}$-completion

Graphs of all orders have been classified regarding sign symmetric $P_{0,1}$-completion ([7], Theorem 4.1). A graph has sign symmetric $P_{0,1}$-completion if and only if it is block clique. This theorem states the same is true regarding weakly sign symmetric $P_{0,1}$-completion, but this is false, as shown in ([4], Example 3.1). Any graph of order ≤ 4 has been classified already regarding weakly sign symmetric P_{01}-completion in Chapter 2. In this chapter, we classify graphs of order 5 and 6 as to weakly sign symmetric $P_{0,1}$-completion.

Theorem 3.1 *([7], Theorem 2.4) Any block clique graph has weakly sign symmetric $P_{0,1}$-completion.*

Theorem 3.2 *([4], Theorem 3.10) A pattern whose graph is an n-cycle has weakly sign symmetric P_0-completion if and only if $n \neq 4$ and $n \neq 5$.*

Theorem 3.3 *([4], Theorem 3.9) A pattern whose graph is an n-cycle has weakly sign symmetric P-completion if and only if $n \neq 4$ and $n \neq 5$.*

Lemma 3.4 *(Classification of Patterns of 5×5 matrices regarding weakly sign symmetric $P_{0,1}$-completion). Let Q be a pattern for 5×5 matrices that includes all diagonal positions. The pattern Q has weakly sign symmetric P-completion if and only if each component of its graph is block clique.*

Proof:

Part 1: Graphs which have weakly sign symmetric $P_{0,1}$-completion. Every component of each of the following graphs is block-clique, so the graph has weakly sign symmetric $P_{0,1}$-completion by Theorem 3.1.

q =0

q =1;

q =2; n =1-2;

q =3; n =1-4;

q =4; n =2-6;

q =5; n =2,4-5;

q =6; n =1,3;

q =7; n =3;

q =10.

Part 2: Graphs which do not have (weakly) sign symmetric $P_{0,1}$-completion.

The graphs $q = 4$, $n = 1$; $q = 5$, $n = 1$, 3; $q = 6$, $n = 2$, 4-6; $q = 7$, $n =$ 1-2, 4; $q = 8$, $n =$ 1-2; and $q = 9$ do not have weakly sign symmetric $P_{0,1}$-completion because they contain at least one of the order 4 digraphs $q = 8$, $n = 2$ or $q = 10$, $n = 1$ as an induced subdigraph. The graph $q = 5$, $n = 6$ does not have weakly sign symmetric P-completion by Theorem 3.3. Therefore, it does not have weakly sign symmetric $P_{0,1}$-completion by Corollary 2.3.

Lemma 3.5 *(Classification of Patterns of 6×6 matrices regarding weakly sign symmetric $P_{0,1}$-completion). Let Q be a pattern for 6×6 matrices that includes all diagonal positions. The pattern Q has weakly sign symmetric P-completion if and only if each component of its graph is block-clique or the 6-cycle.*

Proof:

Part 1: Graphs which have weakly sign symmetric $P_{0,1}$-completion.

The following graphs have weakly sign symmetric $P_{0,1}$-completion because each component is block-clique.

q =0

q =1;

q =2; n =1-2;

q =3; n =1-5;

q =4; n =1, 3-9;

q =5; n =1, 3, 6-13, 15;

q =6; n =1, 3, 10, 13-15, 17-18, 20-21;

q =7; n =3, 10, 15, 19, 23;

q =8; n =4, 17, 22;

q =9; n =6;

q =10 n =13;

q =11 n =4;

q =15.

The graph $q = 6$, $n = 7$ (the 6-cycle) has weakly sign symmetric P_0-completion by Theorem 3.2. Therefore, the graph $q = 6$, $n = 7$ has weakly sign symmetric $P_{0,1}$-completion by Theorem 2.1.

Part 2: Graphs which do not have weakly sign symmetric $P_{0,1}$-completion.

The graphs $q = 4$, $n = 2$; $q = 5$, $n = 2$, 4-5, 14, $q = 6$, $n = 2$, 4-6, 8-9, 11-12, 16, 19; $q = 7$, $n = $ 1-2, 4-9, 11-14, 16-18, 20-22, 24; $q = 8$, $n = $ 1-3, 5-16, 18-21, 23-24; $q = 9$, $n = $ 1-5, 7-21; $q = 10$, $n = $ 1-12, 14-15; $q = 11$, $n = $ 1-3, 5-9; $q = 12$, $n = $ 1-5; $q = 13$, $n = $ 1-2; and $q = 14$ do not have weakly $P_{0,1}$-completion because each contains at least one of the order 4 digraphs $q = 8$, $n = 2$ or $q = 10$, $n = 1$ or the order five graph $q = 5$, $n = 6$ as an induced sub(di)graph.

CHAPTER 4. Classification of digraphs of order less than or equal to 4 regarding sign symmetric $P_{0,1}$-completion

In this chapter we give the complete classification of digraphs of order ≤ 4. We begin with the known results we will need to use.

Theorem 4.1 *[11] Any asymmetric pattern that has sign symmetric P-completion also has sign symmetric $P_{0,1}$-completion.*

Theorem 4.2 *[11] Any pattern that has sign symmetric $P_{0,1}$-completion also has sign symmetric P-completion.*

Corollary 4.3 *Any pattern that does not have sign symmetric P-completion does not have $P_{0,1}$-completion.*

Theorem 4.4 *([7], Theorem 2.4) Let G be an undirected block-clique graph. Then any partial sign symmetric $P_{0,1}$-matrix, the graph of whose specified entries is G, has a sign symmetric $P_{0,1}$-matrix completion.*

Theorem 4.5 *([10], Corollary 5.6) A digraph D has sign symmetric $P_{0,1}$-completion if and only if each block of D has sign symmetric $P_{0,1}$-completion.*

We now classify the order 2 and order 3 digraphs. Note the unusual situation that a triangular pattern with each block complete may lack completion.

Lemma 4.6 *All patterns for 2×2 matrices have sign symmetric $P_{0,1}$-matrix completion. A pattern for 3×3 matrices has sign symmetric $P_{0,1}$-completion if and only if its digraph is not one of the following: $q = 3$, $n = 2$; $q = 4$, $n = 2, 3, 4$; $q = 5$.*

$q = 4,\ n = 3$ $\qquad$ $q = 4,\ n = 4$

$$\begin{bmatrix} 1 & 1 & -1 \\ x_{21} & 1 & x_{23} \\ -1 & 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & x_{12} & -1 \\ 1 & 1 & 1 \\ -1 & x_{32} & 1 \end{bmatrix}$$

Figure 4.1 Digraphs that do not have sign symmetric $P_{0,1}$-completion

Proof: Any partial sign symmetric P-matrix specifying either of the order two digraphs $q = 0$; $q = 1$ or any of the order three digraphs $q = 0$; $q = 1$; $q = 2$, $n = 2, 3, 4$; $q = 3$, $n = 3$ may be completed to a sign symmetric P-matrix [4]. Since these digraphs are also asymmetric, they have sign symmetric $P_{0,1}$-completion by Theorem 4.1. Any partial sign symmetric $P_{0,1}$-matrix specifying the order two digraph $q = 2$ or one of the order three digraphs $q = 2$, $n = 1$ or $q = 6$ is either complete or may be completed to a sign symmetric $P_{0,1}$-matrix by Corollary 5.3 of [10] because its connected components are complete. Any partial sign symmetric $P_{0,1}$-matrix specifying the order three digraphs $q = 3$, $n = 1, 4$ and $q = 4$, $n = 1$ may be completed to a sign symmetric $P_{0,1}$-matrix by Theorem 4.5.

The patterns whose digraphs are one of $q = 3$, $n = 2$; $q = 4$, $n = 2$; $q = 5$ do not have sign symmetric P-completion by Lemma 4.1 of [4]. Therefore, they also do not have sign symmetric $P_{0,1}$-completion by Corollary 4.3. Consider the example matrices in Figure 4.1. The determinant of the first matrix is $-2x_{21} - 2x_{23}$, and the determinant of the second matrix is $-2x_{12} - 2x_{32}$. In each case the unspecified entries must be positive, since the symmetrically placed entry is positive. However, then the determinants of the matrices are negative. Therefore, the digraphs $q = 4$, $n =$ 3-4 do not have sign symmetric $P_{0,1}$-completion. □

We now begin the classification of the order 4 digraphs. In order to do the classification, we will need to establish four digraphs that have completion (Lemmas 4.7- 4.9) and six digraphs that do not (Lemmas 3.10-3.13). In Lemmas 4.8- 4.9, we introduce a new completion technique, where an entry x_{ij} is initially set to 0 (violating sign symmetry), but all principal minors are made positive by the choice of the other entries and then x_{ij} is perturbed.

Lemma 4.7 *A* 4×4 *partial sign symmetric* $P_{0,1}$*-matrix specifying the pattern with digraph* $q = 5$, $n = 7$ *has sign symmetric* $P_{0,1}$*-completion.*

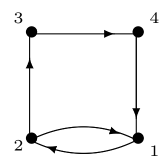

Figure 4.2 $q = 5$, $n = 7$

Proof: Let $A = \begin{bmatrix} 1 & a_{12} & x_{13} & x_{14} \\ a_{21} & 1 & a_{23} & x_{24} \\ x_{31} & x_{32} & 1 & a_{34} \\ a_{41} & x_{42} & x_{43} & 1 \end{bmatrix}$ be a partial sign symmetric $P_{0,1}$-matrix specifying the digraph $q = 5$, $n = 7$ in Figure 4.2. Clearly, the original 2×2 minor, $1 - a_{12}a_{21}$ is nonnegative. We consider three cases: (1) $a_{12}a_{23}a_{34}a_{41} > 0$, (2) $a_{12}a_{23}a_{34}a_{41} < 0$, and (3) $a_{12}a_{23}a_{34}a_{41} = 0$.

Case 1: $a_{12}a_{23}a_{34}a_{41} > 0$. We can assume $a_{12} = a_{23} = a_{34} = 1$ by Theorem 1.1. Then a_{41} is positive. We set $x_{32} = x_{43} = x_{14} = e$ and choose e small and positive. Finally, set $x_{13} = x_{24} = 1$, $x_{31} = ea_{21}$, and $x_{42} = e^2$. Then the determinant of A, shown in Table 4.1, is positive if e is small enough because $1 - a_{21} \geq 0$ and $a_{41} > 0$. Likewise, the 3×3 principal minors of A, also shown in Table 4.1, are nonnegative. The 2×2 principal minors are clearly all positive if e is small enough.

Case 2: $a_{12}a_{23}a_{34}a_{41} < 0$. We can assume $a_{12} = a_{23} = a_{34} = 1$ by Theorem 1.1. Then a_{41} is negative. We replace the symbol a_{41} with $-b$, with b positive. Set $x_{32} = x_{43} = e$ and $x_{14} = -e$, and assign $x_{13} = x_{31} = f > 0$ and $x_{24} = x_{42} = -g$, $g > 0$. We first let e equal zero and choose f and g small enough to make each minor of size three or four positive. Then perturb so that e is positive, and yet small enough to keep all principal minors nonnegative.

Table 4.1 Det$A$ and 3×3 principal minors of A in Case 1

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1 - a_{21} + a_{41} - 2e + 2a_{21}e - 2a_{41}e + e^2 - 2a_{21}e^2 + a_{41}e^2 + 2a_{21}e^3 - a_{21}e^4$
$A(1,2,3)$	$(1 - a_{21})(1 - e)$
$A(1,2,4)$	$1 - a_{21} + a_{41} - a_{41}e - e^2 + a_{21}e^3$
$A(1,3,4)$	$1 + a_{41} - e - a_{21}e - a_{41}e + a_{21}e^3$
$A(2,3,4)$	$1 - 2e + e^2$

The determinant of A and the 3×3 principal minors of A are shown in Table 4.2. The determinant of A is positive if the sum of all the terms which contain an f or g is less in magnitude than b. Each of these minors, as well as all the newly created 2×2 principal minors are obviously positive if f and g are small enough. Because the determinant of A is a continuous function, as long as each of these principal minors and Det$A$ are positive, we can perturb e slightly and yet keep all of the principal minors nonnegative.

Table 4.2 Det$A$ and 3×3 principal minors of A in Case 2 with $e = 0$

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1 - a_{21} + b + f - bf - f^2 - g + bg + a_{21}fg - g^2 + f^2g^2$
$A(1,2,3)$	$1 - a_{21} + f - f^2$
$A(1,2,4)$	$1 - a_{21} + bg - g^2$
$A(1,3,4)$	$1 - bf - f^2$
$A(2,3,4)$	$1 - g - g^2$

Case 3: $a_{12}a_{23}a_{34}a_{41} = 0$. We consider this case in 4 parts. In each part we consider one of the four entries a_{12}, a_{23}, a_{34}, or a_{41} equal to zero.

Subcase 3A: Assume $a_{12} = 0$. Then $a_{21} = 0$ by sign symmetry. Let $x_{14} = e$, $x_{32} = f$, and $x_{43} = g$ and of the same sign as a_{41}, a_{23}, and a_{34} respectively. Set all other unspecified entries equal to zero. The determinant of A and the 3×3 principal minors of A are shown in Table 4.3. Each of these and each of the 2×2 principal minors of A are clearly nonnegative if e, f, and g are small enough.

Subcase 3B: Assume $a_{34} = 0$. Then $x_{43} = 0$ by sign symmetry.

Table 4.3 Det$A$ and 3×3 principal minors of A in Case 3A

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1 - a_{41}e - a_{23}f + a_{23}a_{41}ef - a_{34}g$
$A(1,2,3)$	$1 - a_{23}f$
$A(1,2,4)$	$1 - a_{41}e$
$A(1,3,4)$	$1 - a_{41}e - a_{34}g$
$A(2,3,4)$	$1 - a_{23}f - a_{34}g$

(i) If $a_{12}a_{21} \neq 1$, let $x_{14} = e$ and $x_{32} = f$ be small enough and of the same sign as a_{41} and a_{23} respectively. Set all other unspecified entries equal to zero. The determinant of A and the 3×3 principal minors of A are shown in the second column of Table 4.4. These and all of the 2×2 principal minors are clearly nonnegative if f and e are small enough.

Table 4.4 Det$A$ and 3×3 principal minors of A in Case 3B

$A(\alpha)$	Det$A(\alpha)$ for part (i)	Det$A(\alpha)$ for part (ii)
$A(1,2,3,4)$	$1 - a_{12}a_{21} - a_{41}e - a_{23}f + a_{23}a_{41}ef$	$a_{41}g - g^2 - a_{41}s + gs$
$A(1,2,3)$	$1 - a_{12}a_{21} - a_{23}f$	0
$A(1,2,4)$	$1 - a_{12}a_{21} - a_{41}e$	$a_{41}g - g^2 - a_{41}s + gs$
$A(1,3,4)$	$1 - a_{41}e$	$1 - a_{41}s - a_{34}t$
$A(2,3,4)$	$1 - a_{23}f$	$1 - g^2 - a_{34}t$

(ii) If $a_{12}a_{21} = 1$, without loss of generality, we can make both a_{12} and a_{21} equal to 1. Set $x_{13} = x_{31} = 0$ and set $x_{24} = x_{42} = g$ with $g = \text{sgn}(a_{41})\dfrac{\min\{|a_{41}|, 1\}}{2}$, so $a_{41}g - g^2 > 0$ or $a_{41} = g = 0$. Set $x_{14} = s$ and $x_{43} = t$ very small, matching signs of a_{41} and a_{34} respectively. Then Det$A$ and the 3×3 principal minors of A are displayed in the third column of Table 4.4, and they and the 2×2 principal minors of A are clearly nonnegative with the given choice of g if s and t are made small enough.

Subcase 3C: Assume $a_{23} = 0$. Then x_{32} must be zero also by sign symmetry. Set $x_{14} = e$ and $x_{43} = f$ small enough and of the same sign as a_{41} and a_{34} respectively. Set $x_{42} = a_{12}a_{41}$, and $x_{24} = s$ small enough and of the same sign as $a_{12}a_{41}$. Set all other unspecified entries equal to zero. The determinant of A and the 3×3 principal minors of A are shown in Table 4.5.

Notice that Det$A$ can also be written as $(1 - a_{12}a_{21})(1 - a_{41}e - a_{34}f)$, and that each factor is nonnegative if e and f are small enough. The minor $A(1, 2, 3)$ is equal to the original 2×2 minor, so it is nonnegative. The minor $A(1, 2, 4)$ can also be written as $(1 - a_{12}a_{21})(1 - a_{41}e)$ and is therefore nonnegative if e is small enough. The rest of the 3×3 principal minors and all of the 2×2 are nonnegative as long as e, f, and s are small enough.

Table 4.5 Det$A$ and 3×3 principal minors of A in Case 3C

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1 - a_{12}a_{21} - a_{41}e + a_{12}a_{21}a_{41}e - a_{34}f + a_{12}a_{21}a_{34}f$
$A(1,2,3)$	$1 - a_{12}a_{21}$
$A(1,2,4)$	$1 - a_{12}a_{21} - a_{41}e + a_{12}a_{21}a_{41}e$
$A(1,3,4)$	$1 - a_{41}e - a_{34}f$
$A(2,3,4)$	1 - $a_{34}f - a_{12}a_{41}s$

Subcase 3D: Assume $a_{41} = 0$. Then $x_{14} = 0$ also by sign symmetry. Set $x_{32} = e$ small enough and of the same sign as a_{23}. Let $x_{43} = f$ be small and of the same sign as a_{34}. Set $x_{13} = a_{12}a_{23}$, $x_{31} = s$ small and of the same sign as $a_{12}a_{23}$. Set all other unspecified entries equal to zero.

Now Det$A$, shown in Table 4.6, can also be written as $(1 - a_{12}a_{21})(1 - a_{23}e - a_{34}f)$, which is nonnegative if e and f are small enough. Similarly, the 3×3 principal minors, also shown in Table 4.6 can be shown to be nonnegative. The minor $A(1, 2, 3)$ can be written as $(1 - a_{12}a_{21})(1 - a_{23}e)$, which is also nonnegative is e is made small enough. The minor $A(1, 2, 4)$ is equal to the 2×2 original minor of A, and is therefore nonnegative. The final two principal minors, $A(1, 3, 4)$ and $A(2, 3, 4)$, are nonnegative if f,s, and e are small enough. The 2×2 principal minors of A are small enough if f, e, and s are made small enough.

Lemma 4.8 *A 4×4 partial sign symmetric $P_{0,1}$-matrix specifying the pattern with digraph $q = 5$, $n = 8$ or $q = 5$, $n = 10$ has sign symmetric $P_{0,1}$-completion.*

Table 4.6 Det$A$ and 3×3 principal minors of A in Case 3D

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1 - a_{12}a_{21} - a_{23}e + a_{12}a_{21}a_{23}e - a_{34}f + a_{12}a_{21}a_{34}f$
$A(1,2,3)$	$1 - a_{12}a_{21} - a_{23}e + a_{12}a_{21}a_{23}e$
$A(1,2,4)$	$1 - a_{12}a_{21}$
$A(1,3,4)$	$1 - a_{34}f - a_{12}a_{23}s$
$A(2,3,4)$	$1 - a_{23}e - a_{34}f$

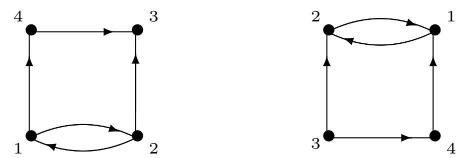

Figure 4.3 The digraphs $q = 5$, $n = 8$ and $q = 5$, $n = 10$

Proof: Let $A = \begin{bmatrix} 1 & a_{12} & x_{13} & a_{14} \\ a_{21} & 1 & a_{23} & x_{24} \\ x_{31} & x_{32} & 1 & x_{34} \\ x_{41} & x_{42} & a_{43} & 1 \end{bmatrix}$ be a partial sign symmetric $P_{0,1}$-matrix specifying the digraph $q = 5$, $n = 8$ in Figure 4.3. Clearly, the original minor, $1 - a_{12}a_{21}$ is nonnegative. We consider three cases: (1) $a_{21}a_{23}a_{43}a_{14} > 0$, (2) $a_{21}a_{23}a_{43}a_{14} < 0$, and (3) $a_{21}a_{23}a_{43}a_{14} = 0$.

Case 1: $a_{21}a_{23}a_{43}a_{14} > 0$. We can assume $a_{12} = a_{23} = a_{43} = 1$ by Theorem 1.1. Then a_{14} is positive. We set $x_{32} = x_{34} = e$, $x_{13} = 1$, $x_{31} = a_{21}e$, $x_{24} = e$, $x_{42} = e$, and $x_{41} = a_{21}e$. Let $e > 0$ be small enough to make the principal minors nonnegative. Then the determinant of A, shown in Table 4.7, can also be written as $(1 - a_{21})(1 - e)^2$, which is nonnegative if e is small enough. Likewise, the 3×3 principal minors of A, also shown in Table 4.7 can be made nonnegative: The minor $A(1,2,3)$ can be written as $(1 - a_{21})(1 - e)$, the minor $A(1,2,4)$ can be written as $(1 - a_{21})\left(1 - e^2\right)$, and the minor $A(1,3,4)$ can also be written as $(1 - e)(1 - a_{21}e)$. Each of the 3×3 principal minors are nonnegative if e is small enough. Also, each of the 2×2 principal minors is nonnegative if e is small enough.

Table 4.7 Det$A$ and 3×3 principal minors of A in Case 1

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1 - a_{21} - 2e + 2a_{21}e + e^2 - a_{21}e^2$
$A(1,2,3)$	$1 - a_{21} - e + a_{21}e$
$A(1,2,4)$	$1 - a_{21} - e^2 + a_{21}e^2$
$A(1,3,4)$	$1 - e - a_{21}e + a_{21}e^2$
$A(2,3,4)$	$1 - 2e + e^2$

Case 2: $a_{21}a_{23}a_{43}a_{14} < 0$. We can assume $a_{12} = a_{23} = a_{43} = 1$ by Theorem 1.1. Then $a_{14} < 0$ and therefore $x_{41} < 0$. We replace the symbol a_{14} with $-b$ with b positive.

Case 2a: $a_{21} \neq 1$. We set $x_{34} = x_{24} = x_{42} = x_{32} = e > 0$, $x_{41} = -e$, $x_{13} = x_{31} = 0$. The determinant of A and the 3×3 principal minors of A corresponding to Case 2a are shown the second column of Table 4.8.

Notice that Det$A(1,2) = 1 - a_{21} > 0$. Therefore, we can make e small enough so that the sum of the remaining terms in each minor is smaller than $1 - a_{12}$. Then all the 3×3 principal minors are positive. Likewise, each of the 2×2 principal minors can be made to be nonnegative if e is made small enough. Since there are only finitely many principal minors of A, it is possible to make e small enough to make each of them nonnegative.

Table 4.8 Det$A$ and 3×3 principal minors of A in Case 2

$A(\alpha)$	Det$A(\alpha)$ for part 2(a)	Det$A(\alpha)$ for part 2(b) with $e = 0$
$A(1,2,3,4)$	$1 - a_{21} - 2e + a_{21}e - be + e^2 + be^2$	$bg - bfg - g^2 + f^2g^2$
$A(1,2,3)$	$1 - a_{21} - e$	0
$A(1,2,4)$	$1 - a_{21} - be - a_{21}be - 2e^2$	$bg - g^2$
$A(1,3,4)$	$1 - e - be$	$1 - bf - f^2$
$A(2,3,4)$	$1 - 2e + e^2$	$1 - f - fg - g^2$

Case 2b: $a_{21} = 1$. We set $x_{34} = e$, $x_{24} = x_{42} = -g$, $x_{41} = -e$, $x_{32} = x_{13} = x_{31} = f$, with e, f, and g positive. We begin by setting $e = 0$. The determinant of A and the 3×3 principal minors of A corresponding to Case 2 are shown in the third column of Table 4.8. We will choose f and g so that each of the nonzero principal minors is positive (see below). Once this

is done, we perturb e slightly, so that it is positive, but the positive principal minors remain positive.

The determinant of A can also be written as $g\left(b\left(1-f\right)-\left(1-f^2\right)g\right)$. We first choose $f<1$ and such that $1-bf-f^2>0$ and then choose $g<1$ so that $\left(1-f^2\right)g<b\left(1-f\right)$. Then Det$A$ is nonnegative. Then the 3×3 principal minors of A are nonnegative if g is small enough. In fact, they can all be made positive except $A(1,2,3)$, which is zero. However, $A(1,2,3)$ has no elements set equal to e, so perturbing e will not make its determinant negative. Also, each new 2×2 minor is positive.

Case 3: $a_{21}a_{23}a_{43}a_{14}=0$.

Case 3a: $a_{12}=a_{21}=0$. We set $x_{41}=e$, $x_{32}=f$, $x_{34}=g$, and $x_{24}=x_{42}=x_{13}=x_{31}=0$. The principal minors of A are shown in Table 4.9. The determinant of A and each of the 3×3 principal minors of A are clearly nonnegative if e, f, and g are small enough. Similarly, the 2×2 principal minors of A are nonnegative if e, f, and g are small enough. Therefore, all of the principal minors of A can be made nonnegative.

Table 4.9 Det$A$ and 3×3 principal minors of A in Case 3a

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1-a_{14}e-a_{23}f-a_{43}g+a_{14}a_{23}ef$
$A(1,2,3)$	$1-a_{23}f$
$A(1,2,4)$	$1-a_{14}e$
$A(1,3,4)$	$1-a_{14}e-a_{43}g$
$A(2,3,4)$	$1-a_{23}f-a_{43}g$

Case 3b: $a_{23}=0$. This implies $x_{32}=0$

Part (i): Let $a_{12}a_{21}\neq 1$, let $x_{31}=x_{13}=x_{24}=x_{42}=x_{32}=0$ and $x_{41}=e$, $x_{34}=f$. The 3×3 principal minors and the determinant of A are shown in the second column of Table 4.10. Each of these principal minors is obviously nonnegative as long as e and f are small enough, since $a_{12}a_{21}\neq 1$, and therefore $1-a_{12}a_{21}>0$.

Part(ii): If $a_{12}a_{21}=1$, then a_{12} and a_{21} can be made equal to 1 through the use of a diagonal similarity. Now, let $x_{31}=x_{13}=0$, $x_{24}=x_{42}=g$, $x_{32}=0$, $x_{41}=e$, and $x_{34}=f$.

Begin with $e = 0$ and perturb it later if $a_{14} \neq 0$. With $e = 0$, the determinant and the new 3×3 principal minors are shown in the third column of Table 4.10. Choose f the same sign as a_{43} and small enough in absolute value so that $1 - a_{43}f > 0$. If $a_{14} = 0$, then e will remain 0 by sign symmetry. In this case, set $g = 0$. Then all of the principal minors of A are nonnegative. If $a_{14} \neq 0$, let g have the same sign as a_{14} and be small enough so that $|g| < \min\{|a_{14}|, 1\}$ and $1 - g^2 - a_{43}f \geq 0$. Then every submatrix which contains e has a positive minor. This may be checked by referring to the principal minors of A. Now we can perturb e slightly so it is the same sign as a_{14}, but not so large as to make any of the principal minors negative. In this way, we make x_{41} and x_{34} the same sign as its respective symmetric pair, so the resulting matrix is a sign symmetric $P_{0,1}$-matrix.

Also, in the case of both Part (i) and Part (ii), the 2×2 principal minors can easily be seen to be nonnegative if e, f, and g are small enough, so all of the principal minors in each of these cases can be made to be nonnegative.

Table 4.10 Det$A$ and 3×3 principal minors of A in Case 3b

$A(\alpha)$	Det$A(\alpha)$ for part (i)	Det$A(\alpha)$ for part (ii) with $e = 0$
$A(1,2,3,4)$	$1 - a_{12}a_{21} - a_{14}e - a_{43}f + a_{12}a_{21}a_{43}f$	$a_{14}g - g^2$
$A(1,2,3)$	$1 - a_{12}a_{21}$	0
$A(1,2,4)$	$1 - a_{12}a_{21} - a_{14}e$	$a_{14}g - g^2$
$A(1,3,4)$	$1 - a_{14}e - a_{43}f$	$1 - a_{43}f$
$A(2,3,4)$	$1 - a_{43}f$	$1 - g^2 - a_{43}f$

Case 3c: $a_{43} = 0$. Then $x_{34} = 0$.

Part (i): If $a_{12}a_{21} \neq 1$, let $x_{31} = x_{13} = x_{24} = x_{42} = x_{34} = 0$, $x_{32} = f$, and $x_{41} = e$. The principal minors of A are shown in the second column of Table 4.11. Each of these minors is obviously nonnegative as long as e and f are small enough, since $a_{12}a_{21} \neq 1$, and therefore $1 - a_{12}a_{21} > 0$.

Part (ii): If $a_{12}a_{21} = 1$, then a_{12} and a_{21} can be made equal to 1 through the use of a diagonal similarity. We let $x_{31} = x_{13} = g_2$, $x_{34} = 0$, $x_{24} = x_{42} = g_1$, $x_{32} = e_2$, $x_{41} = e_1$. Begin with $e_1 = e_2 = 0$. The principal minors of A with $e_1 = e_2 = 0$ are listed in the third column of Table 4.11.

If $a_{14} = 0$, set $g_1 = 0$. Since e_1 and a_{14} are symmetrically placed entries, e_1 remains zero. If a_{14} is nonzero, choose g_1 to have the same sign as a_{41} and small enough so that $|g_1| < \min\{|a_{14}|, 1\}$. If $a_{23} = 0$, set $g_2 = 0$. Since e_2 and a_{23} are symmetrically placed entries, e_2 remains zero. If $a_{23} \neq 0$, then choose g_2 the same sign as a_{23} and small enough so that $|g_2| < \min\{|a_{23}|, 1\}$ and also small enough so that Det$A$ remains positive, given the choice already made for g_1. Now for $i = 1, 2$, each of the 3×3 principal minors of A that depend on e_i is positive if e_i must be nonzero, and so we can perturb e_i slightly so it is nonzero, but the principal minors remain positive. The values for g_1, g_2, e_1, and e_2 have been chosen to that the determinant of A is positive if at least one of a_{14} and a_{32} is nonzero, so in that case, a perturbation can be done leaving Det$A$ positive. Then the principal minors of A are nonnegative.

Table 4.11 Det$A$ and 3×3 principal minors of A in Case 3c

$A(\alpha)$	Det$A(\alpha)$ for part (i)	Det$A(\alpha)$ for part (ii) with $e_1 = e_2 = 0$
$A(1,2,3,4)$	$1 - a_{12}a_{21} - a_{14}e - a_{23}f + a_{14}a_{23}ef$	$a_{14}g_1 - g_1^2 + a_{23}g_2 - a_{14}a_{23}g_1g_2 - g_2^2 + g_1^2g_2^2$
$A(1,2,3)$	$1 - a_{12}a_{21} - a_{23}f$	$a_{23}g_2 - g_2^2$
$A(1,2,4)$	$1 - a_{12}a_{21} - a_{14}e$	$a_{14}g_1 - g_1^2$
$A(1,3,4)$	$1 - a_{14}e$	$1 - g_2^2$
$A(2,3,4)$	$1 - a_{23}f$	$1 - g_1^2$

Case 3d: $a_{14} = 0$. Then $x_{41} = 0$.

Part (i): If $a_{12}a_{21} \neq 1$, let $x_{31} = x_{13} = x_{24} = x_{42} = x_{41} = 0$, $x_{32} = e$ and $x_{34} = f$. Let e be the same sign as a_{23} and let f be the same sign as a_{43}. Then it can be verified by inspection of the principal minors of A shown in Table 4.12, that if e and f are made small enough in absolute value, the 3×3 principal minors of A and the determinant of A will be nonnegative. The 2×2 principal minors of A are nonnegative if e and f are chosen small enough.

Part (ii): If $a_{12}a_{21} = 1$, then a_{12} and a_{21} can be made equal to 1 through the use of a diagonal similarity. Let $x_{31} = x_{13} = g$, $x_{24} = x_{42} = x_{41} = 0$, $x_{32} = e$, and $x_{34} = f$. We begin by setting $e = 0$. Let f have the same sign as a_{43}. The 3×3 principal minors of A and the determinant of A with $e = 0$ are shown in the third column of Table 4.12. This case is analogous to Case 3b Part (ii).

Table 4.12 Det$A$ and 3×3 principal minors of A in Case 3d

$A(\alpha)$	Det$A(\alpha)$ for part (i)	Det$A(\alpha)$ for part (ii) with $e = 0$
$A(1,2,3,4)$	$1 - a_{12}a_{21} - a_{23}e - a_{43}f + a_{12}a_{21}a_{43}f$	$a_{23}g - g^2$
$A(1,2,3)$	$1 - a_{12}a_{21} - a_{23}e$	$a_{23}g - g^2$
$A(1,2,4)$	$1 - a_{12}a_{21}$	0
$A(1,3,4)$	$1 - a_{43}f$	$1 - a_{43}f - g^2$
$A(2,3,4)$	$1 - a_{23}e - a_{43}f$	$1 - a_{43}f$

This concludes the proof of $q = 5$, $n = 8$. Let A be a partial $P_{0,1}$-matrix with digraph $q = 5$, $n = 10$, labeled as in Figure 4.3. A^T is a partial sign symmetric $P_{0,1}$-matrix specifying the digraph $q = 5$, $n = 8$ in Figure 4.3. We can complete this matrix to a $P_{0,1}$-matrix, as in the proof, and then transpose the matrix again. The resulting matrix is a completion of the matrix A. □

Lemma 4.9 *A 4×4 partial sign symmetric $P_{0,1}$-matrix specifying the pattern with digraph $q = 5$, $n = 9$ has sign symmetric $P_{0,1}$-completion.*

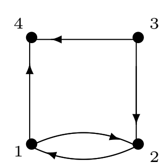

Figure 4.4 $q = 5$, $n = 9$

Consider the digraph in Figure 4.4. Let $A = \begin{bmatrix} 1 & a_{12} & x_{13} & a_{14} \\ a_{21} & 1 & x_{23} & x_{24} \\ x_{31} & a_{32} & 1 & a_{34} \\ x_{41} & x_{42} & x_{43} & 1 \end{bmatrix}$ be a partial sign symmetric $P_{0,1}$-matrix specifying the digraph $q = 5$, $n = 9$. Clearly, the original minor, $1 - a_{12}a_{21}$ is nonnegative. We consider three cases: (1) $a_{21}a_{32}a_{34}a_{14} > 0$, (2) $a_{21}a_{32}a_{34}a_{14} < 0$, and (3) $a_{21}a_{32}a_{34}a_{14} = 0$.

Case 1: $a_{21}a_{32}a_{34}a_{14} > 0$. We can assume $a_{12} = a_{32} = a_{34} = 1$ by Theorem 1.1. Then a_{14} is positive. Set $x_{13} = x_{23} = x_{24} = x_{42} = x_{43} = e$. Set $x_{31} = a_{21}$ and $x_{41} = a_{21}e$. Let $e > 0$.

The determinant of A and the 3×3 principal minors of A are listed in Table 4.13. The determinant of A can also be written as $(1 - a_{21})(1 - e)^2$, which is nonnegative if e is small enough. The minor $A(1, 2, 3)$ can be written as $(1 - a_{21})(1 - e)$, the minor $A(1, 2, 4)$ can be written as $(1 - a_{21})\left(1 - e^2\right)$, the minor $A(1, 3, 4)$ can also be written as $(1 - e)(1 - a_{21}e)$, and the minor $A(2, 3, 4)$ can be written as $(1 - e)^2$ each of which are nonnegative if e is small enough. Each of the 2×2 principal minors are also nonnegative if e is small enough.

Table 4.13 Det$A$ and 3×3 principal minors of A in Case 1

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1 - a_{21} - 2e + 2a_{21}e + e^2 - a_{21}e^2$
$A(1,2,3)$	$1 - a_{21} - e + a_{21}e$
$A(1,2,4)$	$1 - a_{21} - e^2 + a_{21}e^2$
$A(1,3,4)$	$1 - e - a_{21}e + a_{21}e^2$
$A(2,3,4)$	$1 - 2e + e^2$

Case 2: $a_{21}a_{32}a_{34}a_{14} < 0$. We can assume $a_{12} = a_{32} = a_{34} = 1$ by Theorem 1.1. Then a_{14} is negative. We set $a_{14} = -b$ with b positive.

Let $x_{23} = e$, $x_{43} = e$, let $x_{41} = -e$. Then each of these entries has the same sign as their symmetric pair. Let $x_{31} = x_{13} = f$, and $x_{24} = x_{42} = -g$. Begin with $e = 0$ and choose f positive and small enough so that $f < \min\{1, a_{21}\}$. The determinant of A and the principal minors of A with $e = 0$ are shown in Table 4.14. The determinant of A can also be written as

$1 - a_{21} + (a_{21} - f)\, f + a_{21}bg + a_{21}fg - g^2 + f^2g^2$, which can be made positive if g is chosen small enough.

The minor $A(1,2,3)$ can be written as $1 - a_{21} + (a_{21} - f)\, f$, which is positive because $f < a_{21}$, the minor $A(1,2,4)$ can also be written as $1 - a_{21} + (a_{21}b - g)\, g$, which can be made positive if g is chosen positive and small enough, the minor $A(1,3,4)$ is positive because f is less than 1, and the minor $A(2,3,4)$ can be made positive if g is chosen to be less than 1. The 2×2 principal minors are obviously nonnegative if g and f are chosen small enough, and each of them contain only one of g, f, and e, so we need only choose g and f to be small enough to make each of these minors nonnegative. Once we have chosen f and g to be small enough to make the determinant of A and all of the 3×3 principal minors of A positive, we can perturb e slightly so it is positive, but not so large as to make any of the principal minors negative. In this way, each of the principal minors of A can be made to be nonnegative. The 2×2 principal minors of A are nonnegative provided that e, f, and g are small enough.

Table 4.14 Det$A$ and 3×3 principal minors of A in Case 2 with $e = 0$

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$1 - a_{21} + a_{21}f - f^2 + a_{21}bg + a_{21}fg - g^2 + f^2g^2$
$A(1,2,3)$	$1 - a_{21} + a_{21}f - f^2$
$A(1,2,4)$	$1 - a_{21} + a_{21}bg - g^2$
$A(1,3,4)$	$1 - f^2$
$A(2,3,4)$	$1 - g^2$

Case 3(a): $a_{21}a_{32}a_{34}a_{14} = 0$ and $a_{12} = a_{21} = 0$. Set $x_{13} = x_{31} = x_{24} = x_{42} = 0$, and set $x_{41} = e$, $x_{23} = f$, and $x_{43} = g$. The determinant of A and the 3×3 principal minors of A are shown in Table 4.15. Each of the 3×3 principal minors, and the determinant of A can clearly be made nonnegative is e, f, and g are chosen small enough. Likewise, the 2×2 principal minors of A are nonnegative if e, f, and g are chosen small enough.

Case 3(b): $a_{21}a_{32}a_{34}a_{14} = 0$ and $a_{32} = x_{23} = 0$. We may assume a_{21} and a_{12} are nonzero (otherwise, we use Case 3(a)), and $a_{34} \neq 0$ (otherwise use case 3(c) below) so a_{12} and a_{34} may be made equal to 1 by Theorem 1.1. We consider two cases in Case 3(b): (i) $1 - a_{21} > 0$ and

Table 4.15 Det$A$ and 3×3 principal minors of A in Case 3a

$A(\alpha)$	Det$A(\alpha)$ for part (i)
$A(1,2,3,4)$	$1 - a_{14}e - a_{32}f + a_{14}a_{32}ef - a_{34}g$
$A(1,2,3)$	$1 - a_{32}f$
$A(1,2,4)$	$1 - a_{14}e$
$A(1,3,4)$	$1 - a_{34}g - a_{14}e$
$A(2,3,4)$	$1 - a_{34}g - a_{32}e$

(ii) $1 - a_{21} = 0$. Note that $1 - a_{21}$ is not less than zero because it is an original minor.

Part(i): $1 - a_{21} > 0$. Let $x_{13} = x_{31} = x_{24} = x_{42} = 0$, $x_{43} = f$, and $x_{41} = e$. The determinant of A and the 3×3 principal minors of A are shown in the second column of Table 4.16. Clearly, the minors shown in the table are nonnegative if e and f are made small enough. In this way, the completed matrix is a sign symmetric $P_{0,1}$-matrix.

Part(ii): $1 - a_{21} = 0$. Then $a_{21} = 1$. Set $x_{13} = x_{31} = 0$, $x_{24} = x_{42} = g$, and $x_{23} = 0$, $x_{43} = 0.1$, and $x_{41} = e$. Let e be the same sign as a_{14}. Begin with $e = 0$, and let g be as small as needed to make all order 3 and order 4 principal minors which involve x_{43} or x_{41} positive while e is set equal to zero. The determinant of A and the 3×3 principal minors of A with e set equal to zero are shown in the third column of Table 4.16. The principal minors that involve e can clearly be made positive if g is made small enough. Likewise, the 2×2 principal minors of A can clearly be made to be nonnegative. Finally, perturb e so that the principal minors remain nonnegative. Then each element has the same sign as its symmetric pair, and all of the principal minors of A are nonnegative.

Table 4.16 Det$A$ and 3×3 principal minors of A in Case 3b

$A(\alpha)$	Det$A(\alpha)$ for part (i)	Det$A(\alpha)$ for part (ii) with $e = 0$
$A(1,2,3,4)$	$1 - a_{21} - f - a_{14}e + a_{21}f$	$a_{14}g - g^2$
$A(1,2,3)$	$1 - a_{21}$	0
$A(1,2,4)$	$1 - a_{21} - a_{14}e$	$a_{14}g - g^2$
$A(1,3,4)$	$1 - f - a_{14}e$	0.9
$A(2,3,4)$	$1 - f$	$0.9 - g^2$

Case 3(c): $a_{21}a_{32}a_{34}a_{14} = 0$ and $a_{34} = x_{43} = 0$. We may assume a_{21} and a_{12} are nonzero (otherwise, we use Case 3(a)) so a_{12} may be made equal to 1 by Theorem 1.1. We consider two parts in Case 3(c): (i) $1 - a_{21} > 0$ and (ii) $1 - a_{21} = 0$. Note that $1 - a_{21}$ is not negative because it is an original minor.

Part(i): $1 - a_{21} > 0$. Let $x_{13} = x_{31} = x_{24} = x_{42} = 0$ and set $x_{23} = e$, $x_{41} = f$. Let e and f be the same sign as a_{32} and a_{14} respectively. The determinant of A and the 3×3 principal minors of A are shown in the second column of Table 4.17. Clearly, the minors shown in the table are nonnegative if e and f are made small enough. Therefore, the completed matrix is a sign symmetric $P_{0,1}$-matrix.

Part(ii): $1 - a_{21} = 0$. Then $a_{21} = 1$. Set $x_{13} = x_{31} = g_1$, $x_{24} = x_{42} = g_2$, $x_{23} = e_1$, and $x_{41} = e_2$. Begin with $e_1 = e_2 = 0$ and later perturb e_1 to be the same sign as a_{32} and e_2 to be the same sign as a_{14}. The determinant of A and the 3×3 principal minors of A with e_1 and e_2 set equal to zero are shown in column 3 of Table 4.17. If $a_{32} = 0$, set $g_1 = 0$. Then e_1 remains zero by sign symmetry. If $a_{32} \neq 0$, let g_1 be the same sign as a_{32} such that $|g_1| < \min\{1, a_{32}\}$. If $a_{14} = 0$, let $g_2 = 0$. Then e_2 remains zero by sign symmetry. If $a_{14} \neq 0$, let g_2 be the same sign as a_{14} such that $|g_2| < \min\{1, a_{14}\}$.

The determinant of A can also be written as $(a_{14} - g_2)\, g_2 + (a_{32} - g_1)\, g_1 + g_1^2 g_2^2$, the minor $A(1, 2, 3)$ can be written as $(a_{32} - g_1)\, g_1$, and the minor $A(1, 2, 4)$ can be written as $g_2(a_{14} - g_2)$. Then clearly all of the minors listed in the table (as well as the 2×2 principal minors, which are not listed) are positive if g_1 and g_2 are chosen as specified. Then, finally, perturb e_1 and e_2 if necessary so that they are not large enough to cause any of the principal minors of A to become negative. Then each element has the same sign as its symmetric pair, and all of the principal minors of A are nonnegative.

Case 3(d): $a_{21}a_{32}a_{34}a_{14} = 0$ and $a_{14} = x_{41} = 0$. Again, we may assume $a_{21} = a_{34} = 1$ be Theorem 1.1, since otherwise we use Case 3(a) or Case 3(c) respectively. We consider two parts in Case 3(d): (i) $1 - a_{21} > 0$ and (ii) $1 - a_{21} = 0$. Note that $1 - a_{21}$ is an original minor and is therefore nonnegative.

Table 4.17 Det$A$ and 3×3 principal minors of A in Case 3c

$A(\alpha)$	Det$A(\alpha)$ for part (i)	Det$A(\alpha)$ for part (ii) with $e_1 = 0$ and $e_2 = 0$
$A(1,2,3,4)$	$1 - a_{21} - a_{14}f - a_{32}e + a_{14}a_{32}ef$	$a_{14}g_2 - g_2^2 + a_{32}g_1 - g_1^2 + g_1^2g_2^2$
$A(1,2,3)$	$1 - a_{21} - a_{32}e$	$a_{32}g_1 - g_1^2$
$A(1,2,4)$	$1 - a_{21} - a_{14}f$	$a_{14}g_2 - g_2^2$
$A(1,3,4)$	$1 - a_{14}e$	$1 - g_1^2$
$A(2,3,4)$	$1 - a_{32}e$	$1 - g_2^2$

Part(i): $1 - a_{21} > 0$. Let $x_{13} = x_{31} = x_{24} = x_{42} = 0$ and set $x_{23} = e$, $x_{43} = f$. Let e be the same sign as a_{32} and let $f > 0$. The determinant of A and the 3×3 principal minors of A are shown in the second column of Table 4.18. Clearly, the minors shown in the table are nonnegative if e and f are made small enough. Therefore, the completed matrix is a sign symmetric $P_{0,1}$-matrix.

Part(ii): $1 - a_{21} = 0$. Then $a_{21} = 1$. Set $x_{13} = x_{31} = f$, $x_{23} = e$, and $x_{43} = 0.1$. Begin with $e = 0$, and later perturb e to be the same sign as a_{32}. Let f be the same sign as a_{32} and small enough so that $|f| < \min\{1, |a_{32}|\}$. The determinant of A and the 3×3 principal minors of A with e set equal to zero are shown in column 3 of Table 4.18. The determinant of A and the minor $A(1,2,3)$ can be written as $(a_{32} - f)\,f$, so clearly all of the minors listed in the table with the exception of $A(1,2,4)$ are positive. The 2×2 principal minors of A are also positive if f is chosen as specified. The minor $A(1,2,4)$ is nonnegative, and is not dependent on e. Therefore, we may perturb e so that none of the principal minors become negative. Then all of the principal minors of A are nonnegative.

This concludes all three cases of the proof. Therefore, a 4×4 partial sign symmetric $P_{0,1}$-matrix specifying the pattern with digraph $q = 5$, $n = 9$ has sign symmetric $P_{0,1}$-completion.

□

Lemma 4.10 *A 4×4 partial sign symmetric $P_{0,1}$-matrix specifying the pattern with digraph $q = 6$, $n = 4$ does not have sign symmetric $P_{0,1}$-completion.*

Table 4.18 Det$A$ and 3×3 principal minors of A in Case 3d

$A(\alpha)$	Det$A(\alpha)$ for part (i)	Det$A(\alpha)$ for part (ii) with $e = 0$
$A(1,2,3,4)$	$1 - a_{21} - f + a_{21}f - a_{32}e$	$a_{32}f - f^2$
$A(1,2,3)$	$1 - a_{21} - a_{32}e$	$a_{32}f - f^2$
$A(1,2,4)$	$1 - a_{21}$	0
$A(1,3,4)$	$1 - f$	$0.9 - f^2$
$A(2,3,4)$	$1 - f - a_{32}e$	0.9

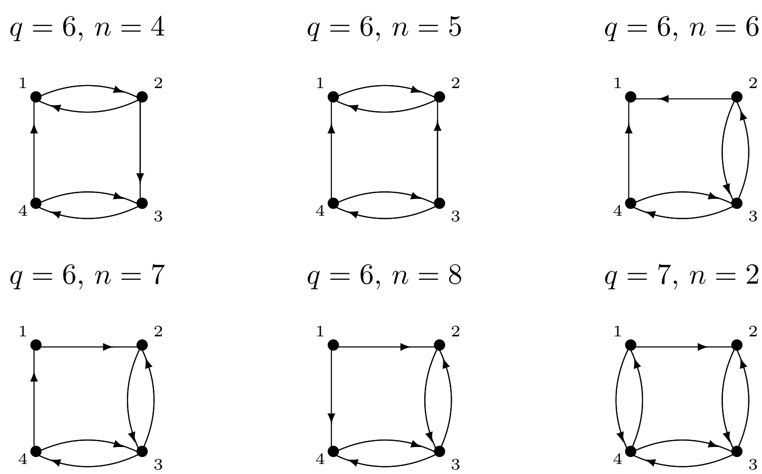

Figure 4.5 The digraphs $q = 6$, $n = 4$-8 and $q = 7$, $n = 2$

Proof: The digraph $q = 6$, $n = 4$ is shown in Figure 4.5. The example matrix A in Figure 4.6 cannot be completed to a sign symmetric $P_{0,1}$-matrix. Consider the submatrix $A(1,2,3)$. Since x_{32} is positive by sign symmetry, the term $-x_{32}$ is negative. The term $-x_{13}x_{31}$ is nonpositive by sign symmetry. Then x_{31} and/or $x_{13}x_{32}$ must be positive in order to make the minor nonnegative. Then x_{31} and x_{13} must be positive by sign symmetry. Next, we consider the principal submatrix $A(1,3,4)$. Since x_{14} is negative by sign symmetry and the term $-x_{13}x_{31}$ is nonpositive by sign symmetry, $-x_{31}$ and/or $x_{13}x_{14}$ must be positive to make the minor nonnegative. Then x_{31} and x_{13} must be negative by sign symmetry. However, this is a contradiction. Therefore, this example matrix cannot be completed to a sign symmetric

$P_{0,1}$-matrix, and so the digraph $q = 6$, $n = 4$ does not have sign symmetric $P_{0,1}$-completion.

The matrix A	The submatrix $A(1,2,3)$	The submatrix $A(1,3,4)$
$\begin{bmatrix} 1 & 1 & x_{13} & x_{14} \\ 1 & 1 & 1 & x_{24} \\ x_{31} & x_{32} & 1 & 1 \\ -1 & x_{42} & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & x_{13} \\ 1 & 1 & 1 \\ x_{31} & x_{32} & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & x_{13} & x_{14} \\ x_{31} & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$
	$x_{31} - x_{13}x_{31} - x_{32} + x_{13}x_{32}$	$-x_{13} + x_{14} - x_{13}x_{31} + x_{14}x_{31}$

Figure 4.6 $q = 6$, $n = 4$ has no sign symmetric $P_{0,1}$-completion

Lemma 4.11 *A* 4×4 *partial sign symmetric* $P_{0,1}$*-matrix specifying the pattern with digraph* $q = 6$*,* $n = 5$ *does not have sign symmetric* $P_{0,1}$*-completion.*

Proof: The digraph $q = 6$, $n = 5$ is shown in Figure 4.5. The example matrix A in Figure 4.7 cannot be completed to a sign symmetric $P_{0,1}$-matrix. Consider the submatrix $A(1,2,3)$. Since x_{23} is positive by sign symmetry, the term $-x_{23}$ is negative. The term $-x_{13}x_{31}$ is nonpositive by sign symmetry. Then x_{13} and/or $x_{23}x_{31}$ must be positive in order to make the minor nonnegative. Then x_{31} and x_{13} must be positive by sign symmetry. Next, we consider the principal submatrix $A(1,3,4)$. Since x_{14} is negative by sign symmetry and the term $-x_{13}x_{31}$ is nonpositive by sign symmetry, $-x_{13}$ and/or $x_{14}x_{31}$ must be positive to make the minor nonnegative. Then x_{31} and x_{13} must be negative by sign symmetry. However, this is a contradiction. Therefore, this example matrix cannot be completed to a sign symmetric $P_{0,1}$-matrix, and so the digraph $q = 6$, $n = 5$ does not have sign symmetric $P_{0,1}$-completion.

Lemma 4.12 *A* 4×4 *partial sign symmetric* $P_{0,1}$*-matrix specifying the pattern with digraph* $q = 6$*,* $n = 6$ *or* $q = 6$*,* $n = 8$ *does not have sign symmetric* $P_{0,1}$*-completion.*

Proof: The digraph $q = 6$, $n = 6$ is shown in Figure 4.5. The example matrix A in Figure 4.8 cannot be completed to a sign symmetric $P_{0,1}$-matrix. Consider the submatrix $A(1,2,3)$. Since x_{12} is positive by sign symmetry, $-x_{12}$ is negative. Also, the term $-x_{13}x_{31}$ is nonpositive by sign symmetry. Then x_{13} and/or $x_{12}x_{31}$ must be positive in order to make

The matrix A	The submatrix $A(1,2,3)$	The submatrix $A(1,3,4)$
$\begin{bmatrix} 1 & 1 & x_{13} & x_{14} \\ 1 & 1 & x_{23} & x_{24} \\ x_{31} & 1 & 1 & 1 \\ -1 & x_{42} & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 1 & x_{13} \\ 1 & 1 & x_{23} \\ x_{31} & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & x_{13} & x_{14} \\ x_{31} & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$
	$x_{13} - x_{23} - x_{13}x_{31} + x_{23}x_{31}$	$-x_{13} + x_{14} - x_{13}x_{31} + x_{14}x_{31}$

Figure 4.7 $q = 6$, $n = 5$ has no sign symmetric $P_{0,1}$-completion

the minor nonnegative. Then x_{31} and x_{13} must be positive by sign symmetry. Next, we consider the principal submatrix $A(1,3,4)$. Since x_{14} is negative by sign symmetry and the term $-x_{13}x_{31}$ is nonpositive by sign symmetry, $-x_{13}$ and/or $x_{14}x_{31}$ must be positive to make the minor nonnegative. Then x_{31} and x_{13} must be negative by sign symmetry. However, this is a contradiction. Therefore, this example matrix cannot be completed to a sign symmetric $P_{0,1}$-matrix, and so the digraph $q = 6$, $n = 6$ does not have sign symmetric $P_{0,1}$-completion.

The matrix A	The submatrix $A(1,2,3)$	The submatrix $A(1,3,4)$
$\begin{bmatrix} 1 & x_{12} & x_{13} & x_{14} \\ 1 & 1 & 1 & x_{24} \\ x_{31} & 1 & 1 & 1 \\ -1 & x_{42} & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & x_{12} & x_{13} \\ 1 & 1 & 1 \\ x_{31} & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & x_{13} & x_{14} \\ x_{31} & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$
	$-x_{21} + x_{13}x_{21} + x_{31} - x_{13}x_{31}$	$-x_{13} + x_{14} - x_{13}x_{31} + x_{14}x_{31}$

Figure 4.8 $q = 6$, $n = 6$ has no sign symmetric $P_{0,1}$-completion

The digraph $q = 6$, $n = 8$ is the digraph of the transpose of a matrix specifying the digraph $q = 6$, $n = 6$, so the digraph $q = 6$, $n = 8$ does not have sign symmetric $P_{0,1}$-completion.

Lemma 4.13 *A 4×4 partial sign symmetric $P_{0,1}$-matrix specifying the pattern with digraph $q = 6$, $n = 7$ or $q = 7$, $n = 2$ does not have sign symmetric $P_{0,1}$-completion.*

Proof: The digraph $q = 6$, $n = 7$ is shown in Figure 4.5. The example matrix A in Figure 4.9 cannot be completed to a sign symmetric $P_{0,1}$-matrix. Consider the submatrix $A(1,2,3)$.

Since x_{21} is positive by sign symmetry, $-x_{21}$ is negative. Also, the term $-x_{13}x_{31}$ is nonpositive by sign symmetry. Then x_{31} and/or $x_{12}x_{13}$ must be positive in order to make the minor nonnegative. Then x_{31} and x_{13} must be positive by sign symmetry. Next, we consider the principal submatrix $A(1,3,4)$. Since x_{14} is negative by sign symmetry and the term $-x_{13}x_{31}$ is nonpositive by sign symmetry, $-x_{13}$ and/or $x_{14}x_{31}$ must be positive to make the minor nonnegative. Then x_{31} and x_{13} must be negative by sign symmetry. However, this is a contradiction. Therefore, this example matrix cannot be completed to a sign symmetric $P_{0,1}$-matrix, and so the digraph $q = 6$, $n = 7$ does not have sign symmetric $P_{0,1}$-completion.

The matrix A

$$\begin{bmatrix} 1 & 1 & x_{13} & x_{14} \\ x_{21} & 1 & 1 & x_{24} \\ x_{31} & 1 & 1 & 1 \\ -1 & x_{42} & 1 & 1 \end{bmatrix}$$

The submatrix $A(1,2,3)$

$$\begin{bmatrix} 1 & 1 & x_{13} \\ x_{21} & 1 & 1 \\ x_{31} & 1 & 1 \end{bmatrix}$$

$$-x_{21} + x_{13}x_{21} + x_{31} - x_{13}x_{31}$$

The submatrix $A(1,3,4)$

$$\begin{bmatrix} 1 & x_{13} & x_{14} \\ x_{31} & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$-x_{13} + x_{14} - x_{13}x_{31} + x_{14}x_{31}$$

Figure 4.9 $q = 6$, $n = 7$ has no sign symmetric $P_{0,1}$-completion

If the digraph $q = 7$, $n = 2$ had sign symmetric $P_{0,1}$-completion, the $q = 6$, $n = 7$ would by Lemma 1.4. Therefore, $q = 7$, $n = 2$ does not have sign symmetric $P_{0,1}$-completion.

Theorem 4.14 *(Classification of Patterns of* 4×4 *matrices regarding sign symmetric* $P_{0,1}$*-completion). Let* Q *be a pattern for* 4×4 *matrices that includes all diagonal positions. The pattern* Q *has sign symmetric* P*-completion if and only if its digraph is one of the following.*

q =0

q =1;

q =2; n =1-5;

q =3; n =1-11, 13;

q =4; n =1-12, 16-19, 21-23, 25-27;

q =5; n =1-5, 7-10, 26-29, 31, 33-34, 36-37;

q =6; n =1-3, 46;

q =7; n =4-5;

q =8; n =1;

q =12.

Proof:

Part 1. Digraphs that have sign symmetric $P_{0,1}$-completion.

The following digraphs have sign symmetric $P_{0,1}$-completion by Theorem 4.5 because each block has sign symmetric $P_{0,1}$-matrix completion by Lemma 4.6: $q = 0$; $q = 1$; $q = 2$, $n =$ 1-5; $q = 3$, $n =$ 1-11, 13; $q = 4$, $n =$1-12, 21-23, 25-27; $q = 5$, $n =$1-5, 26-28; $q = 6$, $n =$1-3; $q = 7$, $n =$ 4-5; $q = 8$, $n = 1$; $q = 12$.

Each of the digraphs $q = 4$, $n =$ 16-19; $q = 5$, $n = 29, 31,$ 33-34, 36-37; $q = 6$, $n =$ 46 are asymmetric and have sign symmetric P-completion [4]. Therefore, any partial sign symmetric $P_{0,1}$-matrix specifying one of these digraphs has sign symmetric $P_{0,1}$-completion by Theorem 4.1.

Any partial sign symmetric sign symmetric $P_{0,1}$-matrix specifying any of the digraphs $q = 5$, $n =$ 7-10 have sign symmetric $P_{0,1}$-matrix completion by Lemma 4.7, Lemma 4.8, and Lemma 4.9.

Part 2. Digraphs that do not have sign symmetric $P_{0,1}$-completion.

The order three digraphs $q = 3$, $n = 2$; $q = 4$, $n = 2, 3, 4$; $q = 5$ do not have sign symmetric $P_{0,1}$-completion by Lemma 4.6. Any digraph which has one of these order 3 digraphs as an induced subdigraph does not have sign symmetric $P_{0,1}$-completion, because a partial sign symmetric $P_{0,1}$-matrix specifying such a digraph then has a partial sign symmetric subdigraph (and therefore a sign symmetric $P_{0,1}$-principal submatrix) which does not have sign symmetric $P_{0,1}$-completion. The following digraphs have at least one of these order 3 digraphs as an induced subdigraph: $q = 3$, $n = 12$; $q = 4$, $n = 13$-15, 20, 24; $q = 5$, $n = 6$, 11-25, 30, 32, 35, 38; $q = 6$, $n = 9$-45, 47-48; $q = 7$, $q = 1$,3, 6-38; $q = 8$, $n =$3-27; $q = 9$, $n = 1$-13; $n = 10$, $q = 2$-5; $q = 11$.

The digraphs $q = 6$, $n = 4$-8 and $q = 7$, $n = 2$ do not have sign symmetric $P_{0,1}$-completion by Lemma 4.10, Lemma 4.11, Lemma 4.12, and Lemma 4.13. The digraphs $q = 8$, $n = 2$; $q = 10$, $n = 1$ do not have sign symmetric $P_{0,1}$-completion by [7].

CHAPTER 5. Classification of order 4 graphs and some order 4 digraphs regarding $P_{0,1}$-completion

In this chapter, we classify the order 4 graphs and some order 4 digraphs regarding $P_{0,1}$-completion. We begin with the known results we will use.

Theorem 5.1 *[7] Every block-clique graph has $P_{0,1}$-completion.*

Theorem 5.2 *[10] If each nonseparable strongly connected induced subdigraph of G has $P_{0,1}$-completion, so does G.*

Lemma 5.3 *[10] All patterns for 3×3 matrices have $P_{0,1}$-completion.*

Corollary 5.4 *If every strongly connected induced subdigraph of digraph G has blocks only of order 3 or less, then G has $P_{0,1}$-completion.*

Corollary 5.5 *Let G be a digraph that has $P_{0,1}$-completion. Let H be a digraph obtained from G by deleting one arc (u, v) such that u and v are contained in at most one clique of order 3 in G. Then H has $P_{0,1}$-completion.*

Equivalently, if G is a digraph obtained from a digraph H by adding one arc (u, v) such that u and v are contained in at most one clique of order 3 in G, and H does not have $P_{0,1}$-completion, then G does not have $P_{0,1}$-completion.

Proof: We establish the first statement. Let A be a partial $P_{0,1}$-matrix specifying H. Choose a value for the unspecified u, v-entry of A to obtain a partial $P_{0,1}$-matrix B specifying G as follows: If u and v are not in any clique of order 3, set the u, v-entry of B to 0. If the subdigraph induced by $\{u, v, w\}$ is a clique in G, choose a value c for the u, v-entry that completes $A(\{u, v, w\})$ to a $P_{0,1}$-matrix (such a c is guaranteed to exist by Lemma 5.3). Then, since G has $P_{0,1}$-completion, we can complete B to a $P_{0,1}$-matrix C, which also completes A. Thus H has $P_{0,1}$-completion. □

Theorem 5.6 *[11] Any pattern that has P_0-completion also has $P_{0,1}$-completion.*

Theorem 5.7 *[11] Any pattern that has $P_{0,1}$-completion also has P-completion.*

Corollary 5.8 *Any pattern that does not have P-completion does not have $P_{0,1}$-completion.*

Theorem 5.9 *[10] The 4-cycle (the graph $q = 4$, $n = 2$ or digraph $q = 8$, $n = 2$) has $P_{0,1}$-completion.*

We begin with the classification of the graph $q = 5$ (digraph $q = 10$, $n = 1$), which is often referred to as the double triangle (See Figure 5.1).

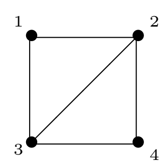

Figure 5.1 The graph $q = 5$

For many classes, the double triangle does not have completion. For instance, the double triangle does not have nonnegative P-completion [7]. It also does not have P_0-completion, as evidenced by the example in [13]: $B = \begin{bmatrix} 1 & 2 & 1 & x_{14} \\ -1 & 0 & 0 & -2 \\ -1 & 0 & 0 & -1 \\ x_{41} & 1 & 1 & 1 \end{bmatrix}$, which is a partial P_0-matrix, but cannot be completed to a P_0-matrix because $\mathrm{Det}B = -1$.

However, all graphs, including the double triangle, have P-completion [13]. The following theorem proves the double triangle also has $P_{0,1}$-completion.

Theorem 5.10 *A 4×4 partial $P_{0,1}$-matrix specifying the pattern with graph $q = 5$ (equivalently, digraph $q = 10$, $n = 1$) has $P_{0,1}$-completion.*

Let A be a partial $P_{0,1}$-matrix specifying graph $q = 5$, which is shown in Figure 5.1. We can assume the diagonal entries of A are equal to 1.

The completion is proven in several cases:

Case 1: $a_{23}a_{32} \neq 1$.

Case 2: $a_{23}a_{32} = 1$.

- Case 2a: $a_{31} = a_{21}$.
- Case 2b: $a_{42} = a_{43}$.
- Case 2c: $a_{31} \neq a_{21}$ and $a_{42} \neq a_{43}$.

The original minors are $\mathrm{Det}A(1,2,3)$, $\mathrm{Det}A(2,3,4)$, $\mathrm{Det}A(1,2)$, $\mathrm{Det}A(2,4)$, $\mathrm{Det}A(3,4)$, $\mathrm{Det}A(1,3)$, and $\mathrm{Det}A(2,3)$. Therefore, it is only necessary to show that $\mathrm{Det}A(1,3,4)$, $\mathrm{Det}A(1,2,4)$, $\mathrm{Det}A(1,4)$, and the determinant of A are nonnegative.

Case 1: $a_{23}a_{32} \neq 1$. Set $x_{14} = x$ and $x_{41} = -x$ for x large enough in absolute value to make the principal minors nonnegative. The principal minors of A that must be checked for Case 1 are shown in the Table 5.1.

Table 5.1 Det$A$ and 3×3 principal minors of A for Case 1

$A(\alpha)$	$\mathrm{Det}A(\alpha)$
$A(1,2,3,4)$	$1-a_{12}a_{21}-a_{13}a_{31}+a_{12}a_{23}a_{31}+a_{13}a_{21}a_{32}-a_{23}a_{32}-a_{24}a_{42}+a_{13}a_{24}a_{31}a_{42}-a_{13}a_{21}a_{34}a_{42}+a_{23}a_{34}a_{42}-a_{12}a_{24}a_{31}a_{43}+a_{24}a_{32}a_{43}-a_{34}a_{43}+a_{12}a_{21}a_{34}a_{43}-a_{12}a_{24}x+a_{13}a_{24}a_{32}x-a_{13}a_{34}x+a_{12}a_{23}a_{34}x+a_{21}a_{42}x-a_{23}a_{31}a_{42}x+a_{31}a_{43}x-a_{21}a_{32}a_{43}x+(1-a_{23}a_{32})\,x^2$
$A(1,2,4)$	$1-a_{12}a_{21}-a_{24}a_{42}-a_{12}a_{24}x+a_{21}a_{42}x+x^2$
$A(1,3,4)$	$1-a_{13}a_{31}-a_{34}a_{43}-a_{13}a_{34}x+a_{31}a_{43}x+x^2$
$A(1,4)$	$1+x^2$

Since $a_{32}a_{23} \neq 1$, the last summand of the determinant of A is positive. Therefore, we can choose x large enough so $\mathrm{Det}A \geq 0$. Similarly, we can choose x large enough so $\mathrm{Det}A(1,2,4)$, $\mathrm{Det}A(1,3,4)$, and $\mathrm{Det}A(1,4)$ are nonnegative.

Case 2: $a_{23}a_{32} = 1$. Since $a_{23}a_{32} = 1$, we can assume $a_{23} = 1$. This is possible by Theorem 1.1. Then $a_{32} = 1$ also.

Case 2a: $a_{31} = a_{21}$. Set $x_{14} = x$ and $x_{41} = -x$ for x large enough in absolute value to make the principal minors nonnegative. Let x be the same sign as $(a_{12} - a_{13})(a_{34} - a_{24})$ if this quantity is nonzero. If it is zero, x can be chosen freely. The principal minors of A which must be checked are listed in Table 5.2. The original minors needed in order to show that they are nonnegative are listed in the same table under the double line.

Table 5.2 Det$A$ and 3×3 principal minors of A in Case 2a

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$(a_{12} - a_{13})a_{21}(-a_{24} + a_{34})a_{42} + (1 - a_{12}a_{21})(-a_{24}a_{42} + a_{34}a_{42} + a_{24}a_{43} - a_{34}a_{43}) + (a_{12} - a_{13})(-a_{24} + a_{34})x$
$A(1,2,4)$	$1 - a_{12}a_{21} - a_{24}a_{42} - a_{12}a_{24}x + a_{21}a_{42}x + x^2$
$A(1,3,4)$	$1 - a_{13}a_{21} - a_{34}a_{43} - a_{13}a_{34}x + a_{21}a_{43}x + x^2$
$A(1,4)$	$1 + x^2$
$A(2,3,4)$	$-a_{24}a_{42} + a_{34}a_{42} + a_{24}a_{43} - a_{34}a_{43}$
$A(1,4)$	$1 - a_{12}a_{21}$

Let us examine Det$A$. If $(a_{12} - a_{13})(a_{34} - a_{24}) \neq 0$, let x be large enough in absolute value to make Det$A \geq 0$. If $(a_{12} - a_{13})(a_{34} - a_{24}) = 0$, then Det$A = (1 - a_{12}a_{21})(-a_{24}a_{42} + a_{34}a_{42} + a_{24}a_{43} - a_{34}a_{43}) =$ Det$A(1,2)$Det$A(2,3,4)$, which is a product of original minors. Therefore, Det$A$ is nonnegative.

The remaining 3×3 principal minors and the 2×2 minor are clearly nonnegative if x is large enough.

Case 2b: $a_{42} = a_{43}$. Set $x_{14} = x$ and $x_{41} = -x$ for x large enough in absolute value to make the principal minors nonnegative. Let x be the same sign as $(a_{12} - a_{13})(a_{34} - a_{24})$ if this quantity is nonzero. It it is zero, x can be chosen freely. The principal minors of A which must be checked are listed in Table 5.3. The original minors needed in order to show that they are nonnegative are listed in the same table under the double line.

Let us examine Det$A$. If $(a_{12} - a_{13})(a_{34} - a_{24}) \neq 0$, let x be large enough in absolute value to make Det$A \geq 0$. If $(a_{12} - a_{13})(a_{34} - a_{24}) = 0$, then Det$A =$ Det$A(1,2,3)$Det$(3,4) \geq 0$. The minors Det$A(1,2,4)$, Det$A(1,3,4)$, and Det$A(1,4)$ are clearly nonnegative if x is chosen large enough.

Table 5.3 Det$A$ and 3×3 principal minors of A in Case 2b

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$(a_{12}-a_{13})a_{31}(-a_{24}+a_{34})a_{43}+(-a_{12}a_{21}+a_{13}a_{21}+a_{12}a_{31}-a_{13}a_{31})(1-a_{34}a_{43})+(a_{12}-a_{13})(-a_{24}+a_{34})x$
$A(1,2,4)$	$1-a_{12}a_{21}-a_{24}a_{43}-a_{12}a_{24}x+a_{21}a_{43}x+x^2$
$A(1,3,4)$	$1-a_{13}a_{31}-a_{34}a_{43}-a_{13}a_{34}x+a_{31}a_{43}x+x^2$
$A(1,4)$	$1+x^2$
$A(3,4)$	$1-a_{34}a_{43}$
$A(1,2,3)$	$-a_{12}a_{21}+a_{13}a_{21}+a_{12}a_{31}-a_{13}a_{31}$

Case 2c: $a_{31} \neq a_{21}$ and $a_{42} \neq a_{43}$. Set $x_{14} = x$ and $x_{41} = -mx$. The determinant of A and the 3×3 principal minors of A which need to be checked are shown in Table 5.4.

Choose m to be a positive constant such that the coefficient $(a_{21}-a_{31})(a_{42}-a_{43})+(-a_{12}a_{24}+a_{13}a_{24}+a_{12}a_{34}-a_{13}a_{34})m$ of x is nonzero. This is possible because $(a_{21}-a_{31})(a_{42}-a_{43})$ is nonzero. So, if $-a_{12}a_{24}+a_{13}a_{24}+a_{12}a_{34}-a_{13}a_{34} \neq 0$, there is a specific value of m such that the coefficient of x is equal to zero; if $-a_{12}a_{24}+a_{13}a_{24}+a_{12}a_{34}-a_{13}a_{34} = 0$, then m is irrelevant. Once m is chosen so that the coefficient of x is nonzero, we can choose x of the same sign as its coefficient and large enough in absolute value to make the determinant of A nonnegative. The 3×3 principal minors and the 2×2 principal minor of A are clearly nonnegative if x is chosen large enough (since $m > 0$).

Table 5.4 Det$A$ and 3×3 principal minors of A in Case 2c

$A(\alpha)$	Det$A(\alpha)$
$A(1,2,3,4)$	$-a_{12}a_{21}+a_{13}a_{21}+a_{12}a_{31}-a_{13}a_{31}-a_{24}a_{42}+a_{13}a_{24}a_{31}a_{42}+a_{34}a_{42}-a_{13}a_{21}a_{34}a_{42}+a_{24}a_{43}-a_{12}a_{24}a_{31}a_{43}-a_{34}a_{43}+a_{12}a_{21}a_{34}a_{43}+((a_{21}-a_{31})(a_{42}-a_{43})+(-a_{12}a_{24}+a_{13}a_{24}+a_{12}a_{34}-a_{13}a_{34})m)x$
$A(1,2,4)$	$1-a_{12}a_{21}-a_{24}a_{42}+a_{21}a_{42}x-a_{12}a_{24}mx+mx^2$
$A(1,3,4)$	$1-a_{13}a_{31}-a_{34}a_{43}+a_{31}a_{43}x-a_{13}a_{34}mx+mx^2$
$A(1,4)$	$1+mx^2$

This concludes both cases. Therefore, the double triangle as $P_{0,1}$-completion. $\square$

Theorem 5.11 *All order 4 graphs have $P_{0,1}$-completion.*

Proof: The double triangle ($q = 5$) has $P_{0,1}$-completion by Theorem 5.10. The 4 cycle ($q = 4$, $n = 2$) has $P_{0,1}$-completion by Theorem 5.9. Each of the remaining graphs has the property that every component is block-clique, and thus has completion by Theorem 5.1.

Theorem 5.12 *All patterns for the digraphs with*

$q = 0$, $q = 1$, $q = 2$, $q = 3$, $q = 4$, $q = 5$, $q = 6$, *and* $q = 12$, *and*

$q = 7$ $n =$ *1-29, 31, 34, 36-37;*

$q = 8$ $n =$ *1-15, 18, 21, 27;*

$q = 9$ $n =$ *1-2, 11;*

$q = 10$ $n =$ *1 ;*

$q = 12$.

have $P_{0,1}$-completion, and there exists at least one matrix satisfying the pattern for the digraphs $q = 9$, $n = 3$; $q = 10$, $n = 5$; and $q = 11$ which cannot be completed to a $P_{0,1}$-matrix.

Proof:

Part 1: All digraphs listed in Theorem 5.12 and not listed below have $P_{0,1}$-completion by Corollary 5.4.

The digraph $q = 8$, $n = 2$ has $P_{0,1}$-completion by Theorem 5.9.

The following digraphs have $P_{0,1}$-completion by Theorem 5.10 and Corollary 5.5: $q = 4$, $n = 16$; $q = 5$, $n =$7, 32, 35; $q = 6$, $n =$4, 7, 22, 28, 30, 31, 33, 34, 37, 42; $q = 7$, $n =$2, 7, 8, 10, 12, 13, 15, 17, 18, 20, 21, 23, 25, 27; $q = 8$, $n =$ 3-9, 11, 13-15; $q = 9$, $n = 1, 2$.

The digraph $q = 6$, $n = 45$ has P_0-completion [2]. Therefore, it also has $P_{0,1}$-completion Theorem 5.6.

Part 2. Digraphs that do not have $P_{0,1}$-completion.

The digraphs $q = 9$, $n = 3$; $q = 10$, $n = 5$; and $q = 11$ do not have P-completion [13], [3], and therefore do not have $P_{0,1}$-completion by Corollary 5.8.

APPENDIX Mathematica Files

The following cell must be run in Mathematica before any of the other Mathematica files in the Appendix can be run.

```
In[1]:= truth[a_] = If[a, 1, 0];

        Clear[M, n, ki, i, kj, j, B, BB, PM, submtx];

        M[B_, n_, ki_, kj_] =
        Table[B[[i + truth[ki <= i], j + truth[kj <= j]]],
            {i, 1, n - 1}, {j, 1, n - 1}];

        cut[A_, set_, PM_] :=
        Block[{k, n, Lset},
          PM = A;
         n = Dimensions[A][[1]];
          Lset = Length[set];
         Do[PM = M[PM, n + 1 - k, set[[k]], set[[k]]], {k, 1, Lset}];
          Print[MatrixForm[PM]]; Print[Det[PM]]];
```

The rest of the files in the Appendix contain the data used to complete the proofs for the classification of the graph of the Double Triangle regarding $P_{0,1}$-completion, the classification of the digraphs $q = 5$, $n = 7$; $q = 5$, $n = 8$; $q = 5$, $n = 9$; $q = 6$, $n = 4$; $q = 6$, $n = 5$; $q = 6$, $n = 6$; $q = 6$, $n = 7$ regarding sign symmetric $P_{0,1}$-completion, and the classification of $q = 7$, $n = 2$ regarding weakly sign symmetric $P_{0,1}$-completion.

Mathematica Files for $q = 7$, $n = 2$ regarding weakly sign symmetric completion

p=4 q=7 n=2 has wssP0,1-completion

Here is how the pattern matrix for q7n2 looks after the diagonal entries have been made equal to 1 by multiplying by a positive diagonal matrix. Label Harary's graph as 4321 beginning in the upper left-hand corner of the graph, and going clockwise around the graph.

```
In[2]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, a34, a41, x42, a43];

        A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
           {x31, x32, 1, a34}, {a41, x42, a43, 1}};

        MatrixForm[
         A]
```

$$Out[2]= \begin{pmatrix} 1 & a12 & x13 & a14 \\ a21 & 1 & a23 & x24 \\ x31 & x32 & 1 & a34 \\ a41 & x42 & a43 & 1 \end{pmatrix}$$

Case 1: a12*a23*a34*a41≤0

```
In[3]:= x13 = a14 * a43; x24 = a21 * a14; x31 = 0; x32 = 0; x42 = 0;

        MatrixForm[A]
```

$$Out[3]= \begin{pmatrix} 1 & a12 & a14\,a43 & a14 \\ a21 & 1 & a23 & a14\,a21 \\ 0 & 0 & 1 & a34 \\ a41 & 0 & a43 & 1 \end{pmatrix}$$

```
In[4]:= Det[A]

Out[4]= 1 - a12 a21 - a14 a41 + a12 a14 a21 a41 -
         a12 a23 a34 a41 - a34 a43 + a12 a21 a34 a43 + a14 a34 a41 a43

In[5]:= Expand[Det[A] - ((1 - a12 * a21) (1 - a14 * a41) (1 - a34 * a43) +
              a12 * a21 * a14 * a41 * a34 * a43 - a12 * a23 * a34 * a41)]
Out[5]= 0
```

3 x 3 minors:

```
In[6]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & a12 & a14\,a43 \\ a21 & 1 & a23 \\ 0 & 0 & 1 \end{pmatrix}$$

1-a12 a21

$$\begin{pmatrix} 1 & a12 & a14 \\ a21 & 1 & a14\,a21 \\ a41 & 0 & 1 \end{pmatrix}$$

1-a12 a21-a14 a41+a12 a14 a21 a41

$$\begin{pmatrix} 1 & a14\,a43 & a14 \\ 0 & 1 & a34 \\ a41 & a43 & 1 \end{pmatrix}$$

1-a14 a41-a34 a43+a14 a34 a41 a43

$$\begin{pmatrix} 1 & a23 & a14\,a21 \\ 0 & 1 & a34 \\ 0 & a43 & 1 \end{pmatrix}$$

1-a34 a43

2 x 2 minors:

```
In[7]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & a34 \\ a43 & 1 \end{pmatrix}$$

1-a34 a43

$$\begin{pmatrix} 1 & a14\,a21 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14 \\ a41 & 1 \end{pmatrix}$$

1-a14 a41

$$\begin{pmatrix} 1 & a23 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14\,a43 \\ 0 & 1 \end{pmatrix}$$

```
1
( 1    a12 )
( a21  1   )
1-a12 a21
```

Case 2: a12*a23*a34*a41>0

```
In[8]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, a34, a41, x42, a43];

        A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
           {x31, x32, 1, a34}, {a41, x42, a43, 1}}; x13 = a14 * a43;
        x24 = a21 * a14 + a23 * a34; x31 = 0; x32 = 0; x42 = 0;

        MatrixForm[A]

         ( 1     a12   a14 a43    a14               )
Out[8]=  ( a21   1     a23        a14 a21 + a23 a34 )
         ( 0     0     1          a34               )
         ( a41   0     a43        1                 )

In[9]:= Det[A]

Out[9]= 1 - a12 a21 - a14 a41 + a12 a14 a21 a41 -
          a34 a43 + a12 a21 a34 a43 + a14 a34 a41 a43

In[10]:= Expand[Det[A] - ((1 - a12 * a21) (1 - a14 * a41) (1 - a34 * a43) +
                 a12 * a21 * a14 * a41 * a34 * a43)]
Out[10]= 0
```

3 x 3 minors

```
In[11]:= Clear[PA]; cut[A, {4}, PA];
( 1    a12   a14 a43 )
( a21  1     a23     )
( 0    0     1       )
1-a12 a21

In[12]:= Clear[PA]; cut[A, {3}, PA];
         Expand[Det[PA] - ((1 - a21 * a12) (1 - a14 * a41) + a12 * a23 * a34 * a41)]
```

$$\begin{pmatrix} 1 & a12 & a14 \\ a21 & 1 & a14\,a21 + a23\,a34 \\ a41 & 0 & 1 \end{pmatrix}$$

1 - a12 a21 - a14 a41 + a12 a14 a21 a41 + a12 a23 a34 a41

```
Out[12]= 0
```

```
In[13]:= Clear[PA]; cut[A, {2}, PA]; Expand[Det[PA] - (1 - a14 * a41) (1 - a34 * a43)]
```

$$\begin{pmatrix} 1 & a14\,a43 & a14 \\ 0 & 1 & a34 \\ a41 & a43 & 1 \end{pmatrix}$$

1 - a14 a41 - a34 a43 + a14 a34 a41 a43

```
Out[13]= 0
```

```
In[14]:= Clear[PA]; cut[A, {1}, PA]
```

$$\begin{pmatrix} 1 & a23 & a14\,a21 + a23\,a34 \\ 0 & 1 & a34 \\ 0 & a43 & 1 \end{pmatrix}$$

1 - a34 a43

2 x 2 minors

```
In[15]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & a34 \\ a43 & 1 \end{pmatrix}$$

1 - a34 a43

$$\begin{pmatrix} 1 & a14\,a21 + a23\,a34 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14 \\ a41 & 1 \end{pmatrix}$$

1 - a14 a41

$$\begin{pmatrix} 1 & a23 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14\,a43 \\ 0 & 1 \end{pmatrix}$$

```
1
( 1    a12 )
( a21  1   )
1-a12 a21
```

Mathematica Files for $q = 5$, $n = 7$ regarding sign symmetric $P_{0,1}$-completion

p=4 q=5 n=7 has ssP0,1-completion

Label the digraph as 3412 starting in the upper left hand corner and goingclockwise. Here is how the pattern matrix looks after the diagonalentries have been made equal to 1 by multiplying by a positive diagonalmatrix.

```
In[16]:= Clear[a12, x13, x14, a21, a23, x24, x31, x32, a34, a41, x42, x43];
         Clear[e, f, g]

         A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
            {x31, x32, 1, a34}, {a41, x42, x43, 1}};

         MatrixForm[
          A]

          ( 1    a12  x13  x14 )
Out[16]= ( a21  1    a23  x24 )
          ( x31  x32  1    a34 )
          ( a41  x42  x43  1   )
```

Case I: a12a23a34a41>0

WLOG (by diagonal similarity) can assume a12, a23, a34 = 1, and thus a41 > 0 also.

We choose x32 small and + so that 1-a23e>0 (strictly), call it e.

We choose x43 small and + so that 1-a34x43>0 (strictly), call it e.

We choose x14 small and + so that 1-a34x14>0 (strictly), call it e.

Note e > 0 and can be chosen as small as desired.

We choose x13=a12a23 = 1, x24=a23a34 = 1, x31=x32a21 = e a21, and x42=x43x32 =eˆ2.

```
In[17]:= Clear[a12, x13, x14, a21, a23, x24, x31, x32, a34, a41, x42, x43];
         Clear[e, f, g]

         A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
            {x31, x32, 1, a34}, {a41, x42, x43, 1}}; x13 = 1;
         x31 = a21 * x32; x14 = e; x24 = 1; x42 = x32 * x43; x32 = e;
         x43 = e; a12 = 1; a23 = 1; a34 = 1;

         MatrixForm[A]
```

Out[17]=

$$\begin{pmatrix} 1 & 1 & 1 & e \\ a21 & 1 & 1 & 1 \\ a21\,e & e & 1 & 1 \\ a41 & e^2 & e & 1 \end{pmatrix}$$

```
In[18]:= Det[A]
```

Out[18]= $1 - a21 + a41 - 2\,e + 2\,a21\,e - 2\,a41\,e + e^2 - 2\,a21\,e^2 + a41\,e^2 + 2\,a21\,e^3 - a21\,e^4$

OK if e is small enough because 1-a12a21≥0 and a41 = a12a23a34a41>0. The rest of the terms have e’s in them, so these terms need only be less than a12a23a34a41 for Det[A]>0

3 x 3 minors:

```
In[19]:= Clear[PA]; cut[A, {4}, PA]
```

$$\begin{pmatrix} 1 & 1 & 1 \\ a21 & 1 & 1 \\ a21\,e & e & 1 \end{pmatrix}$$

1-a21-e+a21 ***e***

```
In[20]:= Expand[Det[PA] - (1 - a21) (1 - e)]
```

Out[20]= 0

OK if e small

```
In[21]:= Clear[PA]; cut[A, {3}, PA];
```

$$\begin{pmatrix} 1 & 1 & e \\ a21 & 1 & 1 \\ a41 & e^2 & 1 \end{pmatrix}$$

1-a21+a41-a41 e-e^2+a21 e^3

OK if e is small enough because 1-a12a21 ≥0 and a41 > 0.

```
In[22]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & 1 & e \\ a21\,e & 1 & 1 \\ a41 & e & 1 \end{pmatrix}$$

1+a41-e-a21 e-a41 e+a21 e^3

OK if e is small enough.

```
In[23]:= Clear[PA]; cut[A, {1}, PA];
```

$$\begin{pmatrix} 1 & 1 & 1 \\ e & 1 & 1 \\ e^2 & e & 1 \end{pmatrix}$$

1-2 e+e^2

OK if e is small enough.

2 x 2 minors:

The following are all OK if e is small enough.

```
In[24]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 1 \\ e & 1 \end{pmatrix}$$

1-e

$$\begin{pmatrix} 1 & 1 \\ e^2 & 1 \end{pmatrix}$$

1-e^2

$$\begin{pmatrix} 1 & e \\ a41 & 1 \end{pmatrix}$$

1-a41 e

$$\begin{pmatrix} 1 & 1 \\ e & 1 \end{pmatrix}$$

1-e

$$\begin{pmatrix} 1 & 1 \\ a21\,e & 1 \end{pmatrix}$$

1-a21 *e*

$$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$$

1-a21

Case II: a12a23a34a41 < 0.

Use a diagonal similarity to make the superdiagonal entries all 1. Then a41 < 0, replace a41 by by -b so the symbol b is positive

set x32=x43=e, x14=-e, assign other values in pairs, f, g small positive

```
In[25]:= Clear[a12, x13, x14, a21, a23, x24, x31, x32, a34, a41, x42, x43];
         Clear[e, f, g];

         A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
            {x31, x32, 1, a34}, {a41, x42, x43, 1}}; a12 = 1;
         a23 = 1; a41 = -b; a34 = 1; x24 = -g; x42 = -g; x32 = e;
         x43 = e; x14 = -e; x13 = f; x31 = f;




         MatrixForm[A]
```

Out[25]=
$$\begin{pmatrix} 1 & 1 & f & -e \\ a21 & 1 & 1 & -g \\ f & e & 1 & 1 \\ -b & -g & e & 1 \end{pmatrix}$$

First let e equal zero and then choose f and g positive & small enough to make each new minor positive.

Then perturb so that e is positive and yet small enough to keep all new minors nonnegative.

```
In[26]:= e = 0;

         MatrixForm[A]
```

Out[26]= $\begin{pmatrix} 1 & 1 & f & 0 \\ a21 & 1 & 1 & -g \\ f & 0 & 1 & 1 \\ -b & -g & 0 & 1 \end{pmatrix}$

```
In[27]:= Det[A]
```

Out[27]= $1 - a21 + b + f - b f - f^2 - g + b g + a21 f g - g^2 + f^2 g^2$

OK, this minor is positive if the sum of all of the terms which contain an f or a g is less in magnitude than a41.

3 x 3 minors:

```
In[28]:= Clear[PA]; cut[A, {4}, PA];
```

$\begin{pmatrix} 1 & 1 & f \\ a21 & 1 & 1 \\ f & 0 & 1 \end{pmatrix}$

$1-a21+f-f^2$

Since 1-a21 is nonnegative, as long as f is less than 1, this minor is positive.

```
In[29]:= Clear[PA]; cut[A, {3}, PA];
```

$\begin{pmatrix} 1 & 1 & 0 \\ a21 & 1 & -g \\ -b & -g & 1 \end{pmatrix}$

$1-a21+b\,g-g^2$

Since 1-a21is nonnegative, this minor is positive as long as g is less than a41.

```
In[30]:= Clear[PA]; cut[A, {2}, PA];
```

$\begin{pmatrix} 1 & f & 0 \\ f & 1 & 1 \\ -b & 0 & 1 \end{pmatrix}$

$1-b\,f-f^2$

OK if f is small enough.

```
In[31]:= Clear[PA]; cut[A, {1}, PA];
```

$$\begin{pmatrix} 1 & 1 & -g \\ 0 & 1 & 1 \\ -g & 0 & 1 \end{pmatrix}$$

$1-g-g^2$

OK if g is small enough

2 x 2 minors:

```
In[32]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & -g \\ -g & 1 \end{pmatrix}$$

$1-g^2$

$$\begin{pmatrix} 1 & 0 \\ -b & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & f \\ f & 1 \end{pmatrix}$$

$1-f^2$

$$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$$

1-a21

All of these 2 x 2s are positive for small f, g except the original minor

Since all new minors are positive, we can perturb the zero entries opposite nonzero entries (by perturbing e to be positive) and yet keep all the minors nonnegative.

Case III: a12a23a34a41 = 0

Then at least one of the off-diagonal entries are zero.

SubCase III A: a12=0

Then a21 is also zero by sign symmetry. Let x14 = e, x 32 = f, x 43 = g be small enough and of same sign as a41, a23, a34 respectively, all others 0.

```
In[33]:= Clear[a12, x13, x14, a21, a23, x24, x31, x32, a34, a41, x42, x43];
         Clear[e, f, g]

         A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
            {x31, x32, 1, a34}, {a41, x42, x43, 1}}; a12 = 0; a21 = 0;
         x13 = 0; x14 = e; x32 = f; x43 = g; x31 = 0; x42 = 0; x24 = 0;

         MatrixForm[A]
```

$$Out[33]= \begin{pmatrix} 1 & 0 & 0 & e \\ 0 & 1 & a23 & 0 \\ 0 & f & 1 & a34 \\ a41 & 0 & g & 1 \end{pmatrix}$$

Then all new minors are OK if e, f, g small enough:

```
In[34]:= Det[A]
```

Out[34]= $1 - a41\,e - a23\,f + a23\,a41\,e\,f - a34\,g$

3 x 3 minors:

```
In[35]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a23 \\ 0 & f & 1 \end{pmatrix}$$

1-a23 ***f***

$$\begin{pmatrix} 1 & 0 & e \\ 0 & 1 & 0 \\ a41 & 0 & 1 \end{pmatrix}$$

1-a41 ***e***

$$\begin{pmatrix} 1 & 0 & e \\ 0 & 1 & a34 \\ a41 & g & 1 \end{pmatrix}$$

1-a41 ***e***-a34 ***g***

$$\begin{pmatrix} 1 & a23 & 0 \\ f & 1 & a34 \\ 0 & g & 1 \end{pmatrix}$$

1-a23 ***f***-a34 ***g***

2 x 2 minors:

```
In[36]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & a34 \\ g & 1 \end{pmatrix}$$

1-a34 ***g***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & e \\ a41 & 1 \end{pmatrix}$$

1-a41 ***e***

$$\begin{pmatrix} 1 & a23 \\ f & 1 \end{pmatrix}$$

1-a23 ***f***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

SucCase III B: a34=0;

Then x43 must also be zero by sign symmetry.

Sub-subCase IIIB i: a12*a21<>1

Let x14 = e, x32 = f be small enough and of same sign as a41, a23, set all others to 0.

```
In[37]:= Clear[a12, x13, x14, a21, a23, x24, x31, x32, a34, a41, x42, x43];
         Clear[e, f, g];

         A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
            {x31, x32, 1, a34}, {a41, x42, x43, 1}}; a34 = 0;
         x13 = 0; x14 = e; x32 = f; x43 = 0; x31 = 0; x42 = 0; x24 = 0;

         MatrixForm[A]
```

Out[37]= $\begin{pmatrix} 1 & a12 & 0 & e \\ a21 & 1 & a23 & 0 \\ 0 & f & 1 & 0 \\ a41 & 0 & 0 & 1 \end{pmatrix}$

Then all new minors are OK if e, f small enough:

```
In[38]:= Det[A]
```

Out[38]= $1 - a12\, a21 - a41\, e - a23\, f + a23\, a41\, e\, f$

3 x 3 minors:

```
In[39]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$\begin{pmatrix} 1 & a12 & 0 \\ a21 & 1 & a23 \\ 0 & f & 1 \end{pmatrix}$

$1 - a12\, a21 - a23\, \boldsymbol{f}$

$\begin{pmatrix} 1 & a12 & e \\ a21 & 1 & 0 \\ a41 & 0 & 1 \end{pmatrix}$

$1 - a12\, a21 - a41\, \boldsymbol{e}$

$\begin{pmatrix} 1 & 0 & e \\ 0 & 1 & 0 \\ a41 & 0 & 1 \end{pmatrix}$

$1 - a41\, \boldsymbol{e}$

$\begin{pmatrix} 1 & a23 & 0 \\ f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

1-a23 ***f***

2 x 2 minors:

```
In[40]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1

$\begin{pmatrix} 1 & e \\ a41 & 1 \end{pmatrix}$

1-a41 ***e***

$\begin{pmatrix} 1 & a23 \\ f & 1 \end{pmatrix}$

1-a23 ***f***

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1

$\begin{pmatrix} 1 & a12 \\ a21 & 1 \end{pmatrix}$

1-a12 a21

Sub-subCase IIIB ii: a12*a21=1

Since a12*a21=1, without loss of generality, we can set them both equal to 1.

Set x13=x31=0.

Set x24=x42=g with g=sign a41*min(|a41|,1)/2, so a41g-gˆ2>0 or a41=g=0.

set x14 =s and x43 = t very small, matching signs of a41, a34.

```
In[41]:= Clear[a12, x13, x14, a21, a23, x24, x31, x32, a34, a41, x42, x43];
         Clear[e, f, g, s, t]

         A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
            {x31, x32, 1, a34}, {a41, x42, x43, 1}}; a12 = 1;
         a21 = 1; a23 = 0; x32 = 0; x24 = g; x42 = g;
         x14 = s; x43 = t; x13 = 0; x31 = 0;

         MatrixForm[A]
```

Out[41]=
$$\begin{pmatrix} 1 & 1 & 0 & s \\ 1 & 1 & 0 & g \\ 0 & 0 & 1 & a34 \\ a41 & g & t & 1 \end{pmatrix}$$

Then using $a41\, g - g^2 > 0$ and $|g| < 1$, if s and t are small enough, all minors are nonnegative:

```
In[42]:= Det[A]
```

Out[42]= $a41\, g - g^2 - a41\, s + g\, s$

3 x 3 minors:

```
In[43]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

0

$$\begin{pmatrix} 1 & 1 & s \\ 1 & 1 & g \\ a41 & g & 1 \end{pmatrix}$$

$a41\, g - g^2 - a41\, s + g\, s$

$$\begin{pmatrix} 1 & 0 & s \\ 0 & 1 & a34 \\ a41 & t & 1 \end{pmatrix}$$

$1 - a41\, s - a34\, t$

$$\begin{pmatrix} 1 & 0 & g \\ 0 & 1 & a34 \\ g & t & 1 \end{pmatrix}$$

$1 - g^2 - a34\, t$

2 x 2 minors:

```
In[44]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$\begin{pmatrix} 1 & a34 \\ t & 1 \end{pmatrix}$

$1 - a34\, t$

$\begin{pmatrix} 1 & g \\ g & 1 \end{pmatrix}$

$1 - g^2$

$\begin{pmatrix} 1 & s \\ a41 & 1 \end{pmatrix}$

$1 - a41\, s$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

1

$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

0

SubCase IIIC: a23=0

Then x32 is also zero by sign symmetry. Set x14=e small enough and of same sign as a41, x34=f small and same sign a34. Set x42=a12*a41, x24=s small same sign a12*a41, all others 0

```
In[45]:= Clear[a12, x13, x14, a21, a23, x24, x31, x32, a34, a41, x42, x43];
         Clear[e, f, g, s];

         A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
            {x31, x32, 1, a34}, {a41, x42, x43, 1}}; a23 = 0;
         x42 = a41 * a12; x14 = e; x32 = 0; x43 = f; x24 = s; x13 = 0; x31 = 0;

         MatrixForm[A]
```

Out[45]= $\begin{pmatrix} 1 & a12 & 0 & e \\ a21 & 1 & 0 & s \\ 0 & 0 & 1 & a34 \\ a41 & a12\, a41 & f & 1 \end{pmatrix}$

```
In[46]:= Det[A]
```

Out[46]= 1 − a12 a21 − a41 e + a12 a21 a41 e − a34 f + a12 a21 a34 f

```
In[47]:= Expand[Det[A] - (1 - a12 * a21) (1 - a41 * e - a34 * f)]
```

Out[47]= 0

OK if e, f small enough.

3 x 3 minors:

The following are all OK if e, f, s small enough.

```
In[48]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$\begin{pmatrix} 1 & a12 & 0 \\ a21 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

1 - a12 a21

$\begin{pmatrix} 1 & a12 & e \\ a21 & 1 & s \\ a41 & a12\, a41 & 1 \end{pmatrix}$

1 - a12 a21 - a41 *e* + a12 a21 a41 *e*

$\begin{pmatrix} 1 & 0 & e \\ 0 & 1 & a34 \\ a41 & f & 1 \end{pmatrix}$

```
1-a41 e-a34 f
(   1      0    s  )
(   0      1   a34 )
( a12 a41  f    1  )
1-a34 f-a12 a41 s
```

2 x 2 minors:

The following are all OK if e, f, s small enough.

```
In[49]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
( 1  a34 )
( f   1  )
1-a34 f
(   1      s )
( a12 a41  1 )
1-a12 a41 s
(  1   e )
( a41  1 )
1-a41 e
( 1  0 )
( 0  1 )
1
( 1  0 )
( 0  1 )
1
(  1   a12 )
( a21   1  )
1-a12 a21
```

SubCase IIID: a41=0 (similar to IIIC a23=0)

Then x14 is also zero by sign symmetry. Set x32=e small enough and of same sign as a23, x43=f small and same sign a34. Set x13=a12*a23, x31=s small same sign a12*a23, all others 0

```
In[50]:= Clear[a12, x13, x14, a21, a23, x24, x31, x32, a34, a41, x42, x43];
         Clear[e, f, g, s];

         A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
            {x31, x32, 1, a34}, {a41, x42, x43, 1}}; a41 = 0;
         x13 = a12 * a23; x14 = 0; x32 = e; x43 = f; x31 = s; x42 = 0; x24 = 0;

         MatrixForm[A]
```

Out[50]=

$$\begin{pmatrix} 1 & a12 & a12\,a23 & 0 \\ a21 & 1 & a23 & 0 \\ s & e & 1 & a34 \\ 0 & 0 & f & 1 \end{pmatrix}$$

All are OK if e, f small enough.

```
In[51]:= Det[A]
```

Out[51]= 1 − a12 a21 − a23 e + a12 a21 a23 e − a34 f + a12 a21 a34 f

```
In[52]:= Expand[Det[A] - (1 - a12 * a21) (1 - a23 * e - a34 * f)]
```

Out[52]= 0

3 x 3 minors:

```
In[53]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & a12 & a12\,a23 \\ a21 & 1 & a23 \\ s & e & 1 \end{pmatrix}$$

1 - a12 a21 - a23 ***e*** + a12 a21 a23 ***e***

$$\begin{pmatrix} 1 & a12 & 0 \\ a21 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1 - a12 a21

$$\begin{pmatrix} 1 & a12\,a23 & 0 \\ s & 1 & a34 \\ 0 & f & 1 \end{pmatrix}$$

1 - a34 ***f*** - a12 a23 ***s***

$$\begin{pmatrix} 1 & a23 & 0 \\ e & 1 & a34 \\ 0 & f & 1 \end{pmatrix}$$

1 - a23 ***e*** - a34 ***f***

2 x 2 minors:

```
In[54]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & a34 \\ f & 1 \end{pmatrix}$$

1 - a34 ***f***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a23 \\ e & 1 \end{pmatrix}$$

1 - a23 ***e***

$$\begin{pmatrix} 1 & a12\,a23 \\ s & 1 \end{pmatrix}$$

1 - a12 a23 ***s***

$$\begin{pmatrix} 1 & a12 \\ a21 & 1 \end{pmatrix}$$

1 - a12 a21

Mathematica Files for $q = 5$, $n = 8$ regarding sign symmetric $P_{0,1}$-completion

p=4 q=5 n=8 has ssP0,1-completion

Label the digraph as 4321 starting in the upper left hand corner and going clockwise. Here is how the pattern matrix looks after the diagonal

entries have been made equal to 1 by multiplying by a positive diagonal matrix.

```
In[55]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];

         A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
            {x31, x32, 1, x34}, {x41, x42, a43, 1}};

         MatrixForm[
          A]
```

Out[55]=

1	a12	x13	a14
a21	1	a23	x24
x31	x32	1	x34
x41	x42	a43	1

Case 1: a21*a23*a43*a14>0

WLOG (by diagonal similarity), we can assume a12, a23, a43 = 1, and thus a41>0 also.

Let x34 = e >0 and small enough so that 1-a43e ≥ 0.

Let x32 = e > 0 and small enough so that both 1-a23e≥0 and 1-a14*a21*a43*e≥0.

Let x13=a12*a23, x31=a21*e, x24=a23*e, x42=e*a43, and x41=a21*e*a43.

```
In[56]:= Clear[e, f, g];

         a12 = 1; a23 = 1; a43 = 1; x13 = a12 * a23; x31 = a21 * x32;
         x24 = a23 * x34; x42 = x32 * a43; x41 = a21 * x32 * a43; x32 = e; x34 = e;

         MatrixForm[A]
```

Out[56]=

1	1	1	a14
a21	1	1	e
a21 e	e	1	e
a21 e	e	1	1

```
In[57]:= Det[A]
```

Out[57]= $1 - a21 - 2e + 2\,a21\,e + e^2 - a21\,e^2$

```
In[58]:= Expand[Det[A] - ((1 - a21 * a12) (1 - e)^2)]
```

Out[58]= 0

3 x 3 minors

```
In[59]:= Clear[PA]; cut[A, {4}, PA]; Expand[Det[PA] - (1 - a12 * a21) (1 - e)]
```

$$\begin{pmatrix} 1 & 1 & 1 \\ a21 & 1 & 1 \\ a21\,e & e & 1 \end{pmatrix}$$

$1 - a21 - e + a21\, e$

```
Out[59]= 0
```

```
In[60]:= Clear[PA]; cut[A, {3}, PA]; Expand[Det[PA] - (1 - a12 * a21) (1 - e^2)]
```

$$\begin{pmatrix} 1 & 1 & a14 \\ a21 & 1 & e \\ a21\,e & e & 1 \end{pmatrix}$$

$1 - a21 - e^2 + a21\, e^2$

```
Out[60]= 0
```

```
In[61]:= Clear[PA]; cut[A, {2}, PA]; Expand[Det[PA] - (1 - a43 * x34) (1 - a21 * e)]
```

$$\begin{pmatrix} 1 & 1 & a14 \\ a21\,e & 1 & e \\ a21\,e & 1 & 1 \end{pmatrix}$$

$1 - e - a21\, e + a21\, e^2$

```
Out[61]= 0
```

```
In[62]:= Clear[PA]; cut[A, {1}, PA]; Expand[Det[PA] - (1 - e)^2]
```

$$\begin{pmatrix} 1 & 1 & e \\ e & 1 & e \\ e & 1 & 1 \end{pmatrix}$$

$1 - 2\, e + e^2$

```
Out[62]= 0
```

2 x 2 minors

all OK or original for e small

```
In[63]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & e \\ 1 & 1 \end{pmatrix}$$

$1-e$

$\begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix}$

$1-e^2$

$\begin{pmatrix} 1 & a14 \\ a21\,e & 1 \end{pmatrix}$

$1-a14\,a21\,e$

$\begin{pmatrix} 1 & 1 \\ e & 1 \end{pmatrix}$

$1-e$

$\begin{pmatrix} 1 & 1 \\ a21\,e & 1 \end{pmatrix}$

$1-a21\,e$

$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$

$1-a21$

Case 2: a21*a23*a43*a14 < 0

Note this means that a21, a23, a43, a14, x32, x34, x41 are all nonzero.

Do a similarity operation that makes a12=1, a23=1, and a43=1.

Then a41 < 0. We replace the symbol a41 with the symbol -b so that all the symbols in the matrix are positive.

Case 2a: a21<>1 (and a21*a23*a43*a14 < 0)

Let x34=e, x32=e, x24=e, x42=e, x41=-e, x13=0, x31=0.

Note 1 - a21 > 0.

Choose e positive and as small as needed. Since there are finitely many continuous minors to satisfy, the following minors can be made nonnegative.

```
In[64]:= Clear[b, e, f, g];

         a12 = 1; a23 = 1; a43 = 1; x34 = e; x32 = e; x24 = e;
         x42 = e; x41 = -e; x13 = 0; x31 = 0; a14 = -b;

         MatrixForm[A]
```

Out[64]=
$$\begin{pmatrix} 1 & 1 & 0 & -b \\ a21 & 1 & 1 & e \\ 0 & e & 1 & e \\ -e & e & 1 & 1 \end{pmatrix}$$

Since 1-a21 > 0, all OK if e is small enough.

```
In[65]:= Det[A]
```

Out[65]= $1 - a21 - 2\,e + a21\,e - b\,e + e^2 + b\,e^2$

3 x 3 minors:

```
In[66]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & 1 & 0 \\ a21 & 1 & 1 \\ 0 & e & 1 \end{pmatrix}$$

1-a21-e

$$\begin{pmatrix} 1 & 1 & -b \\ a21 & 1 & e \\ -e & e & 1 \end{pmatrix}$$

$1\text{-}a21\text{-}b\,e\text{-}a21\,b\,e\text{-}2\,e^2$

$$\begin{pmatrix} 1 & 0 & -b \\ 0 & 1 & e \\ -e & 1 & 1 \end{pmatrix}$$

$1\text{-}e\text{-}b\,e$

$$\begin{pmatrix} 1 & 1 & e \\ e & 1 & e \\ e & 1 & 1 \end{pmatrix}$$

$1\text{-}2\,e+e^2$

2 x 2 minors:

```
In[67]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & e \\ 1 & 1 \end{pmatrix}$$

1-e

$$\begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix}$$

$1-e^2$

$$\begin{pmatrix} 1 & -b \\ -e & 1 \end{pmatrix}$$

$1-b\,e$

$$\begin{pmatrix} 1 & 1 \\ e & 1 \end{pmatrix}$$

1-e

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$$

1-a21

Case 2b: a21 =1 (and a12=a23=a43=1 and a14 = -b, b>0)

Let x31=f, x13=f, x24=-g, x42=-g, x32=f, x41=-e, and x34=e, with e,f,g > 0.

Note this means that a21, a23, a43, a14, x32, x34, x41 are all nonzero. The matrix below has been set up so that all variables are positive.

```
In[68]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];
         Clear[e, f, g, b, c];

          a12 = 1; a21 = 1; a23 = 1; x31 = f; x13 = f; a43 = 1;
         x24 = -g; x42 = -g; x32 = f; x41 = -e; x34 = e; a14 = -b;

         MatrixForm[A]
```

Out[68]= $\begin{pmatrix} 1 & 1 & f & -b \\ 1 & 1 & 1 & -g \\ f & f & 1 & e \\ -e & -g & 1 & 1 \end{pmatrix}$

Choose f and g first so that the minors are positive when e=0. That is, choose f first and then g so that each are the smallest of that necessary to make each of the minors positive. Then perturb e to make e positive, but not so large that the minors become nonnegative.

```
In[69]:= e = 0;

MatrixForm[A]
```

Out[69]= $\begin{pmatrix} 1 & 1 & f & -b \\ 1 & 1 & 1 & -g \\ f & f & 1 & 0 \\ 0 & -g & 1 & 1 \end{pmatrix}$

```
In[70]:= Det[A]
```

Out[70]= $b\,g - b\,f\,g - g^2 + f^2\,g^2$

```
In[71]:= Expand[Det[A] - (g(b(1 - f) - g(1 - f^2)))]
```

Out[71]= 0

Fix f<1. Then choose g so that g(1-f^2) is less than b(1-f). Then Det[A]>0.

3 x 3 minors:

```
In[72]:= Clear[PA]; cut[A, {4}, PA];
```

$\begin{pmatrix} 1 & 1 & f \\ 1 & 1 & 1 \\ f & f & 1 \end{pmatrix}$

0

```
In[73]:= Clear[PA]; cut[A, {3}, PA]; Expand[Det[PA] - g(b - g)]
```

$\begin{pmatrix} 1 & 1 & -b \\ 1 & 1 & -g \\ 0 & -g & 1 \end{pmatrix}$

$b\,g - g^2$

Out[73]= 0

OK if g < b

In[74]:= **Clear[PA]; cut[A, {2}, PA];**

$$\begin{pmatrix} 1 & f & -b \\ f & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$1 - b\,f - f^2$

OK if f is small enough.

In[75]:= **Clear[PA]; cut[A, {1}, PA];**

$$\begin{pmatrix} 1 & 1 & -g \\ f & 1 & 0 \\ -g & 1 & 1 \end{pmatrix}$$

$1 - f - f\,g - g^2$

OK if f and g are small enough.

2 x 2 minors:

The following are all OK if f, g are small enough.

In[76]:= **Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i − 1}], {i, 1, 4}]**

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & -g \\ -g & 1 \end{pmatrix}$$

$1 - g^2$

$$\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 1 \\ f & 1 \end{pmatrix}$$

1-f

$$\begin{pmatrix} 1 & f \\ f & 1 \end{pmatrix}$$

$1 - f^2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

0

Case 3: a21*a23*a43*a14 =0

This implies at least one of the off-diagonal entries are zero.

Case 3a: a12=a21=0

Set x41=e;x32=f;x34=g; and all others 0

```
In[77]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];
         Clear[e, f, g];

         A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
            {x31, x32, 1, x34}, {x41, x42, a43, 1}}; a12 = 0; a21 = 0;
         x31 = 0; x13 = 0; x24 = 0; x42 = 0; x41 = e; x32 = f; x34 = g;

         MatrixForm[A]
```

Out[77]=
$$\begin{pmatrix} 1 & 0 & 0 & a14 \\ 0 & 1 & a23 & 0 \\ 0 & f & 1 & g \\ e & 0 & a43 & 1 \end{pmatrix}$$

All minors OK if e, f, g small enough:

```
In[78]:= Det[A]
```

Out[78]= $1 - a14\, e - a23\, f + a14\, a23\, e\, f - a43\, g$

3 x 3 minors:

```
In[79]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & a23 \\ 0 & f & 1 \end{pmatrix}$$

1-a23 ***f***

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & 0 \\ e & 0 & 1 \end{pmatrix}$$

1-a14 ***e***

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & g \\ e & a43 & 1 \end{pmatrix}$$

1-a14 ***e***-a43 ***g***

$$\begin{pmatrix} 1 & a23 & 0 \\ f & 1 & g \\ 0 & a43 & 1 \end{pmatrix}$$

1-a23 ***f***-a43 ***g***

2 x 2 minors:

```
In[80]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & g \\ a43 & 1 \end{pmatrix}$$

1-a43 ***g***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14 \\ e & 1 \end{pmatrix}$$

1-a14 ***e***

$$\begin{pmatrix} 1 & a23 \\ f & 1 \end{pmatrix}$$

1-a23 ***f***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

Case 3b: a23=0 => x32 = 0

Case3bi: a12*a21<>1

So 1-a21*a21 > 0.

Set x41 = e, x34 = f, all others 0

```
In[81]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];
         Clear[e, f, g];

         A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
            {x31, x32, 1, x34}, {x41, x42, a43, 1}}; a23 = 0;
         x31 = 0; x13 = 0; x24 = 0; x42 = 0; x32 = 0; x41 = e; x34 = f;

         MatrixForm[A]
```

Out[81]= $\begin{pmatrix} 1 & a12 & 0 & a14 \\ a21 & 1 & 0 & 0 \\ 0 & 0 & 1 & f \\ e & 0 & a43 & 1 \end{pmatrix}$

Since 1-a12 a21 > 0, all minors OK if e, f is small enough.

```
In[82]:= Det[A]
```

Out[82]= 1 - a12 a21 - a14 e - a43 f + a12 a21 a43 f

3 x 3 minors:

```
In[83]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$\begin{pmatrix} 1 & a12 & 0 \\ a21 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

1 - a12 a21

$\begin{pmatrix} 1 & a12 & a14 \\ a21 & 1 & 0 \\ e & 0 & 1 \end{pmatrix}$

```
1-a12 a21-a14 e
```

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & f \\ e & a43 & 1 \end{pmatrix}$$

```
1-a14 e-a43 f
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & a43 & 1 \end{pmatrix}$$

```
1-a43 f
```

2 x 2 minors:

```
In[84]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & f \\ a43 & 1 \end{pmatrix}$$

```
1-a43 f
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
1
```

$$\begin{pmatrix} 1 & a14 \\ e & 1 \end{pmatrix}$$

```
1-a14 e
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
1
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
1
```

$$\begin{pmatrix} 1 & a12 \\ a21 & 1 \end{pmatrix}$$

```
1-a12 a21
```

Case3bii: a12*a21=1 (and a23=0 => x32 = 0)

Without loss of generality, we can make a12=a21=1;

set x31=0; x13=0; x24=g; x42=g; x32=0; x41=e; x34=f;

```
In[85]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];
         Clear[e, f, g];

         A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
            {x31, x32, 1, x34}, {x41, x42, a43, 1}}; a12 = 1; a21 = 1;
         a23 = 0; x31 = 0; x13 = 0; x24 = g; x42 = g; x32 = 0; x41 = e; x34 = f;

         MatrixForm[A]
```

Out[85]= $\begin{pmatrix} 1 & 1 & 0 & a14 \\ 1 & 1 & 0 & g \\ 0 & 0 & 1 & f \\ e & g & a43 & 1 \end{pmatrix}$

Temporarily set e=0.

Choose f small enough so that 1-a43 f > 0.

If a14 = 0, set g=0, all are minors nonnegative

If a14 not 0, make all minors that contain e positive so we can perturb e.

```
In[86]:= e = 0; MatrixForm[A]
```

Out[86]= $\begin{pmatrix} 1 & 1 & 0 & a14 \\ 1 & 1 & 0 & g \\ 0 & 0 & 1 & f \\ 0 & g & a43 & 1 \end{pmatrix}$

```
In[87]:= Det[A]
```

Out[87]= $a14\, g - g^2$

Positive if g < a14, else g=0 and this is 0

3 x 3 minors:

```
In[88]:= Clear[PA]; cut[A, {4}, PA];
```

$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

0

None of these entries contain an entry set to e, so it is OK that it is not positive.

```
In[89]:= Clear[PA]; cut[A, {3}, PA];
```

$$\begin{pmatrix} 1 & 1 & a14 \\ 1 & 1 & g \\ 0 & g & 1 \end{pmatrix}$$

$a14\, g - g^2$

Positive if g < a14, else g=0 and this is 0

```
In[90]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & f \\ 0 & a43 & 1 \end{pmatrix}$$

$1 - a43\, f$

```
In[91]:= Clear[PA]; cut[A, {1}, PA];
```

$$\begin{pmatrix} 1 & 0 & g \\ 0 & 1 & f \\ g & a43 & 1 \end{pmatrix}$$

$1 - a43\, f - g^2$

Positive if g small or 0.

2 x 2 minors:

```
In[92]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & f \\ a43 & 1 \end{pmatrix}$$

$1 - a43\, f$

$$\begin{pmatrix} 1 & g \\ g & 1 \end{pmatrix}$$

$1 - g^2$

$$\begin{pmatrix} 1 & a14 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

0

Case 3c: a43=0 => x34 = 0

Case3ci: a12*a21<>1 so 1-a12a21 > 0

Set x31=0;x13=0;x24=0;x42=0;x32=f;x41=e;

```
In[93]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];
         Clear[e, f, g];

         A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
            {x31, x32, 1, x34}, {x41, x42, a43, 1}}; a43 = 0;
         x31 = 0; x13 = 0; x24 = 0; x42 = 0; x32 = f; x41 = e; x34 = 0;

         MatrixForm[A]
```

Out[93]=
$$\begin{pmatrix} 1 & a12 & 0 & a14 \\ a21 & 1 & a23 & 0 \\ 0 & f & 1 & 0 \\ e & 0 & 0 & 1 \end{pmatrix}$$

Since1-a12 a21 > 0, all minors OK if e is small enough.

```
In[94]:= Det[A]
```

Out[94]= 1 − a12 a21 − a14 e − a23 f + a14 a23 e f

3 x 3 minors:

```
In[95]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & a12 & 0 \\ a21 & 1 & a23 \\ 0 & f & 1 \end{pmatrix}$$

1-a12 a21-a23 ***f***

$$\begin{pmatrix} 1 & a12 & a14 \\ a21 & 1 & 0 \\ e & 0 & 1 \end{pmatrix}$$

1-a12 a21-a14 ***e***

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & 0 \\ e & 0 & 1 \end{pmatrix}$$

1-a14 ***e***

$$\begin{pmatrix} 1 & a23 & 0 \\ f & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1-a23 ***f***

2 x 2 minors:

In[96]:= **Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]**

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14 \\ e & 1 \end{pmatrix}$$

1-a14 ***e***

$$\begin{pmatrix} 1 & a23 \\ f & 1 \end{pmatrix}$$

1-a23 ***f***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a12 \\ a21 & 1 \end{pmatrix}$$

1-a12 a21

Case3cii: a12*a21=1

Without loss of generality, we can make a12=a21=1;

Set x31=g2;x13=g2;x24=g1;x42=g1;x32=e2;x41=e1;

```
In[97]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];
         Clear[e, f, g, e1, e2, g1, g2];

         A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
            {x31, x32, 1, x34}, {x41, x42, a43, 1}}; a12 = 1;
         a21 = 1; a43 = 0; x31 = g2; x13 = g2; x24 = g1; x42 = g1;
         x32 = e2; x41 = e1; x34 = 0;

         MatrixForm[A]
```

Out[97]=

$$\begin{pmatrix} 1 & 1 & g2 & a14 \\ 1 & 1 & a23 & g1 \\ g2 & e2 & 1 & 0 \\ e1 & g1 & 0 & 1 \end{pmatrix}$$

Temporarily set e1 = e2 = 0

```
In[98]:= e1 = 0; e2 = 0; MatrixForm[A]
```

Out[98]=

$$\begin{pmatrix} 1 & 1 & g2 & a14 \\ 1 & 1 & a23 & g1 \\ g2 & 0 & 1 & 0 \\ 0 & g1 & 0 & 1 \end{pmatrix}$$

If a14 = 0, set g1=0 (and e1 stays 0)

If a14 not 0, choose g1 small to make all minors that contain e1 positive so we can perturb e1.

If a23 = 0 set g2=0 (and e2 stays 0)

If a23 not 0, choose g2 small to make all minors that contain e2 positive so we can perturb e2.

```
In[99]:= Det[A]
```

Out[99]= a14 g1 − g1^2 + a23 g2 − a14 a23 g1 g2 − g2^2 + g1^2 g2^2

Positive if a14 or a23 nonzero and g1, g2 small enough.

3 x 3 minors:

In[100]:= **Clear[PA]; cut[A, {4}, PA];**

$$\begin{pmatrix} 1 & 1 & g2 \\ 1 & 1 & a23 \\ g2 & 0 & 1 \end{pmatrix}$$

a23 g2 - g2^2

Can be made positive with small g2 if a23 not 0.

In[101]:= **Clear[PA]; cut[A, {3}, PA];**

$$\begin{pmatrix} 1 & 1 & a14 \\ 1 & 1 & g1 \\ 0 & g1 & 1 \end{pmatrix}$$

a14 g1 - g1^2

Can be made positive if with small g1 if a14 not 0.

In[102]:= **Clear[PA]; cut[A, {2}, PA];**

$$\begin{pmatrix} 1 & g2 & a14 \\ g2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1 - g2^2

In[103]:= **Clear[PA]; cut[A, {1}, PA];**

$$\begin{pmatrix} 1 & a23 & g1 \\ 0 & 1 & 0 \\ g1 & 0 & 1 \end{pmatrix}$$

1 - g1^2

2 x 2 minors:

In[104]:= **Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i − 1}], {i, 1, 4}]**

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & g1 \\ g1 & 1 \end{pmatrix}$$

$1-g1^2$

$$\begin{pmatrix} 1 & a14 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a23 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & g2 \\ g2 & 1 \end{pmatrix}$$

$1-g2^2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

0

Case 3d: a14=0 so x41=0

Case3di: a12*a21 not 1

So 1-a21*a21 > 0.

Set x32=e, x34=f, all others 0

```
In[105]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];
          Clear[e, f, g];

          A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
              {x31, x32, 1, x34}, {x41, x42, a43, 1}}; a14 = 0;
          x31 = 0; x13 = 0; x24 = 0; x42 = 0; x32 = e; x41 = 0; x34 = f;

          MatrixForm[A]
```

$$Out[105]= \begin{pmatrix} 1 & a12 & 0 & 0 \\ a21 & 1 & a23 & 0 \\ 0 & e & 1 & f \\ 0 & 0 & a43 & 1 \end{pmatrix}$$

Since 1-a12 a21 > 0, all minors OK if e, f small enough.

```
In[106]:= Det[A]

Out[106]= 1 - a12 a21 - a23 e - a43 f + a12 a21 a43 f
```

3 x 3 minors:

```
In[107]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & a12 & 0 \\ a21 & 1 & a23 \\ 0 & e & 1 \end{pmatrix}$$

1-a12 a21-a23 ***e***

$$\begin{pmatrix} 1 & a12 & 0 \\ a21 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1-a12 a21

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & a43 & 1 \end{pmatrix}$$

1-a43 ***f***

$$\begin{pmatrix} 1 & a23 & 0 \\ e & 1 & f \\ 0 & a43 & 1 \end{pmatrix}$$

1-a23 ***e***-a43 ***f***

2 x 2 minors:

```
In[108]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & f \\ a43 & 1 \end{pmatrix}$$

1-a43 ***f***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
1
(1  0)
(0  1)
1
(1  a23)
(e   1 )
1-a23 e
(1  0)
(0  1)
1
( 1   a12)
(a21   1 )
1-a12 a21
```

Case3dii: a12*a21=1 (and a14=0 so x41=0)

Without loss of generality, we can make a12=a21=1;

Set x31=g; x13=g; x24=0; x42=0; x32=e; x34=f;

```
In[109]:= Clear[a12, x13, a14, a21, a23, x24, x31, x32, x34, x41, x42, a43];
          Clear[e, f, g, e1, e2];

          A = {{1, a12, x13, a14}, {a21, 1, a23, x24},
              {x31, x32, 1, x34}, {x41, x42, a43, 1}}; a12 = 1; a21 = 1;
          a14 = 0; x31 = g; x13 = g; x24 = 0; x42 = 0; x32 = e; x41 = 0; x34 = f;

          MatrixForm[A]

           1  1   g   0
Out[109]= (1  1  a23  0)
           g  e   1   f
           0  0  a43  1
```

Temporarily set e=0.

Choose f small enough so that 1-a43 f > 0.

If a23 = 0, set g=0, all are minors nonnegative

If a23 not 0, choose g to make all minors that contain e positive so we can perturb e.

```
In[110]:= e = 0; MatrixForm[A]
```

$$Out[110]= \begin{pmatrix} 1 & 1 & g & 0 \\ 1 & 1 & a23 & 0 \\ g & 0 & 1 & f \\ 0 & 0 & a43 & 1 \end{pmatrix}$$

```
In[111]:= Det[A]
```

$Out[111]= a23\, g - g^2$

Positive if g < a23.

3 x 3 minors:

```
In[112]:= Clear[PA]; cut[A, {4}, PA];
```

$$\begin{pmatrix} 1 & 1 & g \\ 1 & 1 & a23 \\ g & 0 & 1 \end{pmatrix}$$

$a23\, g - g^2$

Positive if g < a23.

```
In[113]:= Clear[PA]; cut[A, {3}, PA];
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

0

The zero entries in this matrix will remain zero when e is perturbed. Therefore, this minor is OK.

```
In[114]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & g & 0 \\ g & 1 & f \\ 0 & a43 & 1 \end{pmatrix}$$

$1 - a43\, f - g^2$

OK if a43f < 1 and g small

```
In[115]:= Clear[PA]; cut[A, {1}, PA];
```

$$\begin{pmatrix} 1 & a23 & 0 \\ 0 & 1 & f \\ 0 & a43 & 1 \end{pmatrix}$$

1-a43 *f*

2 x 2 minors:

```
In[116]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & f \\ a43 & 1 \end{pmatrix}$$

1-a43 *f*

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a23 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & g \\ g & 1 \end{pmatrix}$$

$1-g^2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

0

Mathematica Files for $q = 5$, $n = 9$ regarding sign symmetric completion

p=4 q=5 n=9 has ssP0,1-completion

Label the digraph as 4321 starting in the upper left hand corner and going clockwise. Here is how the pattern matrix looks after the diagonal entries have been made equal to 1 by multiplying by a positive diagonal matrix.

```
In[117]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43, e];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
             {x31, a32, 1, a34}, {x41, x42, x43, 1}};

          MatrixForm[
           A]
```

Out[117]=

$$\begin{pmatrix} 1 & a12 & x13 & a14 \\ a21 & 1 & x23 & x24 \\ x31 & a32 & 1 & a34 \\ x41 & x42 & x43 & 1 \end{pmatrix}$$

Case 1: a12a32a34a14 > 0

This implies that the factors and also their pairs are nonzero.

WLOG (by diagonal similarity), we can assume a12=a32=a34=1, and thus a14>0 also. Note that a21>0 by sign symmetry.

Let x43= e >0 and small enough so that both 1-a34x43≥0 and 1-a14*a21*a32*x43≥0.

Let x23 =e >0 and small enough so that both 1-a32x23≥0 (If a32=0,then let x23=0).

Let x13=a12*e, x31=a21, x24=e, x42=e, x41=a21*e.

```
In[118]:= a12 = 1; a32 = 1; a34 = 1; x43 = e; x23 = e; x13 = e;
          x31 = a21 * a32; x24 = e; x42 = e; x41 = a21 * x43;

          MatrixForm[A]
```

Out[118]=

$$\begin{pmatrix} 1 & 1 & e & a14 \\ a21 & 1 & e & e \\ a21 & 1 & 1 & 1 \\ a21\,e & e & e & 1 \end{pmatrix}$$

```
In[119]:= Det[A]
```

Out[119]= $1 - a21 - 2\,e + 2\,a21\,e + e^2 - a21\,e^2$

```
In[120]:= Expand[Det[A] - ((1 - a21 * a12) (1 - e)^2)]
```

Out[120]= 0

3 x 3 minors:

```
In[121]:= Clear[PA]; cut[A, {4}, PA]; Expand[Det[PA] - (1 - a21) (1 - e)]
```

$$\begin{pmatrix} 1 & 1 & e \\ a21 & 1 & e \\ a21 & 1 & 1 \end{pmatrix}$$

$1 - a21 - e + a21\, e$

Out[121]= 0

```
In[122]:= Clear[PA]; cut[A, {3}, PA]; Expand[Det[PA] - (1 - a21) (1 - e^2)]
```

$$\begin{pmatrix} 1 & 1 & a14 \\ a21 & 1 & e \\ a21\, e & e & 1 \end{pmatrix}$$

$1 - a21 - e^2 + a21\, e^2$

Out[122]= 0

```
In[123]:= Clear[PA]; cut[A, {2}, PA]; Expand[Det[PA] - (1 - e) (1 - a21 * e)]
```

$$\begin{pmatrix} 1 & e & a14 \\ a21 & 1 & 1 \\ a21\, e & e & 1 \end{pmatrix}$$

$1 - e - a21\, e + a21\, e^2$

Out[123]= 0

```
In[124]:= Clear[PA]; cut[A, {1}, PA]; Expand[Det[PA] - (1 - e) (1 - x43)]
```

$$\begin{pmatrix} 1 & e & e \\ 1 & 1 & 1 \\ e & e & 1 \end{pmatrix}$$

$1 - 2\, e + e^2$

Out[124]= 0

2 x 2 minors:

```
In[125]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 1 \\ e & 1 \end{pmatrix}$$

1-e

$\begin{pmatrix} 1 & e \\ e & 1 \end{pmatrix}$

1-e^2

$\begin{pmatrix} 1 & a14 \\ a21\,e & 1 \end{pmatrix}$

1-a14 a21 e

$\begin{pmatrix} 1 & e \\ 1 & 1 \end{pmatrix}$

1-e

$\begin{pmatrix} 1 & e \\ a21 & 1 \end{pmatrix}$

1-a21 e

$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$

1-a21

```
In[126]:= Clear[PA];cut[A, {4, 3}, PA]
```

$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$

1-a21

Case 2: a12a32a34a14 < 0

Note this means that a12, x23, a34, x41, a21, a32, x43, and a14 are all nonzero.

Do a diagonal similarity operation that makes a12=a32=a34=1. Then a14<0.

We replace a14 with -b so that all the symbols in the matrix below are positive.

Set x23=e, x43=e, and x41=-e. Set x31=x13=f and x24=x42=-g.

First set e = 0.

Choose f positive so that f is less than a21 and 1. Then choose g positive and small enough to make each 3 by 3 and 4 by 4 minor containing x23, x43 or x41 positive. Then we can perturb e to be positive, and yet small enough to keep each minor nonnegative.

```
In[127]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43];
          Clear[e, f, g];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
             {x31, a32, 1, a34}, {x41, x42, x43, 1}}; a12 = 1;
          a32 = 1; a34 = 1; x13 = f; x31 = f; x24 = -g; x42 = -g;
          x23 = e; x43 = e; x41 = -e; a14 = -b;

          MatrixForm[A]
```

$$\text{Out[127]=} \begin{pmatrix} 1 & 1 & f & -b \\ a21 & 1 & e & -g \\ f & 1 & 1 & 1 \\ -e & -g & e & 1 \end{pmatrix}$$

```
In[128]:= e = 0;

          MatrixForm[A]
```

$$\text{Out[128]=} \begin{pmatrix} 1 & 1 & f & -b \\ a21 & 1 & 0 & -g \\ f & 1 & 1 & 1 \\ 0 & -g & 0 & 1 \end{pmatrix}$$

```
In[129]:= Det[A]
```

$\text{Out[129]=}\ 1 - a21 + a21\, f - f^2 + a21\, b\, g + a21\, f\, g - g^2 + f^2 g^2$

In[130]:= Expand[Det[A] - (1 - a21 + (a21 - f) f + a21 * b * g + a21 * f * g - g^2 + f^2 * g^2)]

Out[130]= 0

We have chosen f to be less than a21. Then we choose g so that the sum of the last terms are less than f (a21-f). Then the determinant is positive, and we can perturb e as stated at the beginning of this case.

3 x 3 minors:

```
In[131]:= Clear[PA]; cut[A, {4}, PA]; Expand[Det[PA] - (1 - a21 + f (a21 - f))]
```

$$\begin{pmatrix} 1 & 1 & f \\ a21 & 1 & 0 \\ f & 1 & 1 \end{pmatrix}$$

```
1-a21+a21 f-f^2

Out[131]= 0
```

Positive if f is less than a21.

```
In[132]:= Clear[PA]; cut[A, {3}, PA]; Expand[Det[PA] - (1 - a21 + (a21 * b - g) g)]
```

$$\begin{pmatrix} 1 & 1 & -b \\ a21 & 1 & -g \\ 0 & -g & 1 \end{pmatrix}$$

```
1-a21+a21 b g-g^2

Out[132]= 0
```

Positive if g is less than b*a21.

```
In[133]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & f & -b \\ f & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

```
1-f^2
```

Positive if f is less than 1.

```
In[134]:= Clear[PA]; cut[A, {1}, PA];
```

$$\begin{pmatrix} 1 & 0 & -g \\ 1 & 1 & 1 \\ -g & 0 & 1 \end{pmatrix}$$

```
1-g^2
```

Positive if g is less than 1.

2 x 2 minors:

```
In[135]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

```
1
```

$$\begin{pmatrix} 1 & -g \\ -g & 1 \end{pmatrix}$$

$1 - g^2$

$$\begin{pmatrix} 1 & -b \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & f \\ f & 1 \end{pmatrix}$$

$1 - f^2$

$$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$$

1-a21

Case 3a: a12=a21=0

Set x41=e;x23=f;x43=g; and all others 0

```
In[136]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43];
          Clear[e, f, g];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
             {x31, a32, 1, a34}, {x41, x42, x43, 1}}; a12 = 0; a21 = 0;
          x13 = 0; x31 = 0; x24 = 0; x42 = 0; x41 = e; x23 = f; x43 = g;

          MatrixForm[A]
```

Out[136]=
$$\begin{pmatrix} 1 & 0 & 0 & a14 \\ 0 & 1 & f & 0 \\ 0 & a32 & 1 & a34 \\ e & 0 & g & 1 \end{pmatrix}$$

Let e , f, and g be the same sign as a14,a32, and a34 respectively. All minors are OK if e, f, and g are small enough.

```
In[137]:= Det[A]
```

Out[137]= 1 − a14 e − a32 f + a14 a32 e f − a34 g

3 x 3 minors:

```
In[138]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & f \\ 0 & a32 & 1 \end{pmatrix}$$

1-a32 ***f***

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & 0 \\ e & 0 & 1 \end{pmatrix}$$

1-a14 ***e***

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & a34 \\ e & g & 1 \end{pmatrix}$$

1-a14 ***e***-a34 ***g***

$$\begin{pmatrix} 1 & f & 0 \\ a32 & 1 & a34 \\ 0 & g & 1 \end{pmatrix}$$

1-a32 ***f***-a34 ***g***

2 x 2 minors:

```
In[139]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & a34 \\ g & 1 \end{pmatrix}$$

1-a34 ***g***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14 \\ e & 1 \end{pmatrix}$$

1-a14 ***e***

$$\begin{pmatrix} 1 & f \\ a32 & 1 \end{pmatrix}$$

1-a32 ***f***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

```
1
(1  0)
(0  1)
1
```

Case 3b: a32=0 (and a12 = 1 and a34=1 (not case a, not case c)). Then x23=0.

Case 3b(i): 1-a12a21<>0.

Set x41=e, a43=f, and set all others equal to zero.

Let e and f be the same sign as a14 and a34 respectively. All minors are OK if e and f are small enough.

```
In[140]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43];
          Clear[e, f, g];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
              {x31, a32, 1, a34}, {x41, x42, x43, 1}}; a12 = 1; a32 = 0;
          x13 = 0; a34 = 1; x31 = 0; x24 = 0; x42 = 0; x23 = 0; x41 = e; x43 = f;

          MatrixForm[A]
```

$$Out[140]= \begin{pmatrix} 1 & 1 & 0 & a14 \\ a21 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ e & 0 & f & 1 \end{pmatrix}$$

```
In[141]:= Det[A]

Out[141]= 1 - a21 - a14 e - f + a21 f
```

3 x 3 minors:

```
In[142]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & 1 & 0 \\ a21 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
1-a21
```

$$\begin{pmatrix} 1 & 1 & a14 \\ a21 & 1 & 0 \\ e & 0 & 1 \end{pmatrix}$$

1-a21-a14 ***e***

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & 1 \\ e & f & 1 \end{pmatrix}$$

1-a14 ***e***-f

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & f & 1 \end{pmatrix}$$

1-f

2 x 2 minors:

```
In[143]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 1 \\ f & 1 \end{pmatrix}$$

1-f

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14 \\ e & 1 \end{pmatrix}$$

1-a14 ***e***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$$

1-a21

Case 3b(ii): 1-a12a21=0 (and a23=0 => x32=0 and a34=1)

Since a12=1, a21=1 also. Set x13=x31=0 equal to zero, x24= x42 = g, x43=0.1, and x41=e. Let e be the same sign as a14.

```
In[144]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43];
          Clear[e, f, g, s];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
             {x31, a32, 1, a34}, {x41, x42, x43, 1}}; a21 = 1;
          a12 = 1; a32 = 0; x13 = 0; x31 = 0; x24 = g; x42 = g;
          x23 = 0; x43 = 0.1; x41 = e; a34 = 1;

          MatrixForm[A]
```

Out[144]=

$$\begin{pmatrix} 1 & 1 & 0 & a14 \\ 1 & 1 & 0 & g \\ 0 & 0 & 1 & 1 \\ e & g & 0.1 & 1 \end{pmatrix}$$

Fix f less than

Temporarily set e=0.

Let g be as small as needed so all minors that involve e are positive if a14 not 0 while e is set equal to zero. Then perturb e slightly so that the minors remain nonnegative.

```
In[145]:= e = 0;

          MatrixForm[A]
```

Out[145]=

$$\begin{pmatrix} 1 & 1 & 0 & a14 \\ 1 & 1 & 0 & g \\ 0 & 0 & 1 & 1 \\ 0 & g & 0.1 & 1 \end{pmatrix}$$

```
In[146]:= Det[A]
```

Out[146]= $a14\, g - g^2$

Positive if a14 not 0 and g is less than a14.

```
In[147]:= Clear[PA]; cut[A, {4}, PA];
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

0

This minor does not involve e, so it is OK that it is zero.

```
In[148]:= Clear[PA]; cut[A, {3}, PA];
```

$$\begin{pmatrix} 1 & 1 & a14 \\ 1 & 1 & g \\ 0 & g & 1 \end{pmatrix}$$

$a14\, g - g^2$

Positive if g is less than a14 and a14 not 0.

```
In[149]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & 1 \\ 0 & 0.1 & 1 \end{pmatrix}$$

0.9‘

Positive if f small

```
In[150]:= Clear[PA]; cut[A, {1}, PA];
```

$$\begin{pmatrix} 1 & 0 & g \\ 0 & 1 & 1 \\ g & 0.1 & 1 \end{pmatrix}$$

$0.9 - g^2$

Positive if g is small

2 x 2 minors:

```
In[151]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 1 \\ 0.1 & 1 \end{pmatrix}$$

0.9‘

$$\begin{pmatrix} 1 & g \\ g & 1 \end{pmatrix}$$

$1-g^2$

$$\begin{pmatrix} 1 & a14 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

0

Case 3c: a34=0 => x43=0 (can assume a12 = 1)

Case 3c(i): a12a21<>1, so 1-a12a21>0.

Set x13=x31=x24=x42 =0, x23=e, and x41=f. Let e and f be the same sign as a32 and a14 respectively.

Let e and f be as small as needed.

```
In[152]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43];
          Clear[e, f, g];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
             {x31, a32, 1, a34}, {x41, x42, x43, 1}}; a12 = 1; a34 = 0;
          x13 = 0; x31 = 0; x24 = 0; x42 = 0; x23 = e; x43 = 0; x41 = f;

          MatrixForm[A]
```

Out[152]=

$$\begin{pmatrix} 1 & 1 & 0 & a14 \\ a21 & 1 & e & 0 \\ 0 & a32 & 1 & 0 \\ f & 0 & 0 & 1 \end{pmatrix}$$

All minors are OK if e and f are small enough in absolute value.

In[153]:= **Det[A]**

Out[153]= 1 - a21 - a32 e - a14 f + a14 a32 e f

3 x 3 minors:

In[154]:= **Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]**

$$\begin{pmatrix} 1 & 1 & 0 \\ a21 & 1 & e \\ 0 & a32 & 1 \end{pmatrix}$$

1-a21-a32 ***e***

$$\begin{pmatrix} 1 & 1 & a14 \\ a21 & 1 & 0 \\ f & 0 & 1 \end{pmatrix}$$

1-a21-a14 ***f***

$$\begin{pmatrix} 1 & 0 & a14 \\ 0 & 1 & 0 \\ f & 0 & 1 \end{pmatrix}$$

1-a14 ***f***

$$\begin{pmatrix} 1 & e & 0 \\ a32 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1-a32 ***e***

2 x 2 minors:

In[155]:= **Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]**

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & a14 \\ f & 1 \end{pmatrix}$$

1-a14 ***f***

$$\begin{pmatrix} 1 & e \\ a32 & 1 \end{pmatrix}$$

1-a32 ***e***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$$

1-a21

Case 3c(ii): 1-a12a21=0

Without loss of generality, a21=a12=1. Set x13=x31=g1, x24=x42=g2, x23=e1, and x41=e2.

```
In[156]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43];
          Clear[e, f, g];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
              {x31, a32, 1, a34}, {x41, x42, x43, 1}}; a21 = 1;
          a12 = 1; a34 = 0; x13 = g1; x31 = g1; x24 = g2; x42 = g2;
          x23 = e1; x43 = 0; x41 = e2;

          MatrixForm[A]
```

Out[156]=
$$\begin{pmatrix} 1 & 1 & g1 & a14 \\ 1 & 1 & e1 & g2 \\ g1 & a32 & 1 & 0 \\ e2 & g2 & 0 & 1 \end{pmatrix}$$

Let e1 be the same sign as a32 and e2 the same sign as a14.Temporarily set e1=e2=0.

```
In[157]:= e1 = 0; e2 = 0; MatrixForm[A]
```

Out[157]=
$$\begin{pmatrix} 1 & 1 & g1 & a14 \\ 1 & 1 & 0 & g2 \\ g1 & a32 & 1 & 0 \\ 0 & g2 & 0 & 1 \end{pmatrix}$$

If a32=0, set g1=0 (and e1 stays zero)

If a32 not 0, let g1 be the same sign as a14 and as small as needed to make all order 3 and 4 minors which involve it positive while e1 and e2 are set equal to zero.

If a14=0, let g2=0 (and e2 stays zero)

If a14 not 0, let g2 be the same sign as a14 and as small as needed to make all order 3 and 4 minors which involve it positive while e1 and e2 are set equal to zero.

Then perturb e1 and e2 so they are positive but the minors remain nonnegative.

```
In[158]:= Det[A]
```

Out[158]= $a32\, g1 - g1^2 + a14\, g2 - g2^2 + g1^2\, g2^2$

Positive if a14 or a23 nonzero and g1 and g2 are small enough.

```
In[159]:= Expand[Det[A] - (g2(a14 - g2) + g1(a32 - g1) + g1^2g2^2)]

Out[159]= 0

In[160]:= Clear[PA]; cut[A, {4}, PA];
```

$$\begin{pmatrix} 1 & 1 & g1 \\ 1 & 1 & 0 \\ g1 & a32 & 1 \end{pmatrix}$$

$a32\, g1 - g1^2$

Can be made positive with small g1 if a32 not 0.

```
In[161]:= Clear[PA]; cut[A, {3}, PA];
```

$$\begin{pmatrix} 1 & 1 & a14 \\ 1 & 1 & g2 \\ 0 & g2 & 1 \end{pmatrix}$$

$a14\, g2 - g2^2$

Can be made positive with small g2 if a14 not 0.

```
In[162]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & g1 & a14 \\ g1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$1 - g1^2$

Postive if g1 is less than 1.

```
In[163]:= Clear[PA]; cut[A, {1}, PA];
```

$$\begin{pmatrix} 1 & 0 & g2 \\ a32 & 1 & 0 \\ g2 & 0 & 1 \end{pmatrix}$$

$1 - g2^2$

Positive if g2 is less than 1.

2 x 2 minors:

```
In[164]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & g2 \\ g2 & 1 \end{pmatrix}$$

$1 - g2^2$

$$\begin{pmatrix} 1 & a14 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ a32 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & g1 \\ g1 & 1 \end{pmatrix}$$

$1 - g1^2$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

0

Case d: a14=x41=0 and and a12 = a34=1 (not case a, not case c)).

Case d(i): a12a21 not 1

So 1-a21a12>0

Set x23=e, x43=f, all others 0.

```
In[165]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43];
          Clear[e, f, g];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
             {x31, a32, 1, a34}, {x41, x42, x43, 1}}; a12 = 1; a14 = 0;
          a34 = 1; x13 = 0; x31 = 0; x24 = 0; x42 = 0; x23 = e; x43 = f; x41 = 0;

          MatrixForm[A]
```

$$\text{Out[165]=} \begin{pmatrix} 1 & 1 & 0 & 0 \\ a21 & 1 & e & 0 \\ 0 & a32 & 1 & 1 \\ 0 & 0 & f & 1 \end{pmatrix}$$

Since 1-a12a21>0, all minors OK of e,f small enough.

```
In[166]:= Det[A]

Out[166]= 1 - a21 - a32 e - f + a21 f
```

3 x 3 minors:

```
In[167]:= Do[{Clear[P], cut[A, {i}, P]}, {i, 4, 1, -1}]
```

$$\begin{pmatrix} 1 & 1 & 0 \\ a21 & 1 & e \\ 0 & a32 & 1 \end{pmatrix}$$

1-a21-a32 *e*

$$\begin{pmatrix} 1 & 1 & 0 \\ a21 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1-a21

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & f & 1 \end{pmatrix}$$

1-f

$$\begin{pmatrix} 1 & e & 0 \\ a32 & 1 & 1 \\ 0 & f & 1 \end{pmatrix}$$

1-a32 ***e***-f

2 x 2 minors:

```
In[168]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 1 \\ f & 1 \end{pmatrix}$$

1-f

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & e \\ a32 & 1 \end{pmatrix}$$

1-a32 ***e***

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 1 \\ a21 & 1 \end{pmatrix}$$

1-a21

Case d(ii): a12a21=1 (and a14=0 so x41=0) and not case c, so a34=1

Without loss of generality, we can make a21=a12=1;

We can assume a34 not zero. Otherwise we use Case 3c. WLOG a34=1.

Set x23=e, x43=0.1, x31=x13=f, small and of same sign as a32, and all others equal to 0.

If a32=0, set e1, e2, and f to 0 and all minors are nonnegative.

If a32 not 0, let f be the same sign as a32, and small enough in absolute value so that |f|<min{|a32|,1}.

```
In[169]:= Clear[a12, x13, a14, a21, x23, x24, x31, a32, a34, x41, x42, x43];
          Clear[e, f, g, e1, e2];

          A = {{1, a12, x13, a14}, {a21, 1, x23, x24},
             {x31, a32, 1, a34}, {x41, x42, x43, 1}}; a21 = 1;
          a12 = 1; a34 = 1; a14 = 0; x13 = f; x31 = f; x24 = 0;
          x42 = 0; x23 = e; x43 = 0.1; x41 = 0;

          MatrixForm[A]
```

Out[169]=

$$\begin{pmatrix} 1 & 1 & f & 0 \\ 1 & 1 & e & 0 \\ f & a32 & 1 & 1 \\ 0 & 0 & 0.1 & 1 \end{pmatrix}$$

Temporarily set e to zero.

If a32 is nonzero, we will perturb e to make it positive, but the principal minors remain nonnegative.

```
In[170]:= e = 0; MatrixForm[A]
```

Out[170]=

$$\begin{pmatrix} 1 & 1 & f & 0 \\ 1 & 1 & 0 & 0 \\ f & a32 & 1 & 1 \\ 0 & 0 & 0.1 & 1 \end{pmatrix}$$

```
In[171]:= Det[A]
```

Out[171]= $a32\, f - f^2$

Positive if f is less than a32 in absolute value. If a32=0, then set f equal to zero.

```
In[172]:= Clear[PA]; cut[A, {4}, PA];
```

$$\begin{pmatrix} 1 & 1 & f \\ 1 & 1 & 0 \\ f & a32 & 1 \end{pmatrix}$$

$a32\, f - f^2$

Positive if f is less than a32 in absolute value. If a32=0, then set f equal to zero.

```
In[173]:= Clear[PA]; cut[A, {3}, PA];
```

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

0

```
In[174]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & f & 0 \\ f & 1 & 1 \\ 0 & 0.1 & 1 \end{pmatrix}$$

0.9 - f^2

Positive if f is less than 0.9.

```
In[175]:= Clear[PA]; cut[A, {1}, PA];
```

$$\begin{pmatrix} 1 & 0 & 0 \\ a32 & 1 & 1 \\ 0 & 0.1 & 1 \end{pmatrix}$$

0.9‘

2 x 2 minors:

```
In[176]:= Do[Do[{Clear[P], cut[A, {i, j}, P]}, {j, 1, i - 1}], {i, 1, 4}]
```

$$\begin{pmatrix} 1 & 1 \\ 0.1 & 1 \end{pmatrix}$$

0.9‘

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & 0 \\ a32 & 1 \end{pmatrix}$$

1

$$\begin{pmatrix} 1 & f \\ f & 1 \end{pmatrix}$$

```
1-f²
(1  1)
(1  1)
0
```

Mathematica Files for q = 6, n = 4 regarding sign symmetric $P_{0,1}$-completion

SS P01 Noncompletion of q=6 n=4

Label the digraph as 1234 starting in the upper left hand corner and going clockwise. Here is how the pattern matrix looks after the diagonal entries have been made equal to 1 by multiplying by a positive diagonal matrix.

```
In[177]:= Clear[a12, x13, x14, a21, x23,
             x24, x31, a32, a34, a41, x42, a43, x32];

           A = {{1, a12, x13, x14}, {a21, 1, a23, x24},
               {x31, x32, 1, a34}, {a41, x42, a43, 1}};

           MatrixForm[
             A]

              1    a12   x13   x14
Out[177]=   a21    1    a23   x24
             x31   x32    1    a34
             a41   x42   a43    1
```

Example showing noncompletion:

```
In[178]:= a21 = 1; a12 = 1; a23 = 1; a34 = 1; a43 = 1; a41 = -1; a43 = 1;

           MatrixForm[A]

              1     1    x13   x14
Out[178]=     1     1     1    x24
             x31   x32    1     1
             -1    x42    1     1
```

Contradictory principal minors.

```
In[179]:= Clear[PA]; cut[A, {4}, PA];
```

$$\begin{pmatrix} 1 & 1 & x13 \\ 1 & 1 & 1 \\ x31 & x32 & 1 \end{pmatrix}$$

```
x31 - x13 x31 - x32 + x13 x32
```

x23>0 implies -x23 negative. Also, -x13x31 is nonpositive by sign symmetry. Therefore, in order to make this minor positive, x31 and/or x13x32 must be positive. This implies x13 and x31 must both be positive (by sign symmetry).

```
In[180]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & x13 & x14 \\ x31 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

```
-x13 + x14 - x13 x31 + x14 x31
```

x14<0 and -x13x31<0 implies that -x13 and/or x14x31 must be positive. Then x13 and x31 must be negative by sign symmetry.

This is a contradiction. Therefore, this example cannot be completed to a sign symmetric P01 matrix.

Mathematica Files for $q = 6$, $n = 5$ regarding sign symmetric $P_{0,1}$-completion

SS P01 Noncompletion of q=6 n=5

Label the digraph as 1234 starting in the upper left hand corner and going clockwise. Here is how the pattern matrix looks after the diagonal entries have been made equal to 1 by multiplying by a positive diagonal matrix.

```
In[181]:= Clear[a12, x13, x14, a21, x23, x24, x31, a32, a34, a41, x42, a43];

          A = {{1, a12, x13, x14}, {a21, 1, x23, x24},
              {x31, a32, 1, a34}, {a41, x42, a43, 1}};

          MatrixForm[
            A]
```

$$Out[181]= \begin{pmatrix} 1 & a12 & x13 & x14 \\ a21 & 1 & x23 & x24 \\ x31 & a32 & 1 & a34 \\ a41 & x42 & a43 & 1 \end{pmatrix}$$

Example showing noncompletion:

```
In[182]:= a21 = 1; a12 = 1; a32 = 1; a34 = 1; a43 = 1; a41 = -1; a43 = 1;

          MatrixForm[A]
```

$$Out[182]= \begin{pmatrix} 1 & 1 & x13 & x14 \\ 1 & 1 & x23 & x24 \\ x31 & 1 & 1 & 1 \\ -1 & x42 & 1 & 1 \end{pmatrix}$$

Contradictory principal minors.

```
In[183]:= Clear[PA]; cut[A, {4}, PA];
```

$$\begin{pmatrix} 1 & 1 & x13 \\ 1 & 1 & x23 \\ x31 & 1 & 1 \end{pmatrix}$$

x13-x23-x13 x31+x23 x31

x23>0 implies -x23 negative. Also, -x13x31 is nonpositive by sign symmetry. Therefore, in order to make this minor positive, x13 and/or x23x31 must be positive. This implies x13 and x31 must both be positive (by sign symmetry).

```
In[184]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & x13 & x14 \\ x31 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

```
-x13+x14-x13 x31+x14 x31
```

x14<0 and -x13x31<0 implies that -x13 and/or x14x31 must be positive. Then x13 and x31 must be negative by sign symmetry.

This is a contradiction. Therefore, this example cannot be completed to a sign symmetric P01 matrix.

Mathematica Files for $q = 6$, $n = 6$ regarding sign symmetric $P_{0,1}$-completion

SS P01 Noncompletion of q=6 n=6

Label the digraph as 1234 starting in the upper left hand corner and going clockwise. Here is how the pattern matrix looks after the diagonal entries have been made equal to 1 by multiplying by a positive diagonal matrix.

```
In[185]:= Clear[x12, x13, x14, a21, a23, x24, x31, a32, a34, a41, x42, a43];

          A = {{1, x12, x13, x14}, {a21, 1, a23, x24},
              {x31, a32, 1, a34}, {a41, x42, a43, 1}};

          MatrixForm[
            A]
```

Out[185]=

$$\begin{pmatrix} 1 & x12 & x13 & x14 \\ a21 & 1 & a23 & x24 \\ x31 & a32 & 1 & a34 \\ a41 & x42 & a43 & 1 \end{pmatrix}$$

Example showing noncompletion:

```
In[186]:= a21 = 1; a32 = 1; a23 = 1; a34 = 1; a43 = 1; a41 = -1;

          MatrixForm[A]
```

```
                1   x12  x13  x14
Out[186]=  (    1    1    1   x24  )
               x31   1    1    1
               -1   x42   1    1
```

Contradictory principal minors.

```
In[187]:= Clear[PA]; cut[A, {4}, PA];
  1   x12  x13
( 1    1    1  )
 x31   1    1
-x12+x13+x12 x31-x13 x31
```

The term -x12 is negative by sign symmetry. Also,-x13x31 is nonpositive by sign symmetry. Therefore, in order to make this minor positive, x13 and/or x12x31 must be positive. This implies x13 and x31 must both be positive by sign symmetry.

```
In[188]:= Clear[PA]; cut[A, {2}, PA];
  1   x13  x14
(x31   1    1  )
 -1    1    1
-x13+x14-x13 x31+x14 x31
```

The term x14 is negative by sign symmetry. Also,-x13x31 is nonpositive by sign symmetry. Therefore, in order to make this minor positive,- x13 and/or x14x31 must be positive. This implies x13 and x31 must both be negative by sign symmetry.

This is a contradiction.Therefore,this example cannot be completed to a sign symmetric P01 matrix, and so q6, n6 does not have sign symmetric P01 completion.

Mathematica Files for $q = 6$, $n = 7$ regarding sign symmetric $P_{0,1}$-completion

SS P01 Noncompletion of q=6 n=7

Label the digraph as 1234 starting in the upper left hand corner and going clockwise. Here is how the pattern matrix looks after the diagonal entries have been made equal to 1 by multiplying by a positive diagonal matrix.

```
In[189]:= Clear[a12, x13, x14, x21, a23, x24, x31, a32, a34, a41, x42, a43];

          A = {{1, a12, x13, x14}, {x21, 1, a23, x24},
              {x31, a32, 1, a34}, {a41, x42, a43, 1}};

          MatrixForm[
            A]
```

$$Out[189]= \begin{pmatrix} 1 & a12 & x13 & x14 \\ x21 & 1 & a23 & x24 \\ x31 & a32 & 1 & a34 \\ a41 & x42 & a43 & 1 \end{pmatrix}$$

Example showing noncompletion:

```
In[190]:= a12 = 1; a32 = 1; a23 = 1; a34 = 1; a43 = 1; a41 = -1;

          MatrixForm[A]
```

$$Out[190]= \begin{pmatrix} 1 & 1 & x13 & x14 \\ x21 & 1 & 1 & x24 \\ x31 & 1 & 1 & 1 \\ -1 & x42 & 1 & 1 \end{pmatrix}$$

Contradictory principal minors:

```
In[191]:= Clear[PA]; cut[A, {4}, PA];
```

$$\begin{pmatrix} 1 & 1 & x13 \\ x21 & 1 & 1 \\ x31 & 1 & 1 \end{pmatrix}$$

-x21+x13 x21+x31-x13 x31

Since x21>0 by sign symmetry, -x21 must be negative. Also by sign symmetry, -x13x31 is nonpositive. Now in order for the minor to be positive, x13x21 and/or x31 must be positive. Therefore, both x13 and x31 must be positive by sign symmetry.

```
In[192]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & x13 & x14 \\ x31 & 1 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

```
-x13+x14-x13 x31+x14 x31
```

Since x14<0 by sign symmetry, x14 must be negative. Also by sign symmetry, -x13x31 is nonpositive. Now in order for the minor to be positive, x14x31 and/or -x13 must be positive. Therefore, both x13 and x31 must be negative by sign symmetry.

Therefore, this matrix does not have sign symmetric P01 completion, and so it follows that the digraph q6, n7 does not have sign symmetric P0,1 completion.

Mathematica Files for the Double Triangle regarding $P_{0,1}$-completion

DOUBLE TRIANGLE p=4 q=10 n=1 has P0,1-completion

Label the digraph as 1243 starting in the upper left hand corner and going clockwise. Here is how the pattern matrix looks after the diagonal entries have been made equal to 1 by multiplying by a positive diagonal matrix.

Case 1: a23*a32 <> 1 .

```
In[193]:= Clear[a12, a13, a14, a21, a23, a24, a31, a32,
            a34, a41, a42, a43, x12, x13, x14, x21, x23, x24,
            x31, x32, x34, x41, x42, x43]; Clear[e, f, g, x]

          A = {{1, a12, a13, x14}, {a21, 1, a23, a24},
              {a31, a32, 1, a34}, {x41, a42, a43, 1}};

          MatrixForm[
            A]
```

```
Out[193]= (  1    a12  a13  x14 )
          ( a21    1   a23  a24 )
          ( a31  a32    1   a34 )
          ( x41  a42  a43    1  )

In[194]:= x14 = x; x41 = -x; MatrixForm[A]

Out[194]= (  1    a12  a13   x  )
          ( a21    1   a23  a24 )
          ( a31  a32    1   a34 )
          ( -x   a42  a43    1  )

In[195]:= Det[A]
```

Out[195]= $1 - a12\,a21 - a13\,a31 + a12\,a23\,a31 + a13\,a21\,a32 - a23\,a32 - a24\,a42 + a13\,a24\,a31\,a42 - a13\,a21\,a34\,a42 + a23\,a34\,a42 - a12\,a24\,a31\,a43 + a24\,a32\,a43 - a34\,a43 + a12\,a21\,a34\,a43 - a12\,a24\,x + a13\,a24\,a32\,x - a13\,a34\,x + a12\,a23\,a34\,x + a21\,a42\,x - a23\,a31\,a42\,x + a31\,a43\,x - a21\,a32\,a43\,x + x^2 - a23\,a32\,x^2$

3 x 3 minors:

```
In[196]:= Clear[PA]; cut[A, {3}, PA];

(  1    a12   x  )
( a21    1   a24 )
( -x   a42    1  )
```

$1 - a12\,a21 - a24\,a42 - a12\,a24\,\boldsymbol{x} + a21\,a42\,\boldsymbol{x} + \boldsymbol{x}^2$

```
In[197]:= Clear[PA]; cut[A, {2}, PA];

(  1    a13   x  )
( a31    1   a34 )
( -x   a43    1  )
```

$1 - a13\,a31 - a34\,a43 - a13\,a34\,\boldsymbol{x} + a31\,a43\,\boldsymbol{x} + \boldsymbol{x}^2$

2 x 2 minors:

```
In[198]:= Clear[PA]; cut[A, {3, 2}, PA];

(  1   x )
( -x   1 )
```

$1 + \boldsymbol{x}^2$

Case 2: a23a32=1

Case 2a: a31=a21

```
In[199]:= a23 = 1; a32 = 1; x14 = x; x41 = -x; a31 = a21; MatrixForm[A]

Out[199]= ( 1    a12  a13  x
            a21  1    1    a24
            a21  1    1    a34
            -x   a42  a43  1 )
```

This is an original minor:

```
In[200]:= Clear[A234]; cut[A, {1}, A234];

( 1    1    a24
  1    1    a34
  a42  a43  1 )

-a24 a42+a34 a42+a24 a43-a34 a43
```

```
In[201]:= Det[A]

Out[201]= -a24 a42 + a13 a21 a24 a42 + a34 a42-
            a13 a21 a34 a42 + a24 a43 - a12 a21 a24 a43 - a34 a43+
            a12 a21 a34 a43 - a12 a24 x + a13 a24 x + a12 a34 x - a13 a34 x
```

```
In[202]:= Expand[Det[A] - ((1 - a12 * a21) Det[A234]+
              a21 * a42 (a34 - a24) (a12 - a13) + x (a34 - a24) (a12 - a13))]
Out[202]= 0
```

If (a13-a12)(a34-a24) not 0 make x large of correct sign.

If a13=a12, then DetA=DetA234DetA12≥0.

3 x 3 minors:

```
In[203]:= Clear[PA]; cut[A, {3}, PA];

( 1    a12  x
  a21  1    a24
  -x   a42  1 )
```

$1-a12\,a21-a24\,a42-a12\,a24\,x+a21\,a42\,x+x^2$

Positive if x is large enough in absolute value.

```
In[204]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & a13 & x \\ a21 & 1 & a34 \\ -x & a43 & 1 \end{pmatrix}$$

$1 - a13\ a21 - a34\ a43 - a13\ a34\ x + a21\ a43\ x + x^2$

Positive if x is large enough in absolute value.

2 x 2 minors:

```
In[205]:= Clear[PA]; cut[A, {3, 2}, PA];
```

$$\begin{pmatrix} 1 & x \\ -x & 1 \end{pmatrix}$$

$1 + x^2$

Case 2b: a42=a43

```
In[206]:= a32 = 1; a23 = 1; x14 = x; x41 = -x; a42 = a43;

          MatrixForm[A]
```

Out[206]= $$\begin{pmatrix} 1 & a12 & a13 & x \\ a21 & 1 & 1 & a24 \\ a31 & 1 & 1 & a34 \\ -x & a43 & a43 & 1 \end{pmatrix}$$

This is an original minor:

```
In[207]:= Clear[A123]; cut[A, {4}, A123];
```

$$\begin{pmatrix} 1 & a12 & a13 \\ a21 & 1 & 1 \\ a31 & 1 & 1 \end{pmatrix}$$

$-a12\ a21 + a13\ a21 + a12\ a31 - a13\ a31$

```
In[208]:= Det[A]
```

Out[208]= $-a12\ a21 + a13\ a21 + a12\ a31 - a13\ a31 - a12\ a24\ a31\ a43 + a13\ a24\ a31\ a43 + a12\ a21\ a34\ a43 - a13\ a21\ a34\ a43 - a12\ a24\ x + a13\ a24\ x + a12\ a34\ x - a13\ a34\ x$

```
In[209]:= Expand[Det[A] - (x (a12 - a13) (a34 - a24) +
            (a12 - a13) (a34 - a24) a31 * a43 + (1 - a43 * a34) Det[A123])]
Out[209]= 0
```

3 x 3 minors:

```
In[210]:= Clear[PA]; cut[A, {3}, PA];
```

$$\begin{pmatrix} 1 & a12 & x \\ a21 & 1 & a24 \\ -x & a43 & 1 \end{pmatrix}$$

$1 - a12\,a21 - a24\,a43 - a12\,a24\,x + a21\,a43\,x + x^2$

Positive if x is large enough in absolute value.

```
In[211]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & a13 & x \\ a31 & 1 & a34 \\ -x & a43 & 1 \end{pmatrix}$$

$1 - a13\,a31 - a34\,a43 - a13\,a34\,x + a31\,a43\,x + x^2$

Positive if x is large enough in absolute value.

2 x 2 minors:

```
In[212]:= Clear[PA]; cut[A, {3, 2}, PA];
```

$$\begin{pmatrix} 1 & x \\ -x & 1 \end{pmatrix}$$

$1 + x^2$

Case IIc: a31<>a21 and a42<>a43

```
In[213]:= a23 = 1; a32 = 1; x14 = x; x41 = -m * x;

          MatrixForm[A]
```

$$Out[213]= \begin{pmatrix} 1 & a12 & a13 & x \\ a21 & 1 & 1 & a24 \\ a31 & 1 & 1 & a34 \\ -m\,x & a42 & a43 & 1 \end{pmatrix}$$

```
In[214]:= Det[A]
```

Out[214]= −a12 a21 + a13 a21 + a12 a31 − a13 a31 − a24 a42 + a13 a24 a31 a42 + a34 a42 − a13 a21 a34 a42 + a24 a43 − a12 a24 a31 a43 − a34 a43 + a12 a21 a34 a43 + a21 a42 x − a31 a42 x − a21 a43 x + a31 a43 x − a12 a24 m x + a13 a24 m x + a12 a34 m x − a13 a34 m x

```
In[215]:= Expand[Det[A]-
    (x((a21 - a31)(a42 - a43) + m(-a12 a24 + a13 a24 + a12 a34 - a13 a34)))]
```

Out[215]= −a12 a21 + a13 a21 + a12 a31 − a13 a31 − a24 a42 + a13 a24 a31 a42 + a34 a42 − a13 a21 a34 a42 + a24 a43 − a12 a24 a31 a43 − a34 a43 + a12 a21 a34 a43

OK if (a21-a31)(a42-a43)<>0 by good choice of m (positive and so coefficient of x is nonzero).

3 x 3 minors:

```
In[216]:= Clear[PA]; cut[A, {3}, PA];
```

$$\begin{pmatrix} 1 & a12 & x \\ a21 & 1 & a24 \\ -m\,x & a42 & 1 \end{pmatrix}$$

$1 - a12\,a21 - a24\,a42 + a21\,a42\,x - a12\,a24\,m\,x + m\,x^2$

This is positive if x is large enough in absolute value and m positive.

```
In[217]:= Clear[PA]; cut[A, {2}, PA];
```

$$\begin{pmatrix} 1 & a13 & x \\ a31 & 1 & a34 \\ -m\,x & a43 & 1 \end{pmatrix}$$

$1 - a13\,a31 - a34\,a43 + a31\,a43\,x - a13\,a34\,m\,x + m\,x^2$

This is positive if x is large enough in absolute value and m positive.

2 x 2 minors:

```
In[218]:= Clear[PA]; cut[A, {3, 2}, PA];
```

$$\begin{pmatrix} 1 & x \\ -m\,x & 1 \end{pmatrix}$$

$1+m\,x^2$

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