Connections to zero forcing: Tree covers and power domination

by

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

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DEDICATION

To my son Weston.

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CHAPTER 1. INTRODUCTION

1.1 Overview

Zero forcing is a game played on a graph (following a certain set of rules which are defined below) that was independently introduced in [2] to study the maximum nullity of a graph and in [9] for the study of quantum physics. The maximum nullity of a graph is defined to be the maximum nullity over a specific family of symmetric matrices (defined precisely in the next section) associated with the graph. This topic, in various forms, and its connections have been widely studied, due partly to its relationship to the well known inverse eigenvalue problem for graphs. Since their introductions, several variants and connections of maximum nullity and zero forcing have been introduced and studied. One popular variant of the standard maximum nullity and zero forcing problem is the well known positive semidefinite maximum nullity and zero forcing problem. The study of the positive semidefinite maximum nullity and zero forcing parameters of a graph led to the study of the tree cover number of a graph [4], which is the focus of Chapter 2 of this thesis. Perhaps the most recently discovered variant/connection to zero forcing is power domination, which was introduced in [19] (independent of zero forcing) as a tool for studying the problem of effectively monitoring electric power networks using Phasor Measurement Units (PMUs). The number of steps required to monitor a network (using the least number of PMUs needed) is called the power propagation time of the network and this is the focus of Chapter 3.

1.2 Basic graph theory

A *graph* is a pair $G = (V, E)$ where V is the vertex set and E is the set of edges (two element subsets of the vertices). Unless otherwise stated, $G = (V, E)$ is a simple graph (no multiple edges or loops). The set $V(G)$ is always nonempty and finite. The *complement* of a graph $G = (V(G), E(G))$, is the graph $\overline{G} = (V(G), \overline{E(G)})$, where $\overline{E(G)} = \{\{u, v\}|u, v \in V(G) \text{ and } \{u, v\} \notin E(G)\}$. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. When there is no ambiguity, we simply write $G = (V, E)$. Two vertices $u, v \in V$ are *adjacent* if $\{u, v\} \in E$. Adjacent vertices u and v are also called *neighbors*. The *neighborhood* of v (also referred to as the *open neighborhood* of v), is given by $N(v) = \{u \in V|\{u, v\} \in E\}$. The *closed neighborhood* of v is given by $N[v] = N(v) \cup \{v\}$. The number of neighbors of a vertex v is the *degree* of v, denoted $\deg(v)$. A vertex of degree one is called a *leaf*. The *minimum degree* of G, denoted $\delta(G)$, is $\min\{\deg(v)|v \in V\}$, and $\Delta(G) = \max\{\deg(v)|v \in V\}$ is the *maximum degree* of G. If $e \in E$ and $v \in V$ is an endpoint of e, then e and v are *incident*. The *adjacency matrix* of G is a $|V| \times |V|$ zero-one matrix A such $a_{ij} = 1$ if and only if $i \neq j$ and i is adjacent to j.

A *subgraph* of $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$. A subgraph $G' = (V', E')$ is an *induced subgraph* of $G = (V, E)$ if $E' = \{\{u, v\}|u, v \in V' \text{ and } \{u, v\} \in E\}$. For $S \subseteq V$ we use $G[S]$ to denoted the induced subgraph whose vertex set is the set S.

For $G = (V, E)$ and $S \subseteq V$, S is an *independent set* if for each $u, v \in S$, $\{u, v\} \notin E$. The *independence number* of G, denoted $\alpha(G)$, is given by $\alpha(G) = \max\limits_{S \subseteq V}\{|S| : S \text{ is an independent set}\}$. A graph H is called a *clique* if for each $u, v \in V(H)$, $\{u, v\} \in E(H)$. The *clique number* of G, denoted $\omega(G)$, is given by $\max\{|V(H)| : H \text{ is a subgraph of } G \text{ and } H \text{ is a clique}\}$. A *clique cover* of G is a set of subgraphs of G such that each subgraph in the set is a clique and every edge of G belongs to at least one of the cliques. The *clique cover number* of G, denoted $\mathrm{cc}(G)$, is the cardinality of a minimum clique cover of G.

The *path* on n vertices, denoted P_n, has vertex set $\{v_1, \ldots, v_n\}$ and edges $\{v_i, v_{i+1}\}$ for $i \in \{1, 2, \ldots, n-1\}$. The *cycle* on n vertices is a path on n vertices with the additional edge $\{v_n, v_1\}$ and is denoted C_n. A graph is *chordal* if it has no induced cycles on four or more vertices. The *complete graph* on n vertices, denoted K_n, is the clique on n vertices.

For a graph $G = (V, E)$ and $S \subseteq V$, $G - S$ is used to denote $G[V \setminus S]$. When $S = \{v\}$, we simply write $G - v$. For an induced subgraph H of G, we use $G - H$ to denote $G[V(G) \setminus V(H)]$. For an edge $e = \{u, v\} \in E$, $G - e$ is the graph obtained from G by deleting e, and the graph obtained by *subdividing* e, denoted G_e, is given by $G_e = (V', E')$ where $V' = V \cup \{w\}$ and $E' = (E \setminus \{u, v\}) \cup \{\{u, w\}, \{v, w\}\}$.

A graph is *connected* if there is a path between any two vertices of the graph. A connected graph is a *tree* if it contains no cycles. The *connectivity* of a connected graph G, denoted $\kappa(G)$, is the smallest number of vertices whose removal results in a disconnected graph. A *cut-vertex* of a connected graph $G = (V, E)$ is a vertex $v \in V$ such that $G - v$ is disconnected. A graph is *nonseparable* if it is connected and does not have a cut-vertex. A *block* of G is a maximal nonseparable induced subgraph of G. A graph $G = (V, E)$ is a *block-clique graph* if every block of G is a clique, and G is a *cactus* if every block is an edge or a cycle.

A graph is *planar* if it has a crossing-free embedding in the plane, and a graph is *outer-planar* if it has a crossing-free embedding in the plane with every vertex on the boundary of the unbounded face.

1.3 Literature Review

1.3.1 Standard maximum nullity, zero forcing, and propagation time

For a graph $G = (V, E)$ on n vertices, let $\mathcal{S}(G)$ denote the set of real $n \times n$ symmetric matrices $A = (a_{ij})$ satisfying $a_{ij} \neq 0$ if and only if $\{i, j\} \in E$, for $i \neq j$, and a_{ii} is any real number. Observe that the adjacency matrix is in $\mathcal{S}(G)$. The *maximum nullity* of G, denoted

$\mathrm{M}(G)$, is given by $\mathrm{M}(G) = \max\{\mathrm{null}(A) | A \in \mathcal{S}(G)\}$, the *minimum rank* of G, denoted $\mathrm{mr}(G)$, is given by $\mathrm{mr}(G) = \min\{\mathrm{rank}(A) | A \in \mathcal{S}(G)\}$, and it follows from the Rank-Nullity Theorem that $\mathrm{M}(G) + \mathrm{mr}(G) = n$.

Note that $\mathcal{S}(G)$ is an infinite family of matrices, and because of this, computing $\mathrm{M}(G)$ and $\mathrm{mr}(G)$ is a difficult task. Many tools have been developed to help compute bounds on these parameters, and arguably the most well known upper bound for the maximum nullity is the zero forcing number. Before we can define the zero forcing number, we first state the color change rule: For a graph $G = (V, E)$, let $B \subseteq V$ be a set of blue vertices and let $V \setminus B$ be colored white. For $b \in B$, if w is the only white neighbor of b in $V \setminus B$, then we can color w blue and say b forces w (often written as $b \to w$). The *final coloring* of a set B (also known in literature as the derived set) is the set of vertices that are colored blue by initially coloring B blue and applying the color change rule until no more changes are possible. A set $B \subseteq V$ is a *zero forcing set* if the final coloring of B is the entire vertex set V. The *zero forcing number* of G, denoted $\mathrm{Z}(G)$ is $\mathrm{Z}(G) = \min\{|B| : B \text{ is a zero forcing set}\}$.

The idea of zero forcing was introduced in [2] and independently in [9]. In [2] the authors give bounds on $\mathrm{M}(G), \mathrm{mr}(G)$, and $\mathrm{Z}(G)$ for an extensive list of graphs, and they prove the following relationship between the maximum nullity and zero forcing number of a graph:

Theorem 1. [2, Proposition 2.4] *Let $G = (V, E)$ be a graph and let $B \subseteq V$ be a zero forcing set. Then $\mathrm{M}(G) \leq |B|$, and thus $\mathrm{M}(G) \leq \mathrm{Z}(G)$.*

The *path cover number* of a graph G, denoted $\mathrm{P}(G)$, is the smallest number of vertex-disjoint paths occurring as induced subgraphs of G that covers all the vertices of G. For any graph G, $\mathrm{P}(G) \leq \mathrm{Z}(G)$ [20, Theorem 2.13], if G is outerplanar, $\mathrm{M}(G) \leq \mathrm{P}(G)$ [30, Theorem 2.8], and for a tree T, $\mathrm{P}(T) = \mathrm{M}(T)$ [13]. The connection between path covers and standard maximum nullity and zero forcing is what motivated the study of the relationship between tree covers and positive semidefinite maximum nullity and zero forcing, which is discussed in Section 1.3.3.

For a more detailed study of $\mathrm{M}(G), \mathrm{mr}(G)$, and $\mathrm{Z}(G)$, see [2], [16], and [17], and for an updated list of graphs and their maxiumum nullity and zero forcing numbers, see the graph catalog given in [1].

In [22], Hogben et al. introduced the propagation time of a zero forcing set. Loosely speaking, the propagation time of a zero forcing set is the number of iterations required to color the entire graph blue (where simultaneous independent applications of the color change rule are allowed). Formally, let $G = (V, E)$ be a graph and B a zero forcing set of G. Define $B^{(0)} = B$, and for $t \geq 0$, $B^{(t+1)}$ is the set of vertices w for which there exists a vertex $b \in \cup_{s=0}^{t} B^{(s)}$ such that w is the only neighbor of b not in $\cup_{s=0}^{t} B^{(s)}$. The *propagation time* of B in G, denoted $\mathrm{pt}(G, B)$, is the smallest integer t_0 such that $V = \cup_{s=0}^{t_0} B^{(s)}$. The *minimum propagation time* of G, denoted $\mathrm{pt}(G)$, is

$$\mathrm{pt}(G) = \min\{\mathrm{pt}(G, B) |\text{ B is a minimum zero forcing set}\},$$

and the *maximum propagation time* of G, denoted $\mathrm{PT}(G)$, is

$$\mathrm{PT}(G) = \max\{\mathrm{pt}(G, B) |\text{ B is a minimum zero forcing set}\}.$$

Remark 2. [22, Remark 1.8] Let G be a graph. Then $\mathrm{PT}(G) \leq |G| - \mathrm{Z}(G)$ because at least one force must be performed at each time step, and $\frac{|G| - \mathrm{Z}(G)}{\mathrm{Z}(G)} \leq \mathrm{pt}(G)$ because using a given zero forcing set B, at most $|B|$ forces can be performed at any one time step.

It is clear that for any graph G, $0 \leq \mathrm{pt}(G) \leq \mathrm{PT}(G) \leq |G| - 1$. Hogben et al. [22] characterize graphs whose propagation times take on the extreme values of $0, 1, |G| - 2$, and $|G| - 1$. We use similar techniques for the propagation time of power domination in Chapter 3.

1.3.2 Positive semidefinite maximum nullity and positive semidefinite zero forcing

A real symmetric $n \times n$ matrix A is *positive semidefinite* if for all nonzero $x \in \mathbb{R}^n$, $x^T Ax \geq 0$. Let $G = (V, E)$ be a graph on n vertices. By restricting ourselves to the positive semidefinite matrices in $\mathcal{S}(G)$, we study the positive semidefinite maximum nullity and positive semidefinite minimum rank of G: Let $\mathcal{S}_+(G)$ denote the set of real $n \times n$ positive semidefinite matrices $A = (a_{ij})$ satisfying $a_{ij} \neq 0$ if and only if $\{i, j\} \in E$, and a_{ii} is any real number. The *maximum positive semidefinite nullity* of G, denoted $\mathrm{M}_+(G)$, is given by $\max\{\mathrm{null}(A)|A \in \mathcal{S}_+(G)\}$, the *minimum positive semidefinite rank* of G, denoted $\mathrm{mr}_+(G)$, is given by $\min\{\mathrm{rank}(A)|A \in \mathcal{S}_+(G)\}$, and it follows that $\mathrm{M}_+(G) + \mathrm{mr}_+(G) = n$. Since $\mathcal{S}_+(G) \subset \mathcal{S}(G)$, it is clear that $\mathrm{mr}(G) \leq \mathrm{mr}_+(G)$ and $\mathrm{M}_+(G) \leq \mathrm{M}(G)$.

The minimum semidefinite rank of G can be computed using orthogonal representations: For a graph $G = (V, E)$ with $V = \{v_1, \ldots, v_n\}$, a $d-$*dimensional orthogonal representation* of G is a set of real vectors $\{\vec{x_1}, \ldots, \vec{x_n}\} \subset \mathbb{R}^d$ satisfying $\vec{x_i} \cdot \vec{x_j} = 0$ (with regard to the usual inner product in $\mathbb{R}^d$) if $\{i, j\} \notin E$. If it also holds that $\{i, j\} \in E$ implies $\vec{x_i} \cdot \vec{x_j} \neq 0$, then the representation is a *faithful orthogonal representation.* Let $\{\vec{x_1}, \ldots, \vec{x_n}\}$ be a faithful orthogonal representation of G and let $X = [\vec{x_1} \cdots \vec{x_n}]$ be the matrix whose columns are the vectors $\vec{x_1}, \ldots, \vec{x_n}$. Then the *Gram matrix* $A = X^T X$ is positive semidefinite, and by construction, $A \in \mathcal{S}_+(G)$. Furthermore, since any real $n \times n$ positive semidefinite matrix A with rank r may be written as $A = B^T B$ for some B where $\mathrm{rank}(B) = r$ [25, Corollary 7.2.11], then we have the next observation.

Observation 3. [21, Observation 1.2] Let $d(G)$ denote the smallest dimension d over all faithful orthogonal representations of G. Then $\mathrm{mr}_+(G) = d(G)$.

For a graph $G = (V, E)$ and $V = \{v_1, \ldots, v_n\}$, let D be the $n \times n$ diagonal matrix with $D_{ii} = \deg(v_i)$. The Laplacian matrix of G, $\mathcal{L}_G$, is given by $\mathcal{L}_G = D - A$, where A is the adjacency matrix of G. It is well known that $\mathcal{L}_G$ is positive semidefinite and that the rank of $\mathcal{L}_G = n - k$, where k is the number of connected components of G. By definition, $\mathcal{L}_G \in \mathcal{S}_+(G)$, so it follows that $0 \le \operatorname{mr}_+(G) \le n-1$. If G has an edge, then it is clear that $1 \le \operatorname{mr}_+(G) \le n-1$.

Theorem 4. [23] *For a graph G on n vertices,* $\operatorname{mr}_+(G) = n-1$ *if and only if G is a tree.*

Theorem 5. [2, Theorem 3.16 and Corollary 3.17] *For a tree T on n vertices that is not a star,* $\operatorname{mr}_+(\overline{T}) = 3$.

Theorem 6. [11, Theorem 3.3] *Let G be a triangle-free graph on n vertices and no isolated vertices. Then* $\operatorname{mr}_+(G) \ge \frac{n}{2}$.

See Table 3.1 of [28] for a list of the maximum positive semidefinite nullity values of several other graphs.

There is an extensive amount of research on the relationships between $\mathrm{M}_+(G)$ and other graph paramaters (see Section 3 of [16]). We now include some of the well known relationships, starting with the upper bound given by the positive semidefinite zero forcing number, $\mathrm{Z}_+(G)$. We first define the positive semidefinite color change rule: For a graph $G = (V, E)$, let $B \subseteq V$ be an initial set of blue vertices and let $G_1, \ldots, G_r$ be the connected components of $G - B$. If for some $b \in B$, $w \in V(G_i)$ is the only white neighbor of b in G_i, then we change w to blue, and we say that b *forces* w. For an initial set of blue vertices B, the *final coloring of* B is the set of blue vertices that results from applying the positive semidefinite color change rule until no more changes are possible. A set B is a *positive semidefinite zero forcing set* if the final coloring of B is the entire set V. The *positive semidefinite zero forcing number* of G, denoted $\mathrm{Z}_+(G)$, is $\min\{|B| : B$ is a positive semidefinite zero forcing set$\}$. The concept of positive semidefinite zero forcing was introduced in [3]. Since any zero forcing set is a positive semidefinite zero forcing set, then $\mathrm{Z}_+(G) \le \mathrm{Z}(G)$ [3]. See [32] for a study of positive semidefinite propagation time.

Theorem 7. [3] *For any graph G,* $\mathrm{M}_+(G) \leq \mathrm{Z}_+(G)$.

In Theorem 4.1 of [14], Ekstrand et al. show that $\mathrm{Z}_+(G) = 2$ if and only if $\mathrm{M}_+(G) = 2$, and in Corollary 4.2, they show that if $\mathrm{Z}_+(G) \leq 3$, then $M_+(G) = \mathrm{Z}_+(G)$. They also observe that for V_8, the Möbius ladder on 8 vertices, $\mathrm{M}_+(V_8) = 3$ and $\mathrm{Z}_+(V_8) = 4$, so these paramaters are not equal in general.

Recall that for a graph G, $\alpha(G)$ is the independence number of G, $\mathrm{cc}(G)$ is the clique cover number of G, and $\kappa(G)$ is the connectivity of G.

Theorem 8. [7, Corollary 2.7] *For any connected graph G, $\alpha(G) \leq \mathrm{mr}_+(G)$.*

It is also well known that if G is a graph with no isolated vertices, then $\alpha(G) \leq \mathrm{mr}_+(G)$. This can be seen by the fact that for any orthogonal representation and corresponding Gram matrix X, the columns of X corresponding to an independent set are nonzero, orthogonal, and therefore linearly independent.

Theorem 9. [17, Observation 3.14] *For any graph G,* $\mathrm{mr}_+(G) \leq \mathrm{cc}(G)$.

Theorem 10. [7, Theorem 3.6] If G is a connected chordal graph, then $\mathrm{mr}_+(G) = \mathrm{cc}(G)$.

Theorem 11. [27, 26, Corollary 1.4] *For any graph G,* $\mathrm{mr}_+(G) \leq n - \kappa(G)$.

We conclude this section with a discussion of effects of graph operations on minimum positive semidefinite rank and maximum positive semidefinite nullity. Let G_e denote the graph obtained from G by subdividing the edge e.

Theorem 12. [24, Lemma 2.11] *For any graph G,* $\mathrm{mr}_+(G_e) = \mathrm{mr}_+(G) + 1$.

Note that since $\mathrm{mr}_+(G) + \mathrm{M}_+(G) = n$ this shows also that subdividing an edge does not change the positive semidefinite maximum nullity of a graph. It is also known that subdividing an edge does not change the positive semidefinite zero forcing number [6, Theorem

5.24]. While $\mathrm{Z}(H) = \mathrm{M}(H)$ when H is the graph obtained from a graph G by subdividing every edge of G [5], in general this equality does not hold for $\mathrm{M}_+(H)$ and $\mathrm{Z}_+(H)$ (since $\mathrm{M}_+(H) = \mathrm{Z}_+(H)$ if and only if $\mathrm{M}_+(G) = \mathrm{Z}_+(G)$).

Theorem 13. [14, Proposition 5.14] *For any graph G,* $\mathrm{M}_+(G)-1 \le \mathrm{M}_+(G-e) \le \mathrm{M}_+(G)+1$.

Theorem 14. [14, Observaton 5.2] *For any graph G,* $\mathrm{M}_+(G) - 1 \le \mathrm{M}_+(G - v)$.

See [4], [7], [16], and [24] for more effects of graph operations on $\mathrm{M}_+(G)$ and $\mathrm{mr}_+(G)$.

1.3.3 The tree cover number of a graph

Barioli et al. [4] define a *tree cover* of G to be a collection of vertex-disjoint simple trees occurring as induced subgraphs of G that cover all the vertices of G. (They allow G to be a multigraph, but we restrict ourselves to simple graphs.) The *tree cover number* of G, denoted $T(G)$, is the cardinality of a minimum tree cover. In [4], the tree cover number is used as a tool for studying the positive semidefinite maximum nullity of a graph.

Theorem 15. [4, Theorem 3.4] *If G is an outerplanar graph, then $T(G) = \mathrm{M}_+(G)$.*

Theorem 16. [4, Proposition 4.2] *If G is a chordal graph, then $T(G) \le \mathrm{M}_+(G)$.*

Conjecture 17. [4] *For any graph G, $T(G) \le \mathrm{M}_+(G)$.*

Ekstrand et al. show in [14] that $T(G) \le \mathrm{Z}_+(G)$. The explanation is included here for completeness: Let B be a positive semidefinite zero forcing set. Perform the forces to color the entire graph blue and list the forces in the order in which they happen. This list is called a *chronological list of forces* of B, denoted $\mathcal{F}$. Given a graph G, positive semidefinite zero forcing set B, chronological list of forces $\mathcal{F}$, and a vertex $b \in B$, define V_b to be the set of vertices w such that there is a sequence of forces $b \to v_1 \to \cdots \to v_k \to w$ in $\mathcal{F}$ (the empty sequence of forces is permitted, i.e., $b \in V_b$). The *forcing tree* T_b is the induced subgraph

$T_b = G[V_b]$. The *forcing tree cover* for the chronological list of forces $\mathcal{F}$ is $\mathcal{T} = \{T_b | b \in B\}$. An *optimal forcing tree cover* is a forcing tree cover from a chronological list of forces of a minimum positive semidefinite zero forcing set [14, Definition 2.3].

Theorem 18. [14, Theorem 2.4] *Assume G is a graph, B is a positive semidefinite zero forcing set of G, $\mathcal{F}$ is a chronological list of forces of B, and $b \in B$. Then*

1. *T_b is a tree.*

2. *The forcing tree cover $\mathcal{T} = \{T_b : b \in B\}$ is a tree cover of G.*

3. *$T(G) \leq \mathrm{Z}_+(G)$.*

For a positive integer k, a k-tree is constructed inductively by starting with a complete graph on $k+1$ vertices and connecting each new vertex to the vertices of an existing clique on k vertices. A *partial k-tree* is a subgraph of a k-tree. The *tree-width* $\mathrm{tw}(G)$ of a graph G is the least positive integer k such that G is a partial k-tree.

Theorem 19. [15, Corollary 3.4] *If G is a partial 2-tree, then $T(G) = \mathrm{M}_+(G) = \mathrm{Z}_+(G)$.*

As a consequence of Theorem 19, Ekstrand et al. note in [15] that since outerplanar graphs are partial 2-trees, then $T(G) = \mathrm{M}_+(G) = \mathrm{Z}_+(G)$ for all outerplanar graphs G. It is shown in Chapter 2 of this thesis that for a connected graph G on n vertices, $T(G) \leq \lceil \frac{n}{2} \rceil$, and a characterization of connected outerplanar graphs that achieve this bound is given.

In some ways, the tree cover number of a graph behaves like the maximum positive semidefinite nullity of a graph:

1. For a graph $G = (V, E)$ and $e \in E$, $T(G) - 1 \leq T(G - e) \leq T(G) + 1$ [8, Theorem 3].

2. For a graph $G = (V, E)$ and $v \in V$, $T(G) - 1 \leq T(G - v)$ (since any tree cover of $G - v$ together with $\{v\}$ is a tree cover of G).

3. For a graph $G = (V, E)$ and $e \in E$, $T(G) = T(G_e)$ [4, Proposition 3.3].

4. For a graph G on n vertices and independence number $\alpha(G)$, $T(G) \leq n - \alpha(G)$ [8, Proposition 2].

It is not always the case that T and M_+ behave the same (see Example 6 of [8]).

1.3.4 Power domination and power propagation time

We now turn our attention to power domination and power propagation time. Phasor Measurement Units (PMUs) are machines used to monitor the electric power network. Motivated by the study of PMUs, Haynes et al.[19] use graphs to model electric power networks and introduce the concept of power domination on graphs. The vertices of a graph represent the electrical nodes and edges represent transmission lines between nodes.

For a graph $G = (V, E)$ and $S \subseteq V$, define $S^{[0]} = S$, $S^{[1]} = N[S]$, and for $t \geq 1$, $S^{[t+1]} = S^{[t]} \cup \{w \in V(G) | \ \exists v \in S^{[t]}, N(v) \setminus S^{[t]} = \{w\}\}$ (we say v forces w). A set S is a *power dominating set* if there exists ℓ such that $S^{[\ell]} = V$. The power domination number of G, denoted $\gamma_P(G)$, is $\gamma_P(G) = \min\{|S| : S \text{ is a power dominating set}\}$. It follows from the definitions that any zero forcing set of G is also a power dominating set of G, so $\gamma_P(G) \leq Z(G)$. Benson et al. make the following connection between power domination and zero forcing:

Observation 20. [6] For $G = (V, E)$ and $S \subseteq V$, S is a power dominating set of G if and only if $N[S]$ is a zero forcing set of G.

Benson et al. [6] observed that for a graph G, $\frac{\mathrm{Z}(G)}{\Delta(G)+1} \leq \gamma_P(G)$, and using Lemma 2 given in [12], they improve this bound:

Theorem 21. [6, Theorem 3.2] *Let G be a graph that has an edge. Then* $\left\lceil \frac{\mathrm{Z}(G)}{\Delta(G)} \right\rceil \leq \gamma_P(G)$, *and this bound is tight.*

Analagous to the propagation time of a zero forcing set, Ferrero et al. [18] define the power propagation time: For $G = (V, E)$ and a power dominating set S, the *power propagation time of S in G*, denoted $\mathrm{ppt}(G, S)$ is the smallest ℓ such that $S^{[\ell]} = V$. The *minimum power propagation time of G*, denoted $\mathrm{ppt}(G)$, is

$$\mathrm{ppt}(G) = \min\{\mathrm{ppt}(G, S) | S \text{ is a minimum power dominating set}\}.$$

Ferrero et al. show in [18] that for a graph G on n vertices and power dominating set S,

$$\mathrm{ppt}(G) \geq \left\lceil \frac{n - \gamma_p(G)}{\gamma_P(G) \cdot \Delta(G)} \right\rceil.$$

In [10], Chang et al. generalize the concept of power domination and introduce $k-$power domination. Let $k \geq 1$. For a set $S \subseteq V(G)$, define the following sets:

1. $S^{[0]} = S, S^{[1]} = N[S]$.

2. For $t \geq 1$, $S^{[t+1]} = S^{[t]} \cup \{w \in V(G) | \, \exists \, v \in S^{[t]}, w \in N(v) \setminus S^{[t]} \text{ and } |N(v) \setminus S^{[t]}| \leq k\}$.

A set S is said to be a *$k-$power dominating set* if there exists an l such that $S^{[l]} = V(G)$. Note that when $k = 1$ the set is a power dominating set. The *$k-$power domination number* of G, denoted $\gamma_{P,k}(G)$, is defined to be the minimum cardinality over all $k-$power dominating sets of G. The $k-$power propagation time is defined in Chapter 3, and results proven for power propagation time are generalized to $k-$power propagation time.

1.4 Thesis organization

Chapters 2 of this thesis consists of a paper that is in preparation for submission and Chapter 3 of this thesis consists of a paper that has been submitted for publication. In each paper, I am the single author. Chapter 2 contains *A note on the positive semidefinite maximum nullity and tree cover number of a graph*. In this paper, I prove bounds on the tree cover number and use these bounds to deduce bounds on the positive semidefinite maximum nullity of connected outerplanar graphs. Connected outerplanar graphs whose tree cover number achieves the upper bound are characterized, certian graphs with $T(G) \leq \mathrm{M}_+(G)$ are given, and relationships between the tree cover number of a graph and other graph parameters are established.

Chapter 3 contains the paper *On the power propagation time of a graph.* In this paper, I give Nordhaus-Gaddum sum upper bounds on the power propagation time of a graph and its complement, I characterize certain graphs whose power propagation time is one, and I consider the effects of graph operations on power propagation time. The generalization of power propagation time, known as $k-$power propagation time is also studied. In addition, Chapter 3 includes an unpublished post script section.

CHAPTER 2. A NOTE ON THE TREE COVER NUMBER AND THE POSITIVE SEMIDEFINITE MAXIMUM NULLITY OF A GRAPH

Chassidy Bozeman

Abstract

For a simple graph $G = (V, E)$, let $\mathcal{S}_+(G)$ denote the set of real positive semidefinite matrices $A = (a_{ij})$ such that $a_{ij} \neq 0$ if $\{i, j\} \in E$, $a_{ij} = 0$ if $\{i, j\} \notin E$, and a_{ii} is any real number. The maximum positive semidefinite nullity of G, denoted $\mathrm{M}_+(G)$, is $\max\{\mathrm{null}(A) | A \in \mathcal{S}_+(G)\}$. A tree cover of G is a collection of vertex-disjoint simple trees occurring as induced subgraphs of G that cover all the vertices of G. The tree cover number of G, denoted $T(G)$, is the minimum cardinality of a tree cover. It is known that the tree cover number of a graph and the maximum positive semidefinite nullity of a graph are equal for outerplanar graphs, and it was conjectured in 2011 that $T(G) \leq \mathrm{M}_+(G)$ for all graphs [Barioli et al., Minimum semidefinite rank of outerplanar graphs and the tree cover number, *Elec. J. Lin. Alg.*, 2011]. We prove bounds on $T(G)$ to show that if G is a connected outerplanar graph on $n \geq 2$ vertices, then $\mathrm{M}_+(G) = T(G) \leq \lceil \frac{n}{2} \rceil$, and if G is a connected outerplanar graph on $n \geq 6$ vertices with no three or four cycle, then $\mathrm{M}_+(G) = T(G) \leq \frac{n}{3}$. We characterize connected outerplanar graphs with $\mathrm{M}_+(G) = T(G) = \lceil \frac{n}{2} \rceil$, and for each cactus graph G, we give a formula for computing $T(G)$ (and therefore $\mathrm{M}_+(G)$). Furthermore, we show that if G is a connected graph on n vertices and at least $2n$ edges, then $T(L(G)) \leq$

$\mathrm{M}_+(L(G))$, where $L(G)$ denotes the line graph of G, and we give inequalities involving $T(G)$ and other graph parameters.

2.1 Introduction

A *graph* is a pair $G = (V, E)$ where V is the vertex set and E is the set of edges (two element subsets of the vertices). All graphs discussed are simple (no loops or multiple edges) and finite. For a graph $G = (V, E)$ on n vertices, we use $\mathcal{S}_+(G)$ to denote the set of real $n \times n$ positive semidefinite matrices $A = (a_{ij})$ satisfying $a_{ij} \neq 0$ if and only if $\{i, j\} \in E$, for $i \neq j$, and a_{ii} is any real number. The maximum positive semidefinite nullity of G, denoted $\mathrm{M}_+(G)$, is defined as $\max\{\mathrm{null}(A) | A \in \mathcal{S}_+(G)\}$. The minimum positive semidefinite rank of G, denoted $\mathrm{mr}_+(G)$, is defined as $\min\{\mathrm{rank}(A) | A \in \mathcal{S}_+(G)\}$, and it follows from the Rank-Nullity Theorem that $\mathrm{M}_+(G) + \mathrm{mr}_+(G) = n$. Barioli et al. [2] define a *tree cover* of G to be a collection of vertex-disjoint simple trees occurring as induced subgraphs of G that cover all the vertices of G. The *tree cover number* of G, denoted $T(G)$, is the cardinality of a minimum tree cover, and it is used as a tool for studying the positive semidefinite maximum nullity of G. (In their paper [2], G is allowed to be a multigraph, but we restrict ourselves to simple graphs.) It was conjectured in [2] that $T(G) \leq \mathrm{M}_+(G)$ for all graphs, and it is shown there that $T(G) = \mathrm{M}_+(G)$ for outerplanar graphs.

In Section 2.2, we prove bounds on $T(G)$ and deduce bounds on $\mathrm{M}_+(G)$ for connected outerplanar graphs. We show that $T(G) \leq \mathrm{M}_+(G)$ for certain families of graphs in Section 2.3. In section 2.4, we characterize connected outerplanar graphs on n vertices having positive semidefinite maximum nullity and tree cover number equal to the upper bound of $\lceil \frac{n}{2} \rceil$, and we give a formula for computing $T(G)$ and $\mathrm{M}_+(G)$ for cactus graphs. In Section 2.5, we give relationships between the tree cover number and other graph parameters.

2.1.1 Graph theory terminology

For a graph $G = (V, E)$ and $v \in V$, the *neighborhood* of v, denoted $N(v)$, is the set of vertices adjacent to v. The degree of v is the cardinality of $N(v)$ and is denoted by $\deg(v)$. A vertex of degree one is called a *leaf*. A set $S \subseteq V$ is *independent* if no two of the vertices of S are adjacent.

The *path* P_n is the graph with vertex set $\{v_1, \ldots, v_n\}$ and edge set $\{\{v_i, v_{i+1}\} | i \in \{1, \ldots, n-1\}\}$. The *cycle* C_n is formed by adding the edge $\{v_n, v_1\}$ to P_n. The *girth* of a graph is the size of the smallest cycle in the graph. We denote the graph on n vertices containing every edge possible by K_n, and we use $K_{s,t}$ to denote the *complete bipartite graph*, the graph whose vertex set may be partitioned into two independent sets X and Y such that $|X| = s, |Y| = t$, for each $x \in X$ and $y \in Y$, $\{x, y\}$ is an edge, and each edge has one endpoint in X and one endpoint in Y. The graph $K_{1,3}$ is referred to as a *claw* and the graph $K_{1,t}$ is called a *star*.

For a graph $G = (V, E)$, a graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. A subgraph G' is an *induced subgraph* of G if $V(G') \subseteq V(G)$ and $E(G') = \{\{u, v\} | \{u, v\} \in E(G)$ and $u, v \in V(G')\}$. If $S \subseteq V(G)$, then we use $G[S]$ to denote the subgraph induced by S. For $S \subseteq V(G)$, we use $G - S$ to denote $G[V(G) \setminus S]$, and for $e \in E$, $G - e$ denotes the graph obtained by deleting e. For a graph G and an induced subgraph H, $G - H$ denotes the graph that results from G by deleting $V(H)$. A graph $H = (V(H), E(H))$ is a *clique* if for each $u, v \in V(H), \{u, v\} \in E(H)$. The *clique number* of G, denoted $\omega(G)$, is $\omega(G) = \max\{|V(H)| : H$ is a subgraph of G and H is a clique$\}$. The *independence number* of G, denoted $\alpha(G)$, is the cardinality of a maximum independent set.

A graph is *connected* if there is a path from any vertex to any other vertex. For a connected graph $G = (V, E)$, an edge $e \in E$ is called a *bridge* if $G - e$ is disconnected. We *subdivide* an edge $e = \{u, w\} \in E$ by removing e and adding a new vertex v_e such that $N(v_e) = \{u, w\}$.

A graph $G = (V, E)$ is *outerplanar* if it has a crossing-free embedding in the plane with every vertex on the boundary of the unbounded face. A *cut-vertex* of a connected graph $G = (V, E)$ is a vertex $v \in V$ such that $G - v$ is disconnected. A graph is *nonseparable* if it is connected and does not have a cut-vertex. A *block* is a maximal nonseparable induced subgraph. A graph G is a *cactus* graph if every block of G is either a cycle or a single edge, and G is a block-clique graph if every block is a clique. Cactus graphs are a well studied family of outerplanar graphs.

Throughout this paper, given a graph $G = (V, E)$ and a tree cover $\mathcal{T}$ of G, we use $T_v \in \mathcal{T}$ to denote the tree containing $v \in V$.

2.1.2 Preliminaries

In this section, we give some preliminary results that will be used throughout the remainder of the paper.

It is shown in Propostion 3.3 of [2] that deleting a leaf of a graph does not affect the tree cover number of the graph and that subdividing an edge does not affect the tree cover number of the graph. These two facts will be used repeatedly in the proofs throughout this paper.

Theorem 22. *[8] Suppose $G_i, i = 1, \ldots, h$ are graphs, there is a vertex v for all $i \neq j$, $G_i \cap G_j = \{v\}$ and $G = \cup_{i=1}^{h} G_i$. Then,*

$$\mathrm{M}_+(G) = \left(\sum_{i=1}^{h} \mathrm{M}_+(G_i) \right) - h + 1.$$

This is known as the *cut-vertex reduction formula.* The authors of [6] give an analogous cut-vertex reduction formula for computing the tree cover number and we use this technique multiple times throughout this paper.

Proposition 23. *[6] Suppose $G_i, i = 1, \ldots, h$, are graphs, there is a vertex v for all $i \neq j$, $G_i \cap G_j = \{v\}$ and $G = \cup_{i=1}^{h} G_i$. Then*

$$T(G) = \left(\sum_{i=1}^{h} T(G_i) \right) - h + 1.$$

In the case that h from Proposition 23 is two, we say G is the *vertex-sum* of G_1 and G_2 and write $G = G_1 \oplus_v G_2$.

Bozeman et al. [4] give a bound on the tree cover number of a graph in terms of the independence number of the graph.

Proposition 24. *[4] Let $G = (V, E)$ be a connected graph, and let $S \subseteq V(G)$ be an independent set. Then, $T(G) \leq |G| - |S|$. In particular, $T(G) \leq |G| - \alpha(G)$, where $\alpha(G)$ is the independence number of G. Furthermore, this bound is tight.*

It is also shown in Proposition 6 of [4] that for a graph $G = (V, E)$ and a bridge $e \in E$, e belongs to some tree in every minimum tree cover. Embedded in the proof of this proposition is the following lemma, and we include the proof of the lemma for completeness.

Lemma 25. *Let $G = (V, E)$ be a connected graph and $e = \{u, v\}$ a bridge in E. Let G_1 and G_2 be the connected components of $G - e$. Then $T(G) = T(G_1) + T(G_2) - 1 = T(G - e) - 1$.*

Proof. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be minimum tree covers of G_1 and G_2, respectively. By adding the edge e, we can form a tree cover of G in which T_u and T_w become one tree connected by e, so $T(G) \leq T(G_1) + T(G_2) - 1$. To see the reverse inequality, let $\mathcal{T}$ be a minimum tree cover of G. Recall that e is required to be in some tree T_{uv} of $\mathcal{T}$. By deleting e from T_{uv}, we form a tree cover of $G - e$ of size $T(G) + 1$, so $T(G) \geq T(G_1) + T(G_2) - 1$. □

For a graph $G = (V, E)$ and an edge $e \in E$, $\mathrm{M}_+(G) - 1 \leq \mathrm{M}_+(G - e) \leq \mathrm{M}_+(G) + 1$ [6] and $T(G) - 1 \leq T(G - e) \leq T(G) + 1$ [4]. It is also known that for $v \in V$, $\mathrm{M}_+(G) - 1 \leq \mathrm{M}_+(G - v) \leq \mathrm{M}_+(G) + \deg(v) - 1$ (see Fact 11 of page 46-11 of [7]). We show that an analogous bound holds for $T(G)$.

Proposition 26. *For a graph $G = (V, E)$ and vertex $v \in V$,*

$$T(G) - 1 \leq T(G - v) \leq T(G) + \deg(v) - 1.$$

Proof. Since any tree cover of $G - v$ together with the tree consisting of the single vertex v is a tree cover for G, then $T(G) \leq T(G - v) + 1$, which gives the lower bound. To see the upper bound, let E_v denote the set of edges incident to v, and let $G - E_v$ denote the graph resulting from deleting the edges in E_v. Note that $|E_v| = \deg(v)$, and that $T(G - E_v) = T(G - v) + 1$. Since the deletion of an edge can raise the tree cover number by at most 1, then $T(G - v) + 1 = T(G - E_v) \leq T(G) + \deg(v)$, and the upper bound follows. □

2.2 Bounds on the tree cover number

In this section, we show that for any connected graph G on $n \geq 2$ vertices, $T(G) \leq \lceil \frac{n}{2} \rceil$ and if G has girth at least 5, then $T(G) \leq \frac{n}{3}$. We use these bounds to deduce bounds on $\mathrm{M}_+(G)$ for connected outerplanar graphs.

Lemma 27. *Let G be a connected graph on $n \geq 3$ vertices. Then there exists an induced subgraph H of G such that $H = K_{1,p}$ for some $p \geq 1$ and $G - H$ is connected.* (See Figure 2.1).

Proof. We prove the lemma by induction. For $n = 3$ the claim holds. Let G be a graph on $n \geq 4$ vertices and suppose the lemma holds for all graphs on $3 \leq k \leq n - 1$ vertices. It is known that every connected graph has at most $n - 2$ cut vertices (since a spanning tree of the graph has at least two leaves and the removal of these leaves will not disconnect the graph). Let v be a vertex in $V(G)$ that is not a cut vertex. By hypothesis, there exists an induced subgraph $H' = K_{1,p}$ for some $p \geq 1$ in $G - v$ whose deletion does not disconnect $G - v$. First we consider the case with $p = 1$, and then we consider the case with $p \geq 2$.

Case 1: Suppose $p = 1$ (i.e., $H' = K_2$), and let a, b be the vertices of H'. If v has a neighbor in $G[V(G) \setminus \{a, b\}]$, then $G[V(G) \setminus \{a, b\}]$ is connected, and the claim holds with $H = H'$. Otherwise v has a neighbor in $\{a, b\}$. Assume first that v is adjacent to exactly one of a and b. Without loss of generality, suppose v is adjacent to a and not adjacent to b. Then, $H = G[a, b, v] = K_{1,2}$ and $G - H$ is connected. Now suppose that v is adjacent to both a and b. Since $G - v$ is connected, then either a or b has a neighbor in $G[V(G) \setminus \{v, a, b\}]$. Without loss of generality, let a have a neighbor in $G[V(G) \setminus \{v, a, b\}]$. Then $H = G[\{v, b\}] = K_{1,1}$ and $G - H$ is connected.

Case 2: Suppose $p \geq 2$. If v has a neighbor in $G[V(G) \setminus V(H')]$, then set $H = H'$ and the claim holds. Otherwise v has neighbors only in $V(H')$. Recall that H' is a star. First suppose that v is adjacent to a leaf $w \in V(H')$. If w is not a cut vertex of $G - v$, then for $H = G[\{v, w\}] = K_{1,1}$, $G - H$ is connected. If w is a cut vertex of $G - v$, then w has a neighbor in $G[V(G) \setminus \{V(H') \cup v\}]$. Then $H = G[V(H') \setminus \{w\}] = K_{1,q}$ for some $q \geq 1$ and $G - H$ is connected. Next suppose that v is not adacent to a leaf in H'. Then it must be adjacent to the center vertex. Then $H = G[V(H') \cup \{v\}]$ is a star, and $G - H$ is connected. This completes the proof. □

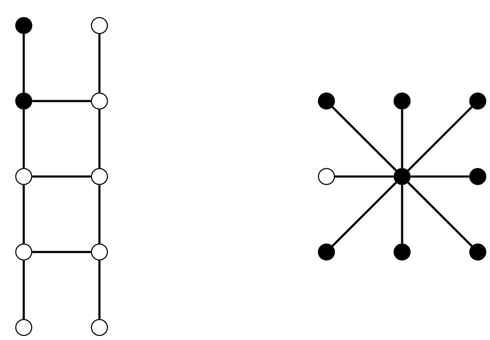


Figure 2.1: Two applications of Lemma 27, where induced subgraphs H are black.

Theorem 28. *For any simple connected graph $G = (V, E)$ on $n \geq 2$ vertices, $T(G) \leq \left\lceil \frac{n}{2} \right\rceil$.*

Proof. The theorem holds for $n = 2$. Let G be a graph on $n \geq 3$ such that the claim holds on all graphs with fewer than n vertices. By Lemma 27, there exists an induced tree $H = K_{1,p}$, for some $p \geq 1$, of G such that $G' = G - H$ is connected. By the induction hypothesis, $T(G') \leq \left\lceil \frac{|G'|}{2} \right\rceil \leq \left\lceil \frac{n-2}{2} \right\rceil$. Then $T(G) \leq T(G') + 1 \leq \left\lceil \frac{n}{2} \right\rceil$. $\square$

It is shown in [5] that for a triangle-free graph G, $\mathrm{M}_+(G) \leq \frac{n}{2}$. The next corollary is a result of Theorem 28 and the fact that $\mathrm{M}_+(G) = \mathrm{T}(G)$ for outerplanar graphs.

Corollary 29. *If G is a connected outerplanar graph on n vertices, then* $\mathrm{M}_+(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

Some examples of graphs with $T(G) = \left\lceil \frac{n}{2} \right\rceil$ are the complete graphs K_n and the well known Friendship graphs (graphs on $n = 2k + 1$ vertices, $k \geq 1$, consisting of exactly k triangles all joined at a single vertex.)

Connected outerplanar graphs having $T(G) = \left\lceil \frac{n}{2} \right\rceil$ are characterized in Section 2.4. For many graphs, the tree cover number is much lower than $\left\lceil \frac{n}{2} \right\rceil$. The next theorem improves this bound for graphs with girth at least 5.

Theorem 30. *Let G be a connected graph on $n \geq 6$ vertices with girth at least 5. Then* $T(G) \leq \frac{n}{3}$.

Proof. The proof is by induction on n. A connected graph on 6 vertices with girth at least 5 is either a tree, C_6, or C_5 with a leaf adjacent to one of the vertices on the cycle. In each case, the tree cover number is at most 2, so the theorem holds. Let $n \geq 7$. If G has a leaf v, then $T(G) = T(G - v) \leq \frac{n-1}{3}$. Suppose G has no leaves. Let $P = (x, y, z)$ be an induced path in G. We consider the connected components of $G - P$ (see Figure 2.2).

Note that since G has no leaves and no three or four cycles, $G - P$ cannot have an isolated vertex as a connected component. We now show that if $G - P$ has a connected component H with $|H| \in \{3, 4, 5\}$, then the theorem holds. Suppose $G - P$ has a connected component H of order 3 (i.e., H is a path on three vertices). Note that $G - H$ is a connected graph (since

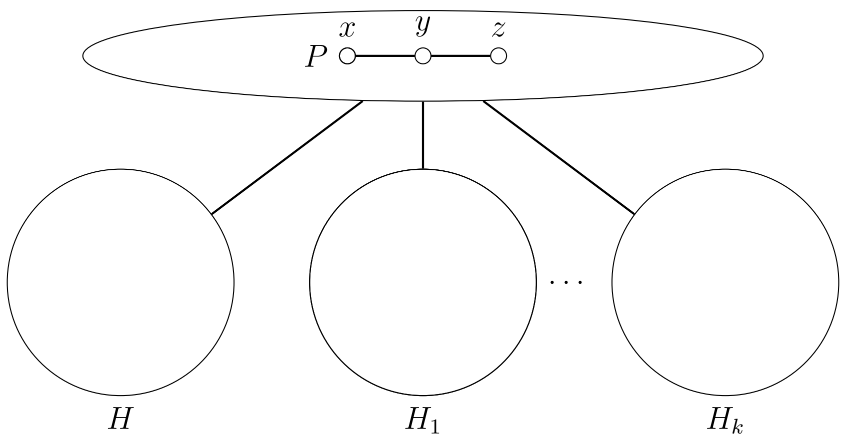


Figure 2.2: The partition described in proof of Theorem 30, where $H, H_1, ..., H_k$ are the connected components of $G - P$.

the remaining components of $G - P$ are all connected to P), so if $|G - H| \geq 6$, by applying the hypothesis to $G - H$ and covering H with a path to get that $T(G) \leq 1 + \frac{n-3}{3} = \frac{n}{3}$. Otherwise, $|G - H| = 5$ since $n \geq 7$ and $G - P$ does not have an isolated vertex as a component, so $G - H - P = K_2$. By assumption G has no leaves and no three or four cycles, so $G - H = C_5$, G is one of the two graphs shown in Figure 2.3 and the theorem holds.

Suppose that $G - P$ has a connected component H of order 4. Then H is a tree. If $|G - H| \geq 6$, then $T(G) \leq 1 + \frac{n-4}{3} = \frac{n-1}{3}$. If $G - H = P$, then $T(G) = 2$, $n = 7$, and the theorem holds. Otherwise $G - H = C_5$, $T(G) \leq 3$ (since $G - H$ may be covered with 2 trees and H is a tree, $n = 9$, and the theorem holds.

Consider $G - P$ having a connected component H of order 5. Then H is either a tree or $H = C_5$. Assume first that H is a tree. If $|G - H| \geq 6$, then $T(G) \leq 1 + \frac{n-5}{3} = \frac{n-2}{3}$. If $G - H = P$, then $T(G) = 2, n = 8$, and the theorem holds. Otherwise, $G - H = C_5$, $T(G) \leq 3$, $n = 10$, and the theorem holds.

Suppose $H = C_5 = (u_1, \ldots, u_5)$, and without loss of generality, assume that u_1 has a neighbor on $P = (x, y, z)$. If $G - H = P$, then $n = 8$ and for $T_1 = G[\{u_2, u_3, u_4, u_5\}]$

and $T_2 = G[\{x, y, z, u_1\}]$, $\mathcal{T} = \{T_1, T_2\}$ is a tree cover of size 2. Otherwise, for path $P' = (u_2, u_3, u_4, u_5)$, $G - P'$ is a connected graph on at least 6 vertices, so $T(G) \leq 1 + \frac{n-4}{3} = \frac{n-1}{3}$.

We may now assume that each component of $G - P$ is K_2 or has at least 6 vertices. If all components of $G - P$ are of order at least 6, then by the induction hypothesis, $T(G) \leq 1 + \frac{n-3}{3} = \frac{n}{3}$. Suppose $G - P$ has exactly one component that is $K_2 = (u, v)$. Since G has no leaves then each of u and v must be adjacent to a vertex of P, and since G has no three or four cycles, then u must be adjacent to x and v must be adjacent to z. Furthermore, since $n \geq 7$, then $G - P$ must have a component H with at least 6 vertices. Note that H has a vertex that is adjacent to some $r \in \{x, y, z\}$. By adding r to H, we partition G into a tree (namely, the tree with vertex set $\{x, y, z, u, v\} \setminus \{r\}$) and connected components of order at least 6. Thus, $T(G) \leq 1 + \frac{n-4}{3} = \frac{n-1}{3}$.

Suppose $G - P$ has $s \geq 2$ components that are K_2. We first show that the vertices of $P = (x, y, z)$ and the vertices of each K_2 can be covered with two trees: recall that since G has no leaves then each endpoint of a K_2 must be adjacent to a vertex of P, and since G has no three or four cycles, then for each K_2, one endpoint must be adjacent to x and the other end must be adjacent to z. Let X be the set of endpoints that are adjacent to x and let Z be the set of of endpoints that are adjacent to z. Then for $T_1 = G[X \cup \{x\}]$ and $T_2 = G[Z \cup \{z, y\}]$, $\mathcal{T} = \{T_1, T_2\}$ is a tree cover of size two that covers the vertices of P and the vertices of $G - P$ belonging to a K_2. We apply the induction hypothesis to each component of $G - P$ with at least 6 vertices to get that $T(G) \leq 2 + \frac{n-3-2s}{3} \leq \frac{n-1}{3}$. $\square$

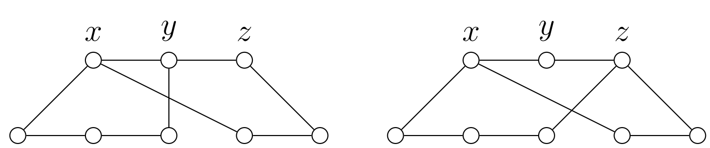


Figure 2.3: Graphs in Proof of Theorem 30

Corollary 31. *If G is a connected outerplanar graph on n vertices with girth at least 5, then* $\mathrm{M}_+(G) \leq \frac{n}{3}$.

Computations in Sage suggest the next conjecture.

Conjecture 32. *For all connected triangle-free graphs,* $T(G) \leq \lceil \frac{n}{3} \rceil$.

2.3 Graphs with $T(G) \leq \mathrm{M}_+(G)$

In this section, we prove that $T(G) \leq \mathrm{M}_+(G)$ for certain line graphs, for $G^{\triangle}$ (defined below) where G is any graph, for graphs whose complements have sufficiently small tree-width, and for graphs with a sufficiently large number of edges.

For a graph $G = (V, E)$, the *line graph* of G, denoted $L(G)$, is the graph whose vertex set is the edge set of G, and two vertices are adjacent in $L(G)$ if and only if the corresponding edges share an endpoint in G.

Theorem 33. *Let G be a graph on n vertices and $m \geq 2n$ edges. Then* $T(L(G)) \leq \mathrm{M}_+(L(G))$.

Proof. The adjacency matrix of $L(G)$ is $A(L(G)) = B^T B - 2I_m$, where B is the vertex-edge incidence matrix of G and I_m is the $m \times m$ identity matrix. Note that $A(L(G)) + 2I_m = B^T B$ is in $S_+(L(G))$ and that $\operatorname{rank}(B) \leq n$. So, $\operatorname{mr}_+(L(G)) \leq \operatorname{rank}(B^T B) = \operatorname{rank}(B) \leq n$. It follows that $\mathrm{M}_+(L(G)) \geq m - n \geq \lceil \frac{m}{2} \rceil \geq T(L(G))$, where the second inequality follows from the fact that $m \geq 2n$ and the last inequality follows from Theorem 28. □

Definition 34. For a graph $G = (V, E)$, let $G^{\triangle}$ be the graph constructed from G such that for each edge $e = \{u, v\} \in E$, add a new vertex w_e where w_e is adjacent to exactly u and v. The vertices w_e are called *edge-vertices* of $G^{\triangle}$.

Theorem 35. *For a connected graph $G = (V, E)$ on n vertices and m edges,* $T(G^{\triangle}) \leq \mathrm{M}_+(G^{\triangle})$.

Proof. We show that $\mathrm{mr}_+(G^\triangle) = \alpha(G^\triangle)$ and then apply Proposition 24. It is always the case that a connected graph H has $\alpha(H) \leq \mathrm{mr}_+(H)$ (see Corollary 2.7 in [3]), so we show that $\mathrm{mr}_+(G^\triangle) \leq \alpha(G^\triangle)$. Let B be the vertex-edge incidence matrix of G, and let $X = \begin{pmatrix} I_m \\ B \end{pmatrix}$, where I_m is the $m \times m$ identity matrix. Then $XX^T = \begin{pmatrix} I_m & B^T \\ B & BB^T \end{pmatrix} \in \mathcal{S}_+(G^\triangle)$, where the first m rows and columns are indexed by the edge-vertices and the last n rows and columns are indexed by the vertices in V. Note that the set of edge-vertices of $G^\triangle$ is an independent set of size m and that the rank of XX^T is m. So $\mathrm{mr}_+(G^\triangle) \leq m \leq \alpha(G)$, and therefore $\mathrm{mr}_+(G^\triangle) = \alpha(G^\triangle)$. By Proposition 24, $\mathrm{T}(G^\triangle) \leq m + n - \mathrm{mr}_+(G^\triangle) = \mathrm{M}_+(G^\triangle)$. □

The next theorem shows that the conjecture $T(G) \leq \mathrm{M}_+(G)$ holds true for graphs with a large number of edges.

Theorem 36. *Let G be a graph on n vertices and $m \leq \frac{3n}{2} - 4$ edges. Then $T(\overline{G}) \leq \mathrm{M}_+(\overline{G})$.*

Proof. Let T be a spanning tree of G. Then $\mathrm{M}_+(\overline{T}) \geq n - 3$ (see [1, Theorem 3.16 and Corollary 3.17]). Note that G can be obtained from T by adding at most $m - (n-1)$ edges, so $\overline{G}$ can be obtained from $\overline{T}$ by deleting at most $m - (n-1)$ edges. Since edge deletion decreases the positive semidefinite maximum nullity by at most 1, then

$$\mathrm{M}_+(\overline{G}) \geq \mathrm{M}_+(\overline{T}) - (m - (n-1)) \geq (n-3) - (m - (n-1)) \geq \frac{n}{2},$$

where the last inequality follows from the fact that $m \leq \frac{3n}{2} - 4$. Since $\mathrm{M}_+(G)$ is an integer, by Theorem 28, we have that $\mathrm{M}_+(\overline{G}) \geq T(\overline{G})$. □

The *tree-width* of a graph G, denoted $\mathrm{tw}(G)$, is a widely studied parameter, and there are multiple ways in which it is defined. Here we define the tree-width in terms of chordal completions. A graph is *chordal* if it has no induced cycle on four or more vertices. If G

is a subgraph of H such that $V(G) = V(H)$ and H is chordal, then H is called a *chordal completion* of G. The tree-width of G is defined as

$$\operatorname{tw}(G) = \min\{\omega(H) - 1 | H \text{ is a chordal completion of } G\}.$$

Proposition 37. *Let G be a graph on n vertices with $\operatorname{tw}(G) \leq \frac{n-4}{2}$. Then $T(\overline{G}) \leq \mathrm{M}_+(\overline{G})$.*

Proof. If $\operatorname{tw}(G) \leq k$, then $\operatorname{mr}_+(\overline{G}) \leq k+2$ [9], i.e., $\mathrm{M}_+(\overline{G}) \geq n-k-2$. For $k = \frac{n-4}{2}$, it follows that $\mathrm{M}_+(\overline{G}) \geq \frac{n}{2}$. Since $\mathrm{M}_+(\overline{G})$ is an integer, we have that $\mathrm{M}_+(\overline{G}) \geq \lceil \frac{n}{2} \rceil \geq T(\overline{G})$, where the last inequality follows from Theorem 28. □

2.4 M_+ and T for connected outerplanar graphs

We now turn our attention specifically to connected outerplanar graphs on $n \geq 2$. We have seen that $\mathrm{M}_+(G) = T(G) \leq \lceil \frac{n}{2} \rceil$, and in this section we characterize graphs that achieve this upper bound. Let $\mathcal{F}$ denote the block-clique graphs such that each clique is K_3 (see Figure 2.4), and observe that every graph in $\mathcal{F}$ has an odd number of vertices. We begin by stating the results that provide this characterization.

Theorem 38. *Let G be a connected outerplanar graph of odd order $n \geq 3$. Then $T(G) = \lceil \frac{n}{2} \rceil$ if and only if $G \in \mathcal{F}$.*

Corollary 39. *Let G be a connected outerplanar graph of odd order n. Then $\mathrm{M}_+(G) = \lceil \frac{n}{2} \rceil$ if and only if $G \in \mathcal{F}$.*

Observe that the only connected graph of order $n = 2$, K_2, has tree cover number $\frac{n}{2} = 1$.

Theorem 40. *For a connected outerplanar graph $G = (V, E)$ of even order $n \geq 4$, $T(G) = \frac{n}{2}$ if and only if one of the following holds:*

(1) G is obtained from some $G' \in \mathcal{F}$ by adding one leaf.

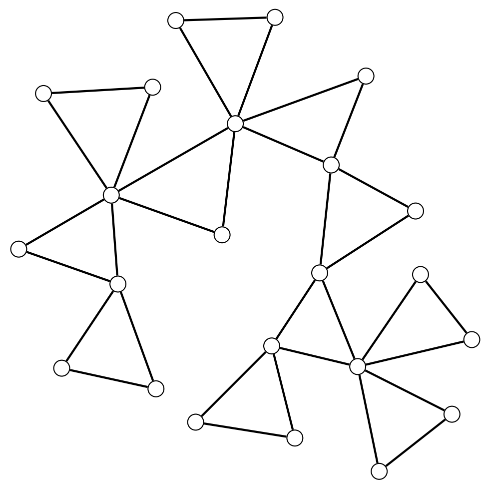


Figure 2.4: A block-clique graph such that each clique is K_3.

(2) G is obtained from some $G_1, G_2 \in \mathcal{F}$ by connecting them with a bridge.

(3) G is constructed from the following iterative process: Start with $G^{[0]} \in \{C_4, K_4 - e, C_r^{\triangle}($ for some $r \geq 3)\}$. For $i \geq 1$, pick a $v \in V(G^{[i-1]})$ and let $G^{[i]} = G^{[i-1]} \oplus_v K_3$.

Corollary 41. *Let $G = (V, E)$ be a connected outerplanar graph of even order n. Then $\mathrm{M}_+(G) = \frac{n}{2}$ if and only if one of (1), (2), (3) of Theorem 40 holds.*

To prove Theorem 38, we use the next observation and the next two lemmas.

Observation 42. *If $G = (V, E)$ is a block-clique graph on $n \geq 5$ vertices such that each clique is a K_3, then it can be seen by induction that $G = G' \oplus_v K_3$ for some $v \in V$, where G' is a block-clique graph on $n - 2$ vertices such that each clique is a K_3.*

Lemma 43. *If G is a connected graph with $n \geq 3$ vertices and $T(G) = \left\lceil \frac{n}{2} \right\rceil$, then there exist adjacent vertices $u, v \in V(G)$ such that $G' = G[V(G) \setminus \{u, v\}]$ remains connected. Furthermore, $T(G') = \left\lceil \frac{n-2}{2} \right\rceil$.*

Proof. By Lemma 27, we may remove an induced subgraph $H = K_{1,p}$ such that $G - H$ remains connected. First note that if $p \geq 3$, then $T(G) \leq 1 + \left\lceil \frac{n-4}{2} \right\rceil < \left\lceil \frac{n}{2} \right\rceil$, which is a contradiction, so $p \leq 2$. If $p = 1$, then we are done. Suppose $p = 2$ (i.e., H is a path (x, y, z)). If x and z are both leaves in G, then $T(G) = T(G - \{x, z\}) \leq \left\lceil \frac{n-2}{2} \right\rceil$, which is a contradiction to $T(G) = \frac{n}{2}$. So without loss of generality, x has a neighbor in $G - H$, and the theorem holds with $u = y$ and $v = z$.

It is easy to see that if $T(G') < \left\lceil \frac{n-2}{2} \right\rceil$, then $T(G) < \left\lceil \frac{n}{2} \right\rceil$. So, $T(G') = \left\lceil \frac{n-2}{2} \right\rceil$. $\square$

Lemma 44. *Let $G = (V, E)$ be a connected graph and suppose $u, v \in V$ are adjacent vertices such that $G' = G[V \setminus \{u, v\}]$ is connected. Let $\mathcal{T}'$ be a minimum tree cover of G', and suppose there exists $w \in V(G')$ such that*

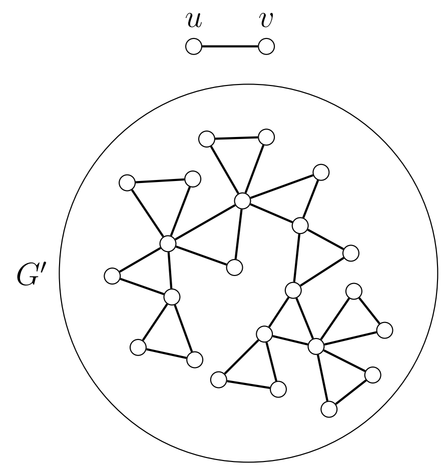
(1.) $V(T_w) = \{w, x\}$

(2.) $\exists y \in N(w) \cap N(x)$ *such that* $N(x) \cap V(T_y) = \{y\}$.

If u is adjacent to w and v is not adjacent to w, then $T(G) \leq T(G')$.

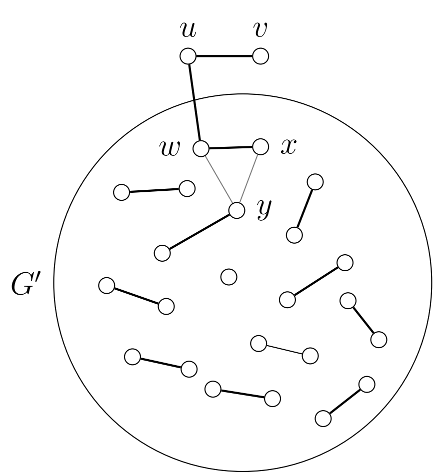
Proof. For $\mathcal{T} = (\mathcal{T}' \setminus \{T_w \cup T_y\}) \cup G[\{u, v, w\}] \cup G[V(T_y) \cup \{x\}], \mathcal{T}$ is a tree cover of G of size $T(G')$. $\square$

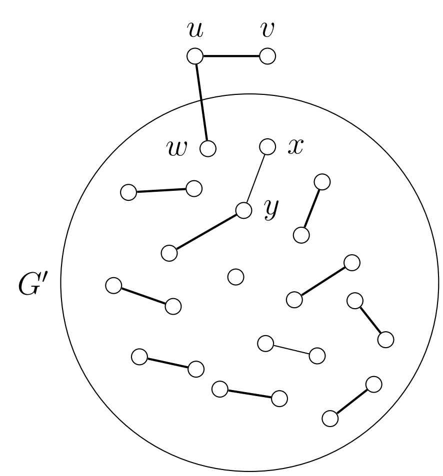
Proof of Theorem 38. Let G be a graph on $n = 2k+1$ vertices and first suppose $T(G) = \left\lceil \frac{n}{2} \right\rceil$. We prove that $G \in \mathcal{F}$ by induction on k. If $k = 1$, then $G = K_3$. Let $n = 2k + 1$ where $k \geq 2$ and suppose that the claim holds for graphs with $2(k - 1) + 1$ vertices. By Lemma 43, we can delete an induced subgraph $H = K_{1,1}$ such that $G - H$ is connected. Note that since $T(G) = \left\lceil \frac{n}{2} \right\rceil$, then $T(G - H) = \left\lceil \frac{n-2}{2} \right\rceil$. By the induction hypothesis, $G - H \in \mathcal{F}$ (see the next figure).



Furthermore, by using Lemma 43, $G-H$ has a minimum tree cover, $\mathcal{T}$, such that one tree has exactly one vertex and the remaining trees have exactly two vertices. Let $V(H) = \{u, v\}$. We show that $G \in \mathcal{F}$ by showing 1) if u is adjacent to a vertex $w \in V(G-H)$, then v must also be adjacent to w and 2) u (and therefore v) is adjacent to exactly one vertex in $V(G-H)$.

To see 1), suppose u is adjacent to $w \in V(G-H)$ and v is not adjacent to w. If $V(T_w) = \{w\}$, then $\mathcal{T}' = (\mathcal{T} \setminus T_w) \cup G[\{w, v, u\}]$ is a tree cover of G of size $\lceil \frac{n-2}{2} \rceil$, which is a contradiction. Otherwise, $T_w = P_2 = (w, x)$ for some $x \in V(G-H)$. Since each edge of $G-H$ belongs to a triangle, then there exists $y \in V(G-H)$ such that $y \in N(w) \cap N(x)$. Since any two triangles in $G-H$ share at most one vertex, it follows from Lemma 44 that $T(G) \leq \lceil \frac{n-2}{2} \rceil$. (See the next figure.) So v must be adjacent to w.





To see 2), suppose that v and u were adjacent to $x, y \in V(G-H)$. We show that G is not outerplanar. Since $G-H$ is connected, then there is a path, P, in $G-H$ with endpoints x to y. Then $G[V(P) \cup \{v, u\}]$ has a K_4-minor, which contradicts G being outerplanar.

We show the converse by induction on k. Let $G \in \mathcal{F}$. For $k = 1$, $G = K_3$, so $T(G) = \left\lceil \frac{n}{2} \right\rceil$. For $k \geq 2$, by Observation 42, $G = G' \oplus_v K_3$ for some $v \in V$, where G' is a graph on $n-2$ vertices and $G' \in \mathcal{F}$. It follows from the induction hypothesis and Proposition 23 that $T(G) = T(G') + T(K_3) - 1 = \left\lceil \frac{n-2}{2} \right\rceil + 1 = \left\lceil \frac{n}{2} \right\rceil$. $\square$

To prove Theorem 40, we use an additional lemma.

Lemma 45. *Let $G = (V, E)$ be a connected graph of even order n with $T(G) = \frac{n}{2}$ that satisfies the following conditions:*

(a) G does not have a bridge.

(b) $\delta(G) \geq 2$, and if $z \in V$ is a vertex such that $N(z) = \{z', z''\}$, then z' and z'' are adjacent.

Let $u, v \in V$ be adjacent vertices in G such that $G' = G[V \setminus \{u, v\}]$ remains connected and $T(G') = \frac{n-2}{2}$. If G' does not have a leaf, then one of the following holds:

1. *G' satisfies (a) and (b).*

2. *G' satisfies (3) of Theorem 40.*

3. *G satisfies (3) of Theorem 40.*

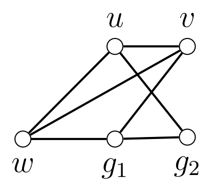
Proof. Assume G' has no leaves. Suppose first that $e = \{g_1, g_2\}$ is a bridge in G' (i.e., G' does not satisfy (a)) and let G_1, G_2 be the connected components of $G' - e$, where $g_1 \in V(G_1)$ and $g_2 \in V(G_2)$. We show that G satisfies (3). By hypothesis, $T(G') = \frac{n-2}{2}$ and by Lemma

25, $T(G') = T(G_1) + T(G_2) - 1$. It follows that $|G_i|$ is odd and $T(G_i) = \left\lceil \frac{|G_i|}{2} \right\rceil$ for $i = 1, 2$. Note that since G' has no leaves, $|G_i| \geq 3$ for $i = 1, 2$, and by Theorem 38, $G_i \in \mathcal{F}$.

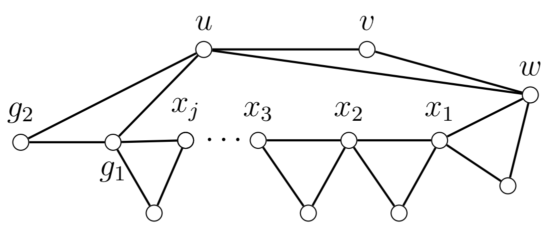
Let W be the set of vertices in $V(G') \setminus \{g_1, g_2\}$ that are adjacent to either u or v. We first show that for each $w \in W$, w is adjacent to both u and v. Without loss of generality, let u have a neighbor w in W, suppose v is not adjacent to w, and suppose that $w \in V(G_1)$. Let $\mathcal{T}'$ be a tree cover of G' such that each tree has exactly two vertices ($\mathcal{T}'$ is guaranteed by Lemma 43) and let $T_w = G[\{w, x\}]$ be the tree containing w. Note that $x \in V(G_1)$ since $w \neq g_1$. Since $G_1 \in \mathcal{F}$, w and x have a common neighbor y. Let $V(T_y) = \{y, y'\}$, and note that $y' \notin N(x)$ (if $y' \in V(G_1)$ then this follows from the fact that $G_1 \in \mathcal{F}$, and if $y' \in V(G_2)$, then this follows from the fact that x has no neighbor in $V(G_2)$). By Lemma 44, $T(G) \leq T(G') = \frac{n-2}{2}$, which contradicts $T(G) = \frac{n}{2}$.

Thus, v is adjacent to w, and this shows that u and v have the same set of neighbors in $V(G') \setminus \{g_1, g_2\}$. Furthermore, since G is outerplanar, it follows that $|W| \leq 1$. If $W = \emptyset$, then $G[\{u, v, g_1, g_2\}]$ is $K_4 - e$ (since u and v are not leaves, G is outerplanar, e is not a bridge in G, and the neighbors of a degree two vertex in G must be adjacent). Since $G_1, G_2 \in \mathcal{F}$, it follows that G satisfies (3) with $G^{[0]} = K_4 - e$.

Consider $|W| = 1$, let $W = \{w\}$, and without loss of generality, suppose $w \in V(G_1)$. Since e is not a bridge in G, then u or v must be adjacent to a vertex in $V(G_2)$, and since $|W| = 1$, this vertex must be g_2. Without loss of generality, suppose u is adjacent to g_2, and note that v cannot also be adjacent to g_2 since G is outerplanar. Suppose first that $N(u) \cap (V(T_{g_2}) \setminus \{g_2\}) = \emptyset$. If $N(v) \cap (V(T_w) \setminus \{w\}) = \emptyset$, then $(\mathcal{T}' \setminus (T_w \cup T_{g_2})) \cup G[V(T_w) \cup \{v\}] \cup G[V(T_{g_2}) \cup \{u\}]$ is a tree cover of G of size $\frac{n-2}{2}$, which contradicts $T(G) = \frac{n}{2}$. Thus, $N(v) \cap (V(T_w) \setminus \{w\}) \neq \emptyset$, so it must be the case that $T_w = \{w, g_1\}$ and v is adjacent to g_1. But then $G[\{u, v, w, g_1, g_2\}]$ has a K_4 minor (see the next figure), which is a contradiction to G being outerplanar.



So, $N(u) \cap (V(T_{g_2}) \setminus \{g_2\}) \neq \emptyset$, and it must be the case that $T_{g_2} = \{g_1, g_2\}$ and u is adjacent to g_1. Note that since G is outerplanar, v is not adjacent to g_1 nor g_2. It follows that G satisfies (3) with $G^{[0]} = C_r^{\triangle}$, where $C_r = (u, w, x_1, \ldots, x_j, g_1)$ and $(w, x_1, \ldots, x_j, g_1)$ is the path between w and g_1 in G_1 (see the next figure).



Suppose now that G' does not satisfy (b). We show that G' satisfies (3). Since G' does not satisfy (b), then there exists a vertex $z \in V(G')$ of degree 2 whose neighbors z' and z'' are not adjacent. By contracting the edge $\{z, z'\}$, we obtain a graph H from G' on $n-3$ vertices with $T(H) = \lceil \frac{n-3}{2} \rceil$. So, $H \in \mathcal{F}$. Then for the triangle (z', z'', y) in H, (z', z, z'', y) is a 4 cycle in G', and G' satisfies (3) of Theorem 40 with $G^{[0]} = C_4$. $\square$

Proof of Theorem 40. Let G be a graph on $n = 2k$ vertices and first suppose $T(G) = \frac{n}{2}$. If G has a leaf v, then $T(G) = T(G - v)$, and $G - v$ is in $\mathcal{F}$ by Theorem 38. Thus (1) holds. If G has a bridge e and the connected components of $G - e$ are G_1 and G_2, then by Lemma 25, $T(G) = T(G - e) - 1 = T(G_1) + T(G_2) - 1$. Note that $|G_1|$ and $|G_2|$ must both be even or both are odd since n is even. If both are even then $T(G) = T(G_1) + T(G_2) - 1 \leq \frac{|G_1|}{2} + \frac{|G_2|}{2} - 1 = \frac{n}{2} - 1$, which contradicts $T(G) = \frac{n}{2}$. Thus, $|G_1|$ and $|G_2|$ are both odd, and $\frac{n}{2} = T(G) = T(G_1) + T(G_2) - 1 \leq \left\lceil \frac{|G_1|}{2} \right\rceil + \left\lceil \frac{|G_2|}{2} \right\rceil - 1 = \frac{|G_1|+1}{2} + \frac{|G_2|+1}{2} - 1 = \frac{n}{2}$. It follows that $T(G_i) = \left\lceil \frac{|G_i|}{2} \right\rceil$ for $i = 1, 2$, so (2) holds. Suppose G can be obtained from some graph

G' by subdividing an edge of G'. Since subdividing an edge does not change the tree cover number, then $\frac{n}{2} = T(G) = T(G')$ and by Theorem 38, $G' \in \mathcal{F}$. Note that subdividing an edge of a graph in $\mathcal{F}$ results in a graph in (3) with $G^{[0]} = C_4$, so G satisfies (3).

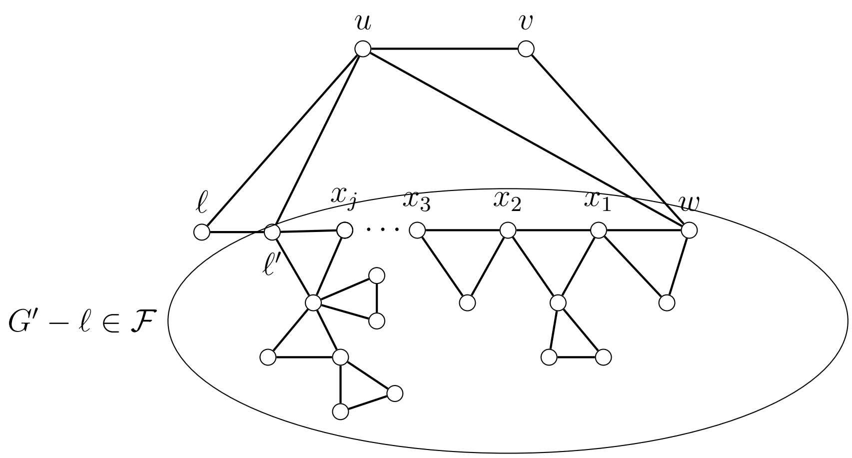
We may now assume that $\delta(G) \geq 2$, G does not have a bridge, and for each $v \in V$ with $\deg(v) = 2$, the neighbors of v are adjacent. For the remainder of the proof, u and v are the adjacent vertices from Lemma 43 such that $G' = G[V(G) \setminus \{u, v\}]$ is connected and $T(G') = \frac{n-2}{2}$. We consider two cases, G' has a leaf and G' does not have a leaf.

Case 1. Suppose G' has a leaf ℓ and let ℓ' be its neighbor. We show G satisfies (3). We first show that u and v have the same set of neighbors in $V(G') \setminus \{l, l'\}$. Note that $T(G' - \ell) = T(G') = \lceil \frac{n-3}{2} \rceil$ and by Theorem 38, $G' - \ell$ is in $\mathcal{F}$. Suppose u has a neighbor w in $V(G') \setminus \{l, l'\}$ and v is not adjacent to w. Let $\mathcal{T}'$ be a tree cover of G' such that each tree has exactly two vertices ($\mathcal{T}'$ is gauranteed by Lemma 43) and let $T_w = \{w, x\}$ be the tree containing w. Since $G' - \ell \in \mathcal{F}$, then w and x have a common neighbor y such that $T_y = \{y, z\} \in \mathcal{T}'$ and $z \notin N(x)$. By Lemma 44, $T(G) \leq \frac{n-2}{2}$, contradicting $T(G) = \frac{n}{2}$. Therefore u and v have the same set of neighbors in $V(G') \setminus \{\ell, \ell'\}$, and since G is outerplanar, this set has cardinality at most one.

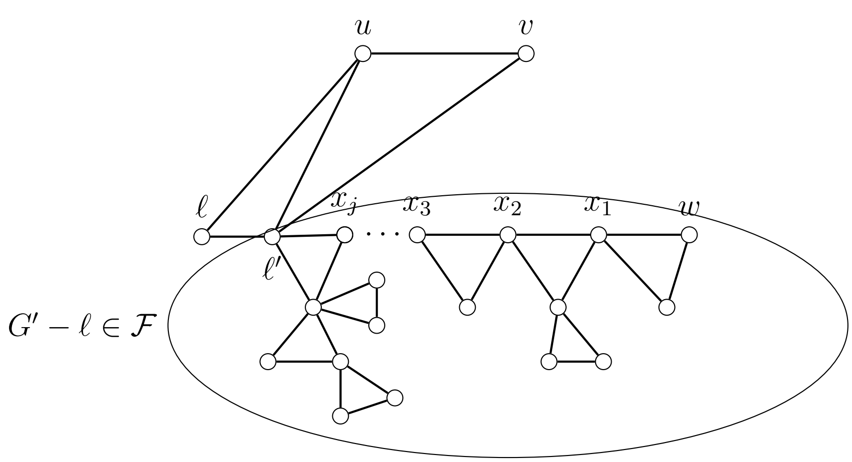
Since G has no leaves, ℓ is not a leaf, so we may assume that u is adjacent to ℓ. Suppose first that v is also adjacent to ℓ. Since $\{\ell, \ell'\}$ is not a bridge in G, then either u or v must have a neighbor in $G' - \ell$, and since G is outerplanar, u and v cannot both have a neighbor in $G' - \ell$ (since if we contract each edge of $G' - \ell$ to obtain a single vertex t, then the graph induced on $\{u, v, \ell, t\}$ would form a K_4). Without loss of generality, we let u have a neighbor $w \in V(G' - \ell)$. Since u and v have the same neighbors in $V(G') \setminus \{\ell, \ell'\}$, then $w = \ell'$, $G[\{u, v, \ell, \ell'\}]$ is $K_4 - e$, and G satisfies (3) with $G^{[0]} = K_4 - e$.

Assume v is not adjacent to ℓ. Since ℓ has degree 2 in G, by hypothesis u is adjacent to ℓ', and since G has no leaves, v has a neighbor $w \in V(G' - \ell)$. If $w \neq \ell'$, we have already seen that u must also be adjacent to w and that $N(u) \cap (V(G) \setminus \{\ell, \ell'\}) = N(v) \cap (V(G) \setminus \{\ell, \ell'\}) = \{w\}$.

Also note that if $w \neq \ell'$, then v cannot also be adjacent to ℓ' since G is outerplanar, so $N(u) = \{v, \ell, \ell', w\}$ and $N(v) = \{u, w\}$. To see that G satisfies (3), let $(\ell', x_1, \ldots, x_j, w)$ be the shortest path from ℓ' to w in G' (see the next figure). Since $G' - \ell \in \mathcal{F}$, it follows that G satisfies (3) with $G^{[0]} = C_r^{\triangle}$ and $C_r = (u, \ell, \ell', x_1, \ldots, x_j, w)$.

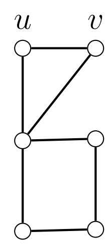


If v is adjacent to ℓ', then u and v share the same set of neighbors in $V(G') \setminus \{\ell\}$, and since G is outerplanar we must have that $N(u) = \{v, \ell, \ell'\}, N(v) = \{u, \ell'\}$ (see the next figure). Thus, G satisfies (3) with $G^{[0]} = K_4 - e$.

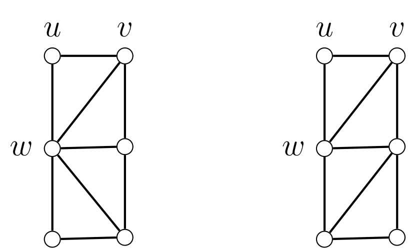


Case 2: Suppose G' does not have a leaf. We prove this case by induction on n. Let $n = 6$. Then G' is a graph on four vertices with tree cover number two. Since G' does not have a leaf, then G' is $K_4 - e$ or C_4.

Suppose first that $G' = C_4$. If u has a neighbor $w \in V(C_4)$ and v is not adjacent to w, then for $T_1 = G[\{u, v, w\}]$ and $T_2 = G[V(C_4) \setminus \{w\}]$, $\{T_1, T_2\}$ is a tree cover of G of size 2, contradiction $T(G) = 3$. So u and v have the same set of neighbors in $V(C_4)$, and since G is outerplanar, u and v have exactly one neighbor in $V(C_4)$ (see next figure), and (3) holds.



Consider $G' = K_4 - e$. It is well known that an outerplanar graph on n vertices has at most $2n-3$ edges (this can be proven by deleting a vertex of degree two and using induction on n). Thus G has at most nine edges. Since there are five edges in $K_4 - e$ and one edge between u and v, there are at most three edges between the sets $\{u, v\}$ and $V(K_4 - e)$, so either u or v has degree two (since G has no leaves). Suppose $N(u) = \{v, w\}$ for some $w \in V(K_4 - e)$. By hypothesis, v and w are adjacent. Note that since G has at most nine edges, v can have at most one additional neighbor. Suppose v has an additional neighbor in $V(K_4 - e)$. Then G is one of the graphs given in the next figure, and $T(G) = 2$, contradicting $T(G) = 3$. Thus, v has no additional neighbors, and G satisfies (3) with $G^{[0]} = K_4 - e$.



Let $n \geq 8$. Since G' has no leaves, by Lemma 45 we either have that G satisfies (3) (in which case the proof is complete), G' has no bridge and is not a subdivision, or G' satisfies (3).

Suppose that G' has no bridge and is not a subdivision. We show that G' satisfies (3). Let G'' be the graph obtained from G' after one more application of Lemma 43. If G'' has a leaf, G' satisfies (3) by case 1. If G'' does not have a leaf, then by the induction hypothesis G' satisfies (3).

We now use the fact that G' satisfies (3) to show that G satisfies (3) by showing that $N(u) = \{v, w\}$ and $N(v) = \{u, w\}$, for some $w \in V(G')$ (i.e., $G = K_3 \oplus_w G'$). Since u is not a leaf, let $w \in V(G')$ be a neighbor of u and suppose first that v is not adjacent to w. We show that this contradicts $T(G) = \frac{n}{2}$. Let $\mathcal{T}'$ be a minimum tree cover of G' with each tree having exactly two vertices and let $T_w = \{w, x\}$. We consider two cases, there exists $y \in N(w) \cap N(x)$ and $N(w) \cap N(x) = \emptyset$. Let $y \in N(w) \cap N(x)$ and let $T_y = \{y, z\}$. If x is not adjacent to z, by Lemma 44, $T(G) \leq \frac{n-2}{2}$, so x is adjacent to z and $G[\{w, x, y, z\}]$ is $K_4 - e$. Since G is outerplanar and v is not adjacent to w, then it can be seen by examination that $G[\{u, v, w, x, y, z\}]$ can be covered with two trees, contradicting $T(G) = \frac{n}{2}$.

Consider $N(w) \cap N(x) = \emptyset$. Note that if G' satisfies (3) with $G^{[0]} \in \{K_4 - e, C_r^{\triangle}\}$, then every edge of G' would belong to a triangle, so $N(w) \cap N(x) = \emptyset$ implies that G' satisfies (3) with $G^{[0]} = C_4$. Furthermore, every edge of G' that is not an edge of C_4 belongs to a triangle, so $\{w, x\}$ is an edge on C_4. Since v is not adjacent to w, we may cover $G[V(C_4) \cup \{u, v\}]$ with two trees, contradicting $T(G) = \frac{n}{2}$. So, v must be adjacent to w, which shows that u and v have the same set of neighbors on G'. Since G is outerplanar, u and v must have exactly one common neighbor in G', which shows that G satisfies (3).

We now show the converse. The removal of a leaf does not affect the tree cover number of a graph, so if G satisfies (1), then $T(G) = \frac{n}{2}$. If G satisfies (2), then by Lemma 25, $T(G) = T(G_1) + T(G_2) - 1 = \left\lceil \frac{|G_1|}{2} \right\rceil + \left\lceil \frac{|G_2|}{2} \right\rceil - 1 = \frac{n}{2}$. For a graph G satisfying (3), $\frac{n}{2} = T(G') = T(G)$ since subdividing does not affect tree cover number. Suppose G satisfies (3). If $G \in \{C_4, K_4 - e, C_r^{\triangle}\}$, then $T(G) = \frac{n}{2}$. Let $G = G^{[k]}$ for some $k \geq 1$. By Proposition 23, and by induction, $T(G) = T(G^{[k-1]}) + T(K_3) - 1 = \frac{n}{2}$. $\square$

The next theorem gives a formula for computing $T(G)$ (and therefore $\mathrm{M}_+(G)$) of cactus graphs.

Theorem 46. *Let $G = (V, E)$ be a cactus graph on $n \geq 3$ vertices. Then $\mathrm{M}_+(G) = T(G) = k + 1$, where k is the number of cycles in G.*

Proof. Let $E' \subset E$ be the set of bridges of G. By Lemma 25, the deletion of a bridge increases the tree cover number by exactly one, so $T(G) = T(G - E') - |E'|$. Let $H_1, \ldots, H_r$ be the connected components of $G - E'$. Observe that H_i does not have a bridge for $i = 1, ..., r$. We first show that for each i, $T(H_i) = c_i + 1$, where c_i is the number of cycles in H_i. Note that H_i is either a single vertex, or it is a cactus graph where each block is a cycle (see Figure 2.5 for example). If H_i is a single vertex, then $T(H_i) = 1 = c_i + 1$. Suppose H_i is a cactus graph where each block is a cycle. Note that any two cycles in H_i share at most one common vertex. It follows from Proposition 23 and induction on c_i that $T(H_i) = c_i + 1$. Thus

$$T(G) = T(G - E') - |E'| = \left(\sum_{i=1}^{r} T(H_i)\right) - |E'| = \left(\sum_{i=1}^{r} c_i\right) + r - |E'| = \left(\sum_{i=1}^{r} c_i\right) + 1,$$

where the last equality follows from the fact that $r - |E'| = 1$. Furthermore, since bridges are not edges of cycles, we have that G and $G - E'$ have the same number of cycles (i.e., $\left(\sum_{i=1}^{r} c_i\right) = k$), so $T(G) = k + 1$. $\square$

Example 47. For the graph G given in Figure 2.5, $T(G) = 6$.

2.5 Tree cover number and other graph parameters

In this secton, we give relationships between the tree cover number and other graph parameters.

Proposition 48. *Let $G = (V, E)$ be a claw-free graph on n vertices and diameter d. Then $\lceil \frac{n}{d+1} \rceil \leq T(G)$.*

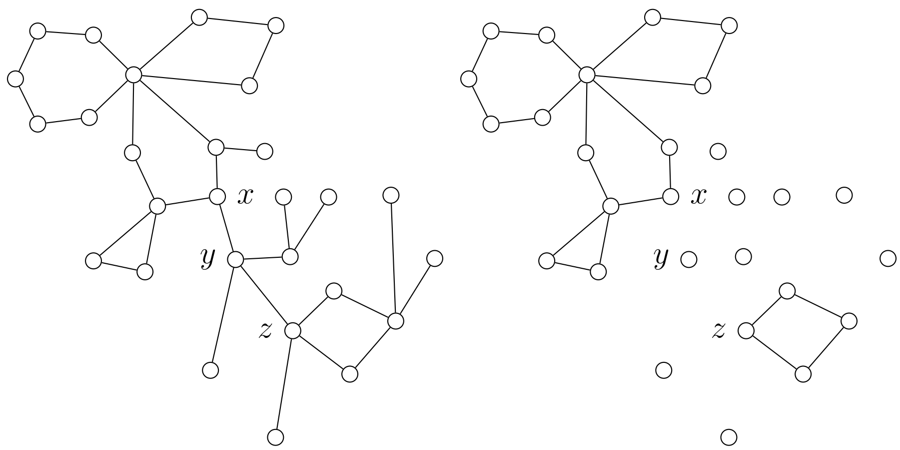


Figure 2.5: Graph G and $G - E'$ of Example 47

Proof. Any induced tree in G is a path. Let $\mathcal{T} = \{T_1, \dots, T_k\}$ be a minimum tree cover of G. Then $n = \sum_{1}^{k} |T_i| \leq (d+1)k$. $\square$

For a graph G, recall that $\omega(G)$ denotes the size of the largest clique in G.

Proposition 49. *For any graph $G = (V, E)$, $\left\lceil \frac{\omega(G)}{2} \right\rceil \leq T(G)$.*

Proof. Any tree in a tree cover of G can contain at most 2 vertices from a clique. $\square$

Corollary 50. *For a graph $G = (V, E)$ with maximum degree $\Delta(G)$ and line graph $L(G)$, $\left\lceil \frac{\Delta(G)}{2} \right\rceil \leq T(L(G))$.*

Proof. For a vertex v in $V(G)$, the set of edges incident to v corresponds to a clique in the line graph $L(G)$. So $\Delta(G) \leq \omega(G)$. The result follows from Theorem 49. $\square$

Proposition 51. *Let $G = (V, E)$ be triangle-free and let $\gamma(G)$ be the domination number of G. Then $T(G) \leq \gamma(G)$.*

Proof. Let $D \subseteq V$ be a dominating set of G, and for each $d_i \in D$, let V_i be the set of vertices dominated by d_i (we require that each $v \in V$ is assigned to exactly one V_i). Then $G[V_i \cup \{d_i\}]$ is a tree (in particular, a star) since G is triangle free, so $T(G) \leq \gamma(G)$. $\square$

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CHAPTER 3. ON THE POWER PROPAGATION TIME OF A GRAPH

A paper submitted for publication.

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Abstract

In this paper, we give Nordhaus-Gaddum upper and lower bounds on the sum of the power propagation time of a graph and its complement, and we consider the effects of edge subdivisions and edge contractions on the power propagation time of a graph. We also study a generalization of power propagation time, known as $k-$power propagation time, by characterizing all simple graphs on n vertices whose $k-$power propagation time is $n-1$ or $n-2$ (for $k \geq 1$) and $n-3$ (for $k \geq 2$). We determine all trees on n vertices whose power propagation time ($k=1$) is $n-3$, and give partial characterizations of graphs whose $k-$power propagation time is equal to 1 (for $k \geq 1$).

3.1 Introduction

Phasor Measurement Units (PMUs) are machines used by energy companies to monitor the electric power grid. They are placed at selected electrical nodes (locations at which transmission lines, loads, and generators are connected) within the system. Due to the high cost of the machines, an extensive amount of research has been devoted to minimizing the number of PMUs needed while maintaining the ability to observe the entire system. In [9], Haynes et al. studied this problems in terms of graphs.

An electric power grid is modeled by a graph by letting vertices represent the electrical nodes and edges represent transmissions lines between nodes. The *power domination process* is defined as follows [9]: A PMU placed at a vertex measures the voltage and phasor angle at that vertex, at the incident edges, and at the vertices at the endpoints of the incident edges. These vertices and edges are said to be *observed*. The rest of the system is observed according to the following propagation rules:

1. Any vertex that is incident to an observed edge is observed.

2. Any edge joining two observed vertices is observed.

3. If a vertex is incident to a total of $t > 1$ edges and if $t-1$ of these edges are observed, then all t of these edges are observed.

Here we give an equivalent formulation of the power domination process using our notation as done in [8]. Let $G = (V, E)$ be a simple graph and $v \in V(G)$. The set of neighbors of v is denoted $N(v)$. For a set S of vertices, the *open neighborhood* of S is given by $N(S) = \cup_{s \in S} N(s)$ and the *closed neighborhood* of S is $N[S] := S \cup N(S)$. Given a set $S \subseteq V(G)$, define the following sets:

1. $S^{[0]} = S, S^{[1]} = N[S]$.

2. For $t \geq 1$, $S^{[t+1]} = S^{[t]} \cup \{w \in V(G) | \, \exists \, v \in S^{[t]}, N(v) \setminus S^{[t]} = \{w\}\}$.

For vertices w and v given in (2) we say v *forces* w. A set S is said to be a *power dominating set* if there exists an ℓ such that $S^{[\ell]} = V(G)$. The *power domination number* of G, denoted $\gamma_P(G)$, is the minimum cardinality over all power dominating sets of G. Computing $S^{[1]}$ is the *domination step* and the computations of $S^{[t+1]}$ (for $t \geq 1$) are the *propagation steps*. The authors of [8] defined the power propagation time: the *power propagation time of*

G with S, denoted $\operatorname{ppt}(G,S)$, is the smallest ℓ such that $S^{[\ell]} = V(G)$. The *power propagation time of G*, denoted $\operatorname{ppt}(G)$, is given by

$$\operatorname{ppt}(G) = \min\{\operatorname{ppt}(G,S) | S \text{ is a minimum power dominating set}\}.$$

A minimum power dominating set S of a graph G is *efficient* if $\operatorname{ppt}(G,S) = \operatorname{ppt}(G)$.

In Section 3.3, we give Nordhaus-Gaddum upper and lower bounds for the sum of the power propagation time of a graph and its complement, and in Section 3.4 we study the effects of edge subdivision and edge contraction on power propagation time. In Sections 3.5.1 and 3.5.2, we characterize graphs with low and high $k-$power propagation times, respectively. (Note that by letting $k = 1$, we characterize graphs with low and high power propagation times.)

Power domination is closely related to the well known domination problem in graph theory. A set $S \subseteq V(G)$ is a *dominating set* if $N[S] = V(G)$. The *domination number* of a graph G, denoted $\gamma(G)$, is the minimum cardinality over all dominating sets of G. Note that each dominating set is a power dominating set, so $\gamma_P(G) \leq \gamma(G)$ [9].

3.1.1 Zero Forcing

The zero forcing problem from combinatorial matrix theory is also closely related to power domination, and in Sections 3.3 and 3.5.1 we use results from zero forcing theory to prove statements about power domination. *Zero forcing* is a game played on a graph using the following *color change rule:* Let B be a set of vertices of G that are colored blue with $V \setminus B$ colored white. If v is a blue vertex and u is the only neighbor of v that is colored white, then change the color of u to blue. In this case, we say u forces v and write $u \rightarrow v$. For a set B of vertices that are initially colored blue, the set of blue vertices that results from applying the color change rule until no more color changes are possible is the *final coloring of B*. A set B is said to be a *zero forcing set* if the final coloring of B is the entire vertex set

$V(G)$. The minimum cardinality over all zero forcing sets of G is the *zero forcing number* of G, denoted $\mathrm{Z}(G)$. The zero forcing number was first introduced in [1] as an upper bound on the linear algebraic parameter of a graph known as the maximum nullity, and independently in [3] to study the control of quantum systems.

Observation 52. [2] A set S is a power dominating set of G if and only if $N[S]$ is a zero forcing set of G. It follows that $N(S) \setminus S$ is a zero forcing set of $G \setminus S$.

The authors of [10] introduced the propagation time of a zero forcing set of a graph. Due to the close relationship between zero forcing and power domination, many of the questions studied in this paper were motivated by results of the propagation time of a zero forcing set.

3.1.2 More notation and terminology

We use P_n, C_n, and K_n to denote the path, cycle, and complete graph on n vertices, respectively. The notation $K_n - e$ represents the complete graph on n vertices minus an edge, and $K_{s,t}$ is the complete bipartite graph with bipartition X, Y where $|X| = s$ and $|Y| = t$. The graph $L(s,t)$ is the lollipop graph consisting of a complete graph K_s and a path on t vertices where one endpoint of the path is connected to one vertex of K_s via a bridge.

Let $G = (V, E)$ be a graph and $e = uv \in E(G)$. The graph resulting from *subdividing* the edge $e = uv$, denoted G_e, is obtained from G by adding a new vertex w such that $V(G_e) = V(G) \cup \{w\}$ and $E(G_e) = (E(G) \setminus \{uv\}) \cup \{uw, wv\}$. To *contract* the edge $e = uv$ is to identify vertices u and v as a single vertex w such that $N(w) = (N(u) \cup N(v)) \setminus \{u, v\}$. The graph obtained from G by contracting the edge e is denoted by G/e.

A *spider* or *generalized star* is a tree formed from a $K_{1,n}$ (for $n \geq 3$) by subdividing any number of its edges any number of times. We use $sp(i_1, i_2, \ldots, i_n)$ to denote the spider obtained from $K_{1,n}$ by subdividing edge e_j a total of $i_j - 1$ times for $1 \leq j \leq n$. For

$G = sp(i_1, i_2, \ldots, i_n)$ and v the unique vertex in $V(G)$ with degree at least 3, we say that the n paths of $G - v$ are the *legs* of G.

3.2 Preliminaries

In this section, we give preliminary results that will be used throughout the remainder of the paper. In particular, Observation 53 and Lemma 54 are central. We also determine the power propagation time of several families of graphs.

Observation 53. Let G be a graph on n vertices and S a power dominating set of G. Then,

$$\operatorname{ppt}(G, S) \leq n - |S| \tag{3.1}$$

and

$$\operatorname{ppt}(G, S) - 1 \leq n - |N[S]| \tag{3.2}$$

This follows from the fact that at least one vertex must be forced at each step.

Lemma 54. [9] *Let G be a connected graph with $\Delta(G) \geq 3$. Then there exists a minimum power dominating set S of G such that $\deg(s) \geq 3$ for each $s \in S$.*

3.2.1 Power propagation time for families

It is well known and clear that the power domination number of the graphs P_n, C_n, K_n, and the spider $\operatorname{sp}(i_i, i_2, \ldots, i_n)$ is 1. For $G = K_n$, any one vertex is a power dominating set with power propagation time 1. We now determine the power propagation times of the graphs P_n, C_n, and $\operatorname{sp}(i_i, i_2, \ldots, i_n)$.

Proposition 55. *Let P_n be the path on n vertices. Then $\gamma_P(P_n) = 1$ and $\operatorname{ppt}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$.*

Proof. Let $G = P_n$. Any one vertex of G is a minimum power dominating set. Label the vertices of G with $v_1, \ldots, v_n$ where $\{v_i, v_{i+1}\} \in E(G)$ for $i \in \{1, \ldots, n-1\}$. For any vertex v_t, $\operatorname{ppt}(G, \{v_t\}) = \max\{t-1, n-t\}$. It follows that for n odd, $\operatorname{ppt}(G) \geq \frac{n-1}{2}$, and equality is obtained by choosing the power dominating set to be $\{v_t\}$ where $t = \frac{n+1}{2}$. For n even $\operatorname{ppt}(G) \geq \frac{n}{2}$, and equality is obtained by choosing the power dominating set $\{v_t\}$ with $t \in \{\frac{n}{2}, \frac{n+1}{2}\}$. □

The proofs of the next three propositions are similar and omitted.

Proposition 56. *Let C_n be the cycle on n vertices. Then* $\gamma_P(C_n) = 1$ and $\operatorname{ppt}(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proposition 57. *Let* $G = \operatorname{sp}(i_1, i_2, \ldots, i_n)$ for some $n \geq 3$. *Then* $\gamma_P(G) = 1$ and $\operatorname{ppt}(G) = \max\{i_1, i_2, \ldots, i_n\}$.

Proposition 58. *For $s, t \geq 3, \gamma_P(K_{s,t}) = 2$ and $\operatorname{ppt}(K_{s,t}) = 1$, for $s \geq 2$ and $t = 2$, $\gamma_P(K_{s,t}) = 1$ and $\operatorname{ppt}(K_{s,t}) = 2$, and for $s \geq 1$ and $t = 1$, $\gamma_P(K_{s,t}) = 1$ and $\operatorname{ppt}(K_{s,t}) = 1$.*

3.3 Nordhaus-Gaddum sum bounds for power propagation time

In 1956, Nordhaus and Gaddum gave upper and lower bounds on the sum and product of the chromatic number of a graph and its complement. Since then, many similar "Nordhaus-Gaddum" bounds have been studied for other graph parameters. In particular, the Nordhaus-Gaddum sum lower bound for the zero forcing number of a graph on n vertices was established in [7]: $n - 2 \leq \operatorname{Z}(G) + \operatorname{Z}(\overline{G})$. In this section we use this result to show that for all graphs on n vertices, $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) \leq n + 2$. We also conjecture that n is the least upper bound, and demonstrate an infinite family of graphs with $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) = n$ for each G in the family.

The graph $G = K_n$ demonstrates that the Nordhaus-Gaddum sum lower bound is 1. If we require that both G and its complement have edges, then the graph $G = K_{n,n}$ (for $n \geq 3$) demonstrates that Nordhaus-Gaddum sum lower bound is 2.

Proposition 59. *Let G be a graph on n vertices. Then* $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) \leq n + 2$.

Proof. If G has no edges, then $\operatorname{ppt}(G) = 0$ and $\operatorname{ppt}(\overline{G}) = 1$ so the claim holds. Suppose G and $\overline{G}$ have an edge. Let S be an efficient power dominating set of G. Note that $N[S]$ is a zero forcing set of G, but it is not minimum: To see this, consider a fixed $s \in S$ (such that $\deg(s) \geq 1$) and a vertex $v_s \in N(s)$. By removing v_s, $N[S] \setminus \{v_s\}$ is also a zero forcing set, so $\operatorname{Z}(G) + 1 \leq |N[S]|$. Similarly, $\operatorname{Z}(\overline{G}) + 1 \leq |N[S']|$, where S' is an efficient power dominating set of $\overline{G}$. It follows from inequality (3.2) that $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) \leq 2n - (\operatorname{Z}(G) + \operatorname{Z}(\overline{G}))$, and since $n - 2 \leq \operatorname{Z}(G) + \operatorname{Z}(\overline{G})$ ([7]), then $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) \leq n + 2$. $\square$

We have not found a graph with $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) = n + 1$, or one such that $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) = n + 2$. We have computationally checked all connected graphs on at most 10 vertices and found several graphs with $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) = n$. Evidence suggests that this is the least upper bound for all graphs. The next example gives an infinite family of graphs such that $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) = n$ for all graphs in the family.

Example 60. Let G_9 denote the graph given in the Figure 3.1. For $n \geq 10$, let G_n be a graph on n vertices constructed from G_{n-1} by adding an n^{th} vertex and adding the edges $\{v_{n-2}, v_n\}$ and $\{v_{n-1}, v_n\}$. Note that the set $V(G_n) \setminus \{v_2, v_3\}$ is not a zero forcing set of G_n (since $N(v_2) = N(v_3)$, v_2 and v_3 will never be forced). So for every power dominating set S of G_n, $N[S]$ must contain either v_2 or v_3. Also note that the sets $\{v_2\}$ and $\{v_3\}$ are minimum power dominating sets of G_n with $\operatorname{ppt}(G_n, v_2) = \operatorname{ppt}(G_n, v_3) = n - 3$. Thus, $\gamma_P(G) = 1$. For $6 \leq i \leq n$, the set $\{v_i\}$ is not a power dominating set since $v_2, v_3 \notin N[\{v_i\}]$. Furthermore, it follows from inspection that the sets $\{v_1\}, \{v_4\},$ and $\{v_5\}$ are not power dominating sets. Thus, $\operatorname{ppt}(G_n) = n - 3$.

Similarly, we show that $\operatorname{ppt}(\overline{G_n}) = 3$. The sets $\{v_{n-1}\}$ and $\{v_n\}$ are power dominating sets of $\overline{G_n}$ with $\operatorname{ppt}(\overline{G_n}, \{v_{n-1}\}) = \operatorname{ppt}(\overline{G_n}, \{v_n\}) = 3$, and the sets $\{v_2\}$ and $\{v_3\}$ are power dominating sets with $\operatorname{ppt}(\overline{G_n}, \{v_2\}) = \operatorname{ppt}(\overline{G_n}, \{v_3\}) = 4$. Since $N(v_2) \setminus \{v_3\} = N(v_3) \setminus \{v_2\}$, the set $V(\overline{G_n}) \setminus \{v_2, v_3\}$ is not a zero forcing set of $\overline{G_n}$. So for each power dominating set S' of $\overline{G_n}$, $N[S']$ must contain v_2 or v_3. It follows that for $i \in \{1, 4, 5\}$, the set $\{v_i\}$ is not a power dominating set since $v_2, v_3 \notin N[\{v_i\}]$. We now show that for $6 \leq i \leq n-2, \{v_i\}$ is not a power dominating set by showing that $N[\{v_i\}]$ is not a zero forcing set. Note that $N[\{v_i\}] = V(\overline{G_n}) \setminus \{v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}\}$. If $j < i-2, v_j$ is adjacent to v_{i+1} and v_{i+2} (since v_j is not adjacent to v_{i+1} and v_{i+2} in G_n). If $j > i+2, v_j$ is adjacent to v_{i-2} and v_{i-1}. Thus, no vertex in $N[\{v_i\}]$ is able to perform a force, so $N[\{v_i\}]$ is not a zero forcing set. This shows that $\operatorname{ppt}(\overline{G_n}) = 3$, so $\operatorname{ppt}(G_n) + \operatorname{ppt}(\overline{G_n}) = n$.

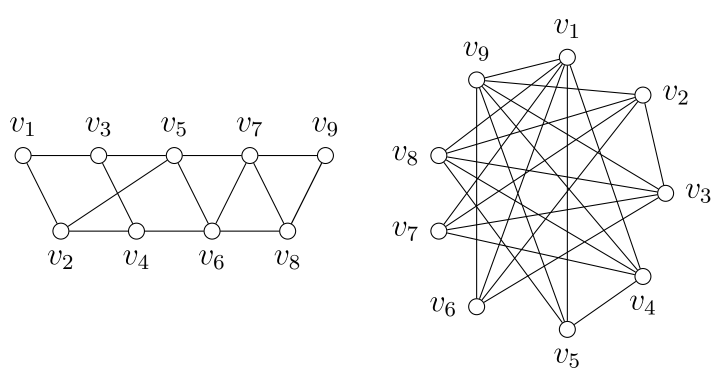

Figure 3.1: Graphs G_9 (left) and $\overline{G_9}$ (right) in Example 60

Conjecture 61. *For all graphs G on n vertices,* $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) \leq n$.

We now show that the conjecture is true for graphs satisfying certain conditions.

Proposition 62. *Let $G \neq P_4$ be a connected graph on n vertices that has a leaf. Then* $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) \leq n - 1$ *and this bound is tight. For $G = P_4$,* $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) = n = 4$.

Proof. The claim holds when $n \leq 2$, so let $n \geq 3$. We first show that $\operatorname{ppt}(\overline{G}) \leq 2$. Let $uv \in E(G)$ such that v is a leaf. If $\deg(u) = n-1$, then $\{v, u\}$ is an efficient power dominating set for $\overline{G}$ and $\operatorname{ppt}(\overline{G}) = 1$. If $\deg(u) \neq n-1$, then $\{v\}$ is an efficient power dominating set for $\overline{G}$, and $\operatorname{ppt}(\overline{G}) = 2$.

Suppose first that $\Delta(G) \geq 3$. By Lemma 54, G has a minimum power dominating set S such that each vertex in S has degree at least 3. Then $|N[S]| \geq 4$, $\operatorname{ppt}(G) \leq n-3$, and $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) \leq n-1$. If $\Delta(G) = 2$, then G is a path. By Proposition 55, $\operatorname{ppt}(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$, so $\operatorname{ppt}(P_n) \leq n-3$ for all $n \geq 6$. For P_3, P_4, P_5, we have by inspection that $\operatorname{ppt}(P_3) + \operatorname{ppt}(\overline{P_3}) = 2$, $\operatorname{ppt}(P_4) + \operatorname{ppt}(\overline{P_4}) = 4$, and $\operatorname{ppt}(P_5) + \operatorname{ppt}(\overline{P_5}) = 4$. Thus, $\operatorname{ppt}(G) + \operatorname{ppt}(\overline{G}) \leq n-1$ for all graphs $G \neq P_4$ containing a leaf. The bound is tight for $G = \operatorname{sp}(1,1,t)$ $(t \geq 2)$ since $\operatorname{ppt}(G) = t = |G| - 3$ by Proposition 57 and $\operatorname{ppt}(\overline{G}) = 2$. □

The *girth* of a graph is defined to be the length of the shortest cycle contained in the graph. If the graph is acyclic, the girth is defined to be infinity. We now show that Conjecture 61 is true for all graph with girth at least 5.

Theorem 63. *Let G be a graph on $n \geq 5$ vertices that has girth at least 5. Then* $\operatorname{ppt}(\overline{G}) \leq 3$.

Proof. Let S' be an efficient power dominating set for $\overline{G}$. We will show that $|N[S']| \geq n-2$. Then it follows from Observation 53 that $\operatorname{ppt}(\overline{G}) \leq 3$.

Assume that $|N[S']| \leq n-3$, so that $V \setminus N[S'] \geq 3$. Let u be in $V \setminus N[S']$ such that u is forced by some $v \in N[S] \setminus S$ in step 2. Recall that in order for v to force u in step 2, u must be the only neighbor of v in $V \setminus N[S']$. Let x and w be two vertices in $V \setminus N[S']$ such that $x \neq u$ and $w \neq u$. We first show that x and w must be adjacent. Since G has no 3 cycles, then for any three vertices in $V(\overline{G})$, two of them must be adjacent. Choose $s \in S'$ such that $v \in N(s)$ (this s is guaranteed since $v \in N[S'] \setminus S'$). Note that $x, w \notin N(s)$, so x and w must be adjacent. Then the graph induced by $\{x, w, s, v\}$ is $K_2 \cup K_2 = \overline{C_4}$. This contradicts the hypothesis that the girth of G is at least 5. So $|N[S']| \geq n-2$ and $\operatorname{ppt}(\overline{G}) \leq 3$.

□

Corollary 64. *Let G be a graph on n vertices with girth at least 5. Then* $\operatorname{ppt}(G)+\operatorname{ppt}(\overline{G}) \leq n$.

Proof. It follows from inspection that the claim holds for $n \leq 4$. Assume $n \geq 5$. By Theorem 63, $\operatorname{ppt}(\overline{G}) \leq 3$. Suppose first that $\Delta(G) \geq 3$, and let G_1 be the connected component of G that has a vertex of degree at least 3. Then there exists a minimum power domination set S_1 of G_1 such that each vertex in S_1 has degree at least 3 (Lemma 54). Therefore, $|N[S_1]| \geq 4$, and for any minimum power dominating set S of G with $S_1 \subseteq S$, $|N[S]| \geq 4$, so $\operatorname{ppt}(G) \leq \operatorname{ppt}(G,S) \leq n-3$ (Observation 53). This gives that $\operatorname{ppt}(G)+\operatorname{ppt}(\overline{G}) \leq n$.

If $\Delta(G) \leq 2$, then G is the union of paths and cycles, and the power propagation time of G is equal to the power propagation time of the path or cycle with the largest number of vertices. This component has at most n vertices, so its power propagation time of this component is at most $\left\lfloor \frac{n}{2} \right\rfloor$ (Propositions 55 and 56). It follows that $\operatorname{ppt}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor$, and since $n \geq 5$, $\operatorname{ppt}(G)+\operatorname{ppt}(\overline{G}) \leq n$. □

Lemma 65. [5] *Let G be a connected graph such that $\Delta(G) \geq 3$. Then there exists a minimum power dominating set S such that each $s \in S$ has at least two neighbors which are not in $N[S \setminus \{v\}]$.*

Proposition 66. *Let G and $\overline{G}$ be connected graphs on n vertices such that $\Delta(G) \geq 3$ and $\Delta(\overline{G}) \geq 3$. Then* $\operatorname{ppt}(G)+\operatorname{ppt}\left(\overline{G}\right) \leq n-(\gamma_P(G)+\gamma_p(\overline{G}))+4$.

Proof. By Lemma 65 and the assumption that $\Delta(G) \geq 3$, there is a minimum power dominating set S of G such that each $s \in S$ has at least one neighbor not in $N[S \setminus \{s\}]$. We first show that $\operatorname{Z}(G) \leq |N[S]| - \gamma_p(G)$. Recall that $N[S]$ is a zero forcing set of G. For each $s \in S$, choose a $v_s \in N(s)$ such that $v_s \notin N[S \setminus \{s\}]$. Then $N[S] \setminus \{v_1, v_2, \ldots, v_{|S|}\}$ is also a zero forcing set since s will force v_s in step one. So, $\operatorname{Z}(G) \leq |N[S]| - \gamma_p(G)$.

By the same argument, we have a minimum power dominating set S' of G such that $\mathrm{Z}(\overline{G}) \leq |N[S']| - \gamma_p(\overline{G})$. Using the bounds $\mathrm{ppt}(G, S) - 1 \leq n - |N[S]|$ and $\mathrm{ppt}(\overline{G}, S') - 1 \leq n - |N[S']|$ (from inequality (3.2)), and $n - 2 \leq \mathrm{Z}(G) + \mathrm{Z}(\overline{G})$ from [7], it follows that

$$\begin{aligned}
\mathrm{ppt}(G) + \mathrm{ppt}(\overline{G}) &\leq \mathrm{ppt}(G, S) + \mathrm{ppt}(\overline{G}, S') \\
&\leq 2n + 2 - (|N[S]| + |N[S']|) \\
&\leq 2n + 2 - (\mathrm{Z}(G) + \mathrm{Z}(\overline{G})) - (\gamma_P(G) + \gamma_P(\overline{G})) \\
&\leq 2n + 2 - (n - 2) - (\gamma_P(G) + \gamma_P(\overline{G})) \\
&= n - (\gamma_P(G) + \gamma_P(\overline{G})) + 4. \quad \square
\end{aligned}$$

Corollary 67. *Let G and $\overline{G}$ be connected graphs on n vertices with $\gamma_P(G) + \gamma_P(\overline{G}) \geq 4$. Then $\mathrm{ppt}(G) + \mathrm{ppt}\left(\overline{G}\right) \leq n$.*

Proof. We first show that $\Delta(G) \geq 3$ and $\Delta(\overline{G}) \geq 3$. If $\Delta(G) \leq 2$ then G is a cycle or a path. By the assumption that G and $\overline{G}$ are connected, $G \notin \{P_2, P_3, C_3, C_4\}$. For $n \geq 4$, $\gamma_P(P_n) = \gamma_P(\overline{P_n}) = 1$ and for $n \geq 5, \gamma_P(C_n) = \gamma_P(\overline{C_n}) = 1$. It follows from the assumption that $\gamma_P(G) + \gamma_P(\overline{G}) \geq 4$ that neither G or $\overline{G}$ is a path or cycle. Thus, $\Delta(G) \geq 3$ and $\Delta(\overline{G}) \geq 3$. By Proposition 66,

$$\mathrm{ppt}(G) + \mathrm{ppt}(\overline{G}) \leq n - (\gamma_P(G) + \gamma_P(\overline{G})) + 4 \leq n.$$

$\square$

3.4 Effects of edge subdivision and edge contraction on power propagation time

Let G_e be a graph obtained from $G = (V, E)$ by subdividing the edge $e \in E$ and let G/e denote the graph resulting from G by contracting the edge e. It is shown in both [2] and

[6] that $\gamma_P(G) - 1 \leq \gamma_P(G/e) \leq \gamma_P(G) + 1$ and in [2] that $\gamma_P(G) \leq \gamma_P(G_e) \leq \gamma_P(G) + 1$. We show that the power propagation time may increase or decrease by any amount when subdividing or contracting an edge.

Proposition 68. *For any $t \geq 0$, there exists a graph $G = (V, E)$ and edge $e \in E$ such that* $\operatorname{ppt}(G_e) \leq \operatorname{ppt}(G) - t$.

Proof. Construct the graph G in the following way: Starting with the path $P_\ell = (v_1, v_2, \ldots, v_\ell)$, $(\ell \geq 7)$, add three leaves to vertex v_1 and add three leaves to vertex v_ℓ. Add one leaf to vertex $v_{\ell-1}$ and add one leaf to vertex $v_{\ell-2}$. (See Figure 3.2.) Then $\{v_1, v_\ell\}$ is the unique efficient power dominating set of G and $\operatorname{ppt}(G) = \ell - 2$. For $e = \{v_{l-2}, v_{l-1}\}$, we consider the graph G_e. Note that $\gamma_p(G_e) = 3$ because $v_1, v_\ell \in S$ for any minimum power dominating set S and $\{v_1, v_\ell\}$ is not a power dominating set. For $S = \{v_1, v_{l-2}, v_\ell\}, \operatorname{ppt}(G_e, S) = \left\lceil \frac{\ell-4}{2} \right\rceil$. By choosing $\ell \geq 2t + 1$, $\operatorname{ppt}(G_e) \leq \operatorname{ppt}(G) - t$.

□

Corollary 69. *For any $t \geq 0$, there exists a graph $H = (V, E)$ and edge $e \in E$ such that* $\operatorname{ppt}(H/e) \geq \operatorname{ppt}(H) + t$.

Proof. From Proposition 68, there exist graphs G and G_e such that $\operatorname{ppt}(G_e) \leq \operatorname{ppt}(G) - t$. Let $H = G_e$ and $H/e = G$. Then $\operatorname{ppt}(H/e) \geq \operatorname{ppt}(H) + t$.

□

Similarly, subdividing an edge can cause the power propagation time to increase by any amount, as demonstrated by the following proposition.

Proposition 70. *For any $t \geq 0$, there exists a graph $G = (V, E)$ and edge $e \in E$ such that* $\operatorname{ppt}(G_e) \geq \operatorname{ppt}(G) + t$.

Proof. Let G be a graph on $n \geq 8$ vertices constructed from the cycle $(v_1, v_2, \ldots, v_{n-4})$ by adding the edges $\{v_1, v_{n-3}\}, \{v_1, v_{n-2}\}, \{v_1, v_{n-1}\}, \{v_2, v_{n-1}\}$, and $\{v_n, v_{n-1}\}$. Let $e =$

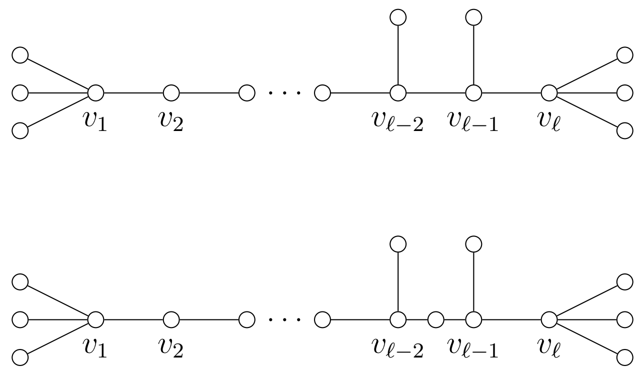

Figure 3.2: Graphs G and G_e in Proposition 68

$\{v_2, v_{n-1}\}$, and consider G_e. The set $\{v_1\}$ is the unique minimum power dominating set of G and $\mathrm{ppt}(G) = \left\lfloor \frac{n-4}{2} \right\rfloor$. The set $\{v_1\}$ is also the unique minimum power dominating set of G_e and $\mathrm{ppt}(G_e) = n - 4$. So, by choosing $n \geq 2t + 4$, $\mathrm{ppt}(G_e) \geq \mathrm{ppt}(G) + t$. $\square$

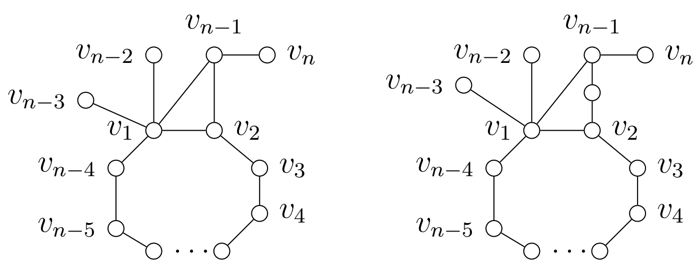

Figure 3.3: Graphs G and G_e in Proposition 70

Corollary 71. *For any $t \geq 0$, there exists a graph $H = (V, E)$ and edge $e \in E$ such that* $\mathrm{ppt}(H/e) \leq \mathrm{ppt}(H) - t$.

Proof. From Proposition 70, there exist graphs G and G_e such that $\mathrm{ppt}(G_e) \geq \mathrm{ppt}(G) + t$. Let $H = G$ and $H/e = G_e$. Then $\mathrm{ppt}(H/e) \leq \mathrm{ppt}(H) - t$.

$\square$

3.5 k-power propagation

The authors of [5] introduced the following generalization of power domination, known as $k-$*power domination.* Let $k \geq 1$. For a set $S \subseteq V(G)$, define the following sets:

1. $S^{[0]} = S, S^{[1]} = N[S]$.

2. For $t \geq 1$, $S^{[t+1]} = S^{[t]} \cup \{w \in V(G) |\, \exists\, v \in S^{[t]}, w \in N(v) \setminus S^{[t]}$ and $|N(v) \setminus S^{[t]}| \leq k\}$.

(For our purposes and convenience, we have defined $S^{[0]} = S$. This is not done in [5].) A set S is said to be a $k-$*power dominating set* if there exists an l such that $S^{[l]} = V(G)$. (Note that when $k = 1$ the set is a power dominating set.) The $k-$*power domination number* of G, denoted $\gamma_{P,k}(G)$, is defined to be the minimum cardinality over all $k-$power dominating sets of G, and $\gamma_{P,k}(G) \leq \gamma_P(G) \leq \gamma(G)$ for all $k \geq 1$ [5].

We define the $k-$power propagation time as follows:

Definition 72. Let S be a $k-$power dominating set. The $k-$*power propagation time of* G *with* S, denoted $\mathrm{ppt}_k(G, S)$, is the smallest ℓ such that $S^{[\ell]} = V(G)$. The $k-$*power propagation time of* G, denoted $\mathrm{ppt}_k(G)$ is given by

$$\mathrm{ppt}_k(G) = \min\{\mathrm{ppt}_k(G, S) | S \text{ is a minimum } k\text{−power dominating set}\}.$$

A minimum $k-$power dominating set S of a graph G is *efficient* if $\mathrm{ppt}_k(G, S) = \mathrm{ppt}_k(G)$.

In this section, we study the $k-$power propagation time of a graph by characterizing graphs with extreme high and extreme low $k-$power propagation times. Note that by letting $k = 1$, we obtain characterizations of graphs with extreme high and extreme low power propagation times.

The next observation and next two propositions are generalizations of Observation 53 and Propositions 55 and 56, and the same arguments hold.

Observation 73. Let G be a graph on n vertices and S a $k-$power dominating set of G. Then,

$$\mathrm{ppt}_k(G,S) \leq n - |S| \tag{3.3}$$

and

$$\mathrm{ppt}_k(G,S) - 1 \leq n - |N[S]| \tag{3.4}$$

Proposition 74. *Let P_n be the path on n vertices. Then* $\mathrm{ppt}_k(P_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Proposition 75. *Let C_n be the cycle on n vertices. Then* $\mathrm{ppt}_k(C_n) = \left\lfloor \frac{n}{2} \right\rfloor$.

Remark 76. It is a well known fact that for a connected graph G of order at least 3, there exists an efficient $k-$power dominating set of G in which every vertex has degree at least 2: For if v is a leaf of an efficient $k-$power dominating set S and $vw \in E(G)$, then w is not a leaf since G is connected and $G \neq K_2$. So, $S' = (S \setminus \{v\}) \cup \{w\}$ is a minimum $k-$power dominating set, and $\mathrm{ppt}_k(G,S') \leq \mathrm{ppt}_k(G,S)$. Repeating this process for each leaf in S, we obtain an efficient $k-$power dominating set of G with no leaves.

Lemma 77. [5] *Let $k \geq 1$ and let G be a connected graph with $\Delta(G) \geq k+2$. Then there exists a minimum $k-$power dominating set S of G such that* $\deg(s) \geq k+2$ *for each $s \in S$.*

Note that $\Delta(G) \geq k+2$ does not guarantee that there exists an efficient $k-$power dominating set S such that $\deg(s) \geq k+2$ for each $s \in S$. This is demonstrated in the following example with $k = 1$.

Example 78. Let G be the graph on $n+2$ vertices ($n \geq 5$) obtained from a path $(v_1, v_2, \ldots, v_n)$ by adding a leaf to v_2 and adding a leaf to v_3. Then $S = \{v_2, v_3\}$ is the unique power dominating set such that $\deg(s) \geq 3$ for each $s \in S$, but for $S' = \{v_2, v_4\}$, $n - 4 = \mathrm{ppt}(G,S') < \mathrm{ppt}(G,S) = n - 3$.

Throughout the rest of this paper, we also use the following generalization of Lemma 77:

Lemma 79. For any $3 \leq t \leq k+2$, if G is connected with $\Delta(G) \geq t$, then there exists a minimum $k-$power dominating set S such that every vertex in S has degree at least t.

Proof. Let $3 \leq t \leq k+2$ and let S be a minimum $k-$power dominating set of G. Suppose $s \in S$ and $\deg(s) < t$. Since G is connected, we may choose $v \in V(G)$ such that $\deg(v) \geq t$ and $\deg(u) < t$ for all interior vertices u on the shortest path from s to v. Then $(S \setminus \{s\}) \cup \{v\}$ is also a minimum k-power dominating set. Continuing this process for all vertices in S with degree less than t, we construct a minimum k-power dominating set of G such that every vertex has degree at least t. □

3.5.1 Low k-power propagation time

We first consider graphs with low $k-$propagation time. If G is a graph with k-propagation time 1, then any efficient k-power dominating set of G is also a dominating set, so $\gamma(G) \leq \gamma_{P,k}(G)$. Since it is always true that $\gamma_{P,k}(G) \leq \gamma(G)$, it follows that $\gamma_{P,k}(G) = \gamma(G)$. In this section, we study graphs with $k-$power propagation time equal to 1.

For $k \geq 1$, a vertex v in $V(G)$ is called a *k-strong support vertex* if v is adjacent to $k+1$ or more leaves. A $1-$strong support vertex is also known as a *strong support vertex* and was originally defined in [9].

Remark 80. Note that every k-strong support vertex of a graph G is in every minimum dominating set of G. Also, if S is a $k-$power dominating set of G and v is a k-strong support vertex of G then either v is in S or all but k of the leaves adjacent to v are in S. So $\gamma_{P,k}(G)$ is at least the number of k-strong support vertices in G. Since $\gamma_{P,k}(G) \leq \gamma(G)$, it follows that if S is a dominating set of G such that every vertex in S is a k-strong support vertex, then S is the unique minimum dominating set of G, $\gamma_{P,k}(G) = \gamma(G)$, and $\mathrm{ppt}_k(G) = 1$.

For a minimum $k-$power dominating set S and a vertex v in S, the *private neighborhood* of v with respect to S, denoted $pn[v, S]$, is the set $N[v] \setminus (N[S \setminus \{v\}])$. Every vertex of $pn[v, S]$ is called a *private neighbor* of v with respect to S, and A_v denotes the set $V \setminus (S \cup pn[v, S])$ [9].

The next theorem and proof is a generalization of Theorem 9 given in [9].

Theorem 81. *For $k \geq 1$, let G be a connected graph on at least $k+2$ vertices that does not contain C_3 or $K_{2,k+1}$ as an induced subgraph. Then $\mathrm{ppt}_k(G) = 1$ if and only if G has a minimum dominating set S such that every vertex in S is a k-strong support vertex. Furthermore, S is the unique minimum dominating set of G.*

Proof. If G has a dominating set S such that each vertex in S is a $k-$strong support vertex, then by Remark 80, $\gamma_{P,k}(G) = \gamma(G)$ and $\mathrm{ppt}_k(G) = 1$.

Conversely, let $\mathrm{ppt}_k(G) = 1$ (i.e., $\gamma_{P,k}(G) = \gamma(G)$). To obtain a contradiction, suppose S is a minimum dominating set of G such that there exists a vertex $v \in S$ that is not a $k-$strong support vertex. If $pn[v, S] = \emptyset$, then $S \setminus \{v\}$ is a smaller dominating set. Suppose that $pn[v, S] = \{v\}$. Then $S \setminus \{v\}$ dominates $V \setminus \{v\}$, and since G is connected, v will be forced in step 1. So $S \setminus \{v\}$ is a smaller k-power dominating set. Thus, $pn[v, S]$ contains at least one vertex that is not v.

Suppose first that $pn[v, S]$ contains a vertex $w \neq v$ that is not a leaf. We show again that $S \setminus \{v\}$ is a smaller k-power dominating set. Since w is not a leaf, it is adjacent to a vertex in A_v. To see this, note that w has no neighbor in $pn[v, S]$ (except for v if v is in $pn[v, S]$) since every other vertex in $pn[v, S]$ is also adjacent to v and G contains no 3 cycles. Furthermore, by the definition of $pn[v, S]$, w has no neighbor in $S \setminus \{v\}$. Since w is not a leaf, then w is adjacent to some vertex w_u in A_v. To see that $S \setminus \{v\}$ is a smaller k-power dominating set, first note that w_u is not adjacent to v (since (v, w_u, w) could give a 3 cycle) and $|N(w_u) \cap (pn[v, S] \setminus \{v\})| \leq k$ (since G is $K_{2,k+1}$-free and the vertices of

$N(w_u) \cap (pn[v,S] \setminus \{v\})$ form the induced graph $K_{2,t}$ where $t = |N(w_u) \cap (pn[v,S] \setminus \{v\})|$. It follows that $S \setminus \{v\}$ is a $k-$power dominating set of G since $S \setminus \{v\}$ dominates A_v in step 1, each w in $pn[v,S] \setminus \{v\}$ that is not a leaf is forced by a neighbor w_u from A_v step 2, if necessary any such w can force v in step 3, and since v is adjacent to at most k leaves, then v will force these leaves (if any) in step 4. So each vertex in $pn[v,S]$ that is not v must be a leaf.

Suppose vertices $w_1, \ldots, w_t$ are leaves in $pn[v,S]$, where $1 \leq t \leq k$ since v is not a $k-$strong support vertex. Since G is connected and each w_i is only adjacent to v, v must have a neighbor in $S \setminus \{v\}$ or in A_v. In either case, we show that $S \setminus \{v\}$ is a smaller $k-$power dominating set. If v has a neighbor in $S \setminus \{v\}$, then $S \setminus \{v\}$ dominates $A_v \cup \{v\}$ in step 1, and v will force $\{w_1, \ldots, w_t\}$ in step 2. If v has a neighbor in A_v (and no neighbor in $S \setminus \{v\}$), then $S \setminus \{v\}$ dominates A_v in step 1, v is forced by a neighbor from A_v in step 2, and v forces $w_1, \ldots, w_t$ in step 3. This completes the proof of the first statement in the theorem. Note that we have shown that if $\operatorname{ppt}_k(G) = 1$, then every minimum dominating set of G contains only $k-$strong support vertices. Thus, if S is a minimum dominating set such that each vertex in S is a $k-$strong support vertex, then S is the unique minimum dominating set of G. $\square$

3.5.2 High k-power propagation time

Here we consider graphs with high $k-$power propagation times. First we characterize all graphs on n vertices with $\operatorname{ppt}_k(G) = n-1$ or $\operatorname{ppt}_k(G) = n-2$.

Theorem 82. *For a graph G on n vertices and $k \geq 1$, $\operatorname{ppt}_k(G) = n-1$ if and only if $G = K_1$ or $G = K_2$.*

Proof. Let S be an efficient k-power dominating set of G. Since $\operatorname{ppt}_k(G) = n-1$, then $S = \{s\}$ for some $s \in V(G)$, and G is connected. Note that at most 1 vertex may be forced at each step, including the domination step, so $\deg(s) \leq 1$. By Remark 76, $n \leq 2$, so $G = K_1$ or $G = K_2$. □

Theorem 83. *Let $k \geq 1$ and let G be a graph on n vertices with $\operatorname{ppt}_k(G) = n-2$. Then $G \in \{K_1 \cup K_1, K_1 \cup K_2, P_3, P_4, C_3, C_4\}$.*

Proof. Since $\operatorname{ppt}_k(G) = n-2$, then for any minimum $k-$power dominating set S, $|S| \leq 2$ and $|N[S]| \leq 3$. Suppose $\Delta(G) \geq 3$ and let G_1 be a connected component of G with $\Delta(G_1) \geq 3$. By Lemma 79, there exists a minimum $k-$power dominating set S_1 of G_1 such that each $s \in S_1$ has degree at least 3. Then for any minimum $k-$power dominating set S of G such that $S_1 \subseteq S$, we have that $|N[S]| \geq 4$, contradicting $|N[S]| \leq 3$. So $\Delta(G) \leq 2$ and G is the union of cycles and paths. Since $|S| \leq 2$, then G has at most 2 components. If G has exactly one component, G is a path or a cycle, and it follows from Propositions 74 and 75 that $G \in \{P_3, P_4, C_3, C_4\}$. Suppose G has 2 components. Since $|N[S]| \leq 3$, one component is K_1, and by Remark 76 (or Theorem 82), the other component is K_1 or K_2. □

Next we consider graphs G whose $k-$power propagation time is $n-3$. The case with $k = 1$ behaves differently than the cases with $k \geq 2$, so we first consider the latter.

We use $\mathfrak{G}$ to denote the family of connected graphs G on 5 vertices with $\Delta(G) = 3$ (see Figure 3.4).

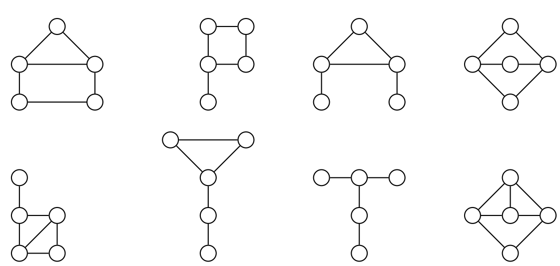

Figure 3.4: $\mathfrak{G}$: Connected graphs G on 5 vertices with $\Delta(G) = 3$

Theorem 84. *Let $k \geq 2$ and let G be a graph on n vertices with* $\operatorname{ppt}_k(G) = n-3$*. Then* $G \in \{P_5, P_6, C_5, C_6, K_{1,3}, L(3,1), K_4 - e, K_4, K_1 \cup P_3, K_1 \cup P_4, K_1 \cup C_3, K_1 \cup C_4, K_2 \cup K_2, \overline{K_3}, \overline{K_2} \cup K_2\} \cup \mathfrak{G}$.

Proof. Since $\operatorname{ppt}_k(G) = n - 3$, then for any minimum $k-$power dominating set S, $|S| \leq 3$ and $|N[S]| \leq 4$. It follows from Lemma 79 that $\Delta(G) \leq 3$.

If $\Delta(G) \leq 2$, then G is the union of paths and cycles. Since $|S| \leq 3$ for any minimum k-power dominating set S, G has at most 3 components. If G is connected, it follows from Propositions 74 and 75 that $G \in \{P_5, P_6, C_5, C_6\}$.

Suppose G has two connected components, G_1 and G_2, and first suppose $|G_1| \geq 3$. By applying Remark 76 to G_1, there exists an efficient $k-$power dominating set S of G such that $|N_{G_1}[S]| \geq 3$, where $N_{G_1}[S] = N[S] \cap V(G_1)$. Since $|N[S]| \leq 4$, we have $G_2 = K_1, \operatorname{ppt}_k(G_1) = |G_1| - 2$, and it follows from Theorem 84 that $G \in \{K_1 \cup P_3, K_1 \cup P_4, K_1 \cup C_3, K_1 \cup C_4\}$. Otherwise, $|G_1| \leq 2$ and $|G_2| \leq 2$, and $G = K_2 \cup K_2$.

If G has 3 connected components, it follows from $|N[S]| \leq 4$ that $G \in \{\overline{K_3}, \overline{K_2} \cup K_2\}$.

Suppose $\Delta(G) = 3$. Let S be a minimum $k-$power dominating set such that every vertex in S has degree at least 3 (S is guaranteed to exist by Lemma 77). Since $|N[S]| \leq 4$, then $S = \{s\}$ and $N[S] = \{s, u_1, u_2, u_3\}$ for some $s, u_1, u_2, u_3 \in V(G)$. If $n = 4$, then $G \in \{K_{1,3}, L(3,1), K_4 - e, K_4\}$.

For $n > 4$, we show that $n = 5$: Since $|N[S]| = 4$ and $\operatorname{ppt}_k(G) = n - 3$, then after the domination step, exactly one force is performed during each step. Without loss of generality, suppose u_1 forces v in step 2.

Claim 1: For $i \in \{1, 2, 3\}$, if $w \in N(u_i)$, then $w \in \{s, u_1, u_2, u_3, v\}$. To see this, recall that $\Delta(G) = 3$. So if u_1 has a neighbor w not in $\{s, u_2, u_3, v\}$, it has exactly one such neighbor. Then u_1 will force w and v in step 2, which contradicts $\operatorname{ppt}_k(G) = n - 3$. Similarly, if

u_i (for $i = 2, 3$) has a neighbor w not in $\{s, u_1, u_2, u_3, v\}$, it has at most two such neighbors, so u_1 will force v in step 2 and u_i will force w in step 2, contradicting $\mathrm{ppt}_k(G) = n - 3$.

Claim 2: Vertex v has no neighbor not in $\{u_1, u_2, u_3\}$. To see this, suppose v has a neighbor w not in $\{u_1, u_2, u_3\}$. Since $\Delta(G) = 3$ and v is adjacent to u_1 by assumption, then v has at most two such neighbors. Then $\{u_1\}$ is a minimum $k-$power dominating set with $\mathrm{ppt}_k(G, \{u_1\}) \leq n - 4$ since u_1 will dominate $\{v, s\}$ in step 1, and if necessary, s will force $\{u_2, u_3\}$ in step 2 and v will force w in step 2.

Therefore, G is a connected graph on 5 vertices with $\Delta(G) = 3$. Also note that all connected graphs on 5 vertices with maximum degree 3 have $\mathrm{ppt}_k(G) = 2$ (for $k \geq 2$). This completes the proof. $\square$

Next we consider graphs with $\mathrm{ppt}(G) = n - 3$. We first characterize all trees with $\mathrm{ppt}(G) = n-3$, then we characterize all graphs with $\mathrm{ppt}(G) = n-3$ and $\gamma_P(G) \in \{2, 3\}$. In Figure 3.5, we provide some graphs with $\mathrm{ppt}(G) = n - 3$ and $\gamma_P(G) = 1$, but characterizing all such graphs is less tractable.

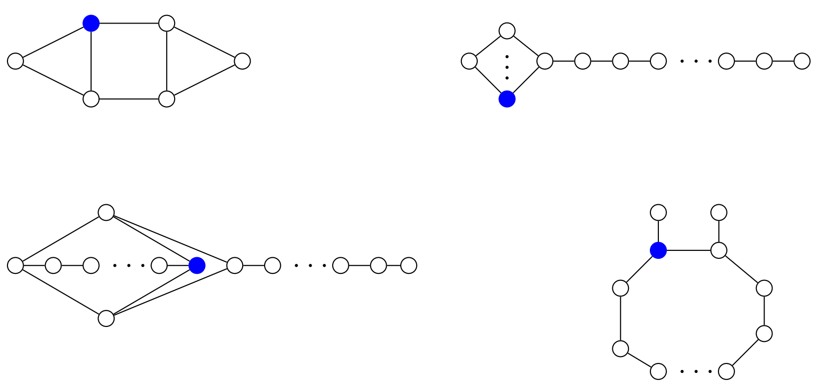

Figure 3.5: Graphs G with $\mathrm{ppt}(G) = n - 3$ and $\gamma_P(G) = 1$. An efficient power dominating set in blue

Proposition 85. *Let T be a tree on n vertices such that* $\mathrm{ppt}(T) = n - 3$. *Then* $T \in \{P_5, P_6, \mathrm{sp}(1, 1, k)$ (for some $k \geq 1$)$\}$.

Proof. Suppose T is a tree on n vertices with $\operatorname{ppt}(T) = n - 3$. If $\Delta(T) \leq 2$, then T must be a path, and by Proposition 55, $T = P_5$ or $T = P_6$. Suppose $\Delta(T) \geq 3$. From Lemma 54, there exists a minimum power dominating set S such that each vertex in S has degree at least 3, so $|N[S]| \geq 4$ and $\operatorname{ppt}(T, S) \leq n - 3$. Since $\operatorname{ppt}(T) = n - 3$ by assumption, then it must be the case that $\operatorname{ppt}(T, S) = n - 3$. Thus, $|S| = 1$ and $|N[S]| = 4$.

Let $S = \{s\}$ and $N[S] = \{s, u_1, u_2, u_3\}$. Note that the path (u_i, s, u_j) (for $i \neq j$) is the unique path from u_i to u_j (since T is a tree), so the graph $T' = T - s$ has 3 connected components T_1, T_2, T_3 with $u_i \in T_i$. By Observation 52, $\{u_1, u_2, u_3\}$ is a zero forcing set for T', and it follows that $\{u_i\}$ is a zero forcing set of T_i. Since T_i has zero forcing number 1, then T_i is a path and u_i is an endpoint of T_i ([11]). This gives that $T = sp(1, 1, k)$ for some $k \geq 1$. □

Theorem 86. *Let G be a graph on n vertices with $\operatorname{ppt}(G) = n-3$ and $\gamma_p(G) \in \{2, 3\}$. Then $G \in \{\overline{K_3}, \overline{K_2} \cup K_2, K_1 \cup C_3, K_1 \cup P_3, K_1 \cup P_4, K_1 \cup C_4, K_2 \cup K_2\}$.*

Proof. For any minimum power dominating set S of G, $|S| \leq 3$ and $|N[S]| \leq 4$. Suppose $\Delta(G) \geq 3$, and let G_1 be a connected component of G containing a vertex of degree at least 3. By Lemma 77, G_1 has a minimum power dominating set S_1 such that each $s \in S_1$ has degree at least 3. Let S be a minimum power dominating of G such that $S_1 \subseteq S$. Since $|S| \in \{2, 3\}$ and each $s \in S_1$ has degree at least 3, it follows that $|N[S]| \geq 5$, which is a contradiction. Thus, $\Delta(G) \leq 2$ and G is the union of paths and cycles. Furthermore, G has at least 2 connected components (since $\gamma_P(G) \neq 1$ then G is not a path or cycle), and G has at most 3 connected components (since $\gamma_P(G) \leq 3$).

Suppose G has only two components, G_1 and G_2, so $\gamma_p(G) = 2$. If G_1 is a path on at least 3 vertices or a cycle, then $G_2 = K_1$ (since $|N[S]| \leq 4$) and $\operatorname{ppt}(G) = \operatorname{ppt}(G_1) = |G_1| - 2$. By Proposition 84, $G_1 \in \{P_3, P_4, C_3, C_4\}$. Otherwise, $G = K_2 \cup K_2$.

If G has three components G_1, G_2, G_3, then $\gamma_p(G) = 3$ and exactly one force is performed at each step. So, $G_2 = G_3 = K_1$, and $\mathrm{ppt}(G) = \mathrm{ppt}(G_1) = |G_1| - 1$. By Proposition 82, $G_1 \in \{K_1, K_2\}$.

$\square$

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3.6 Postscript: a bound on the power propagation time of nonplanar graphs

A graph $G = (V, E)$ is *planar* if it has a crossing-free embedding in the plane. A graph is *nonplanar* if it is not planar.

Let B be a zero forcing set of G. Perform the set of forces to color the entire graph blue, recording the forces in the order in which they are performed. This is the *chronological list of forces.* For a chronological list of forces, a *forcing chain* is a sequence of vertices $(v_1, v_2, \ldots, v_k)$ such that for $i = 1, \ldots, k-1$, v_i forces v_{i+1}. A *maximal forcing chain* is a forcing chain that is not a proper subsequence of another forcing chain. Note that each forcing chain corresponds to an induced path in G, and the maximal forcing chains corresponding to a set of forces partition the vertices of G.

The following theorem gives a tight upper bound on the power propagation time of nonplanar graphs.

Theorem 87. *Let G be a connected graph on n vertices that is not planar. Then* $\mathrm{ppt}(G) \leq n - 4$ *and this bound is tight. Furthermore, if G is connected and* $\mathrm{ppt}(G) = n - 4$, *then* $\gamma_P(G) = 1$.

To prove Theorem 87, we need the next Lemma.

Lemma 88. *Let $G = (V(G), E(G))$ be a connected graph with* $\mathrm{Z}(G) \leq 3$*. Then G is planar.*

Proof. If $\mathrm{Z}(G) = 1$, then $G = P_{|V(G)|}$ [11]. Butler and Young show that if $\mathrm{Z}(G) = 2$, then G is planar [4, Lemma 1]. We use a very similar argument to show that if $\mathrm{Z}(G) = 3$, then G is planar.

Suppose $\mathrm{Z}(G) = 3$, let B be a zero forcing set of G, and let $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$ be the maximal forcing chains of B that partition the vertices of G. Draw the vertices of G in the plane in the following way: draw the vertices of $\mathfrak{C}_1$ on the positive $x-$axis from left to right in consecutive forcing order, draw the vertices of $\mathfrak{C}_2$ on the negative $x-$axis from right to left in consecutive forcing order, and draw the vertices of $\mathfrak{C}_3$ on the positive $y-$axis from bottom to top in consecutive forcing order. Recall that $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$ corresponds to induced paths in G. The remaining edges of G have one endpoint on one of these paths and the other endpoint on another of these paths. Suppose there exist edges $e_1 = v_1v_2$ and $e_2 = u_1u_2$ in $E(G)$ that cross, and without loss of generality let v_1 and u_1 belong to the forcing chain $\mathfrak{C}_1$ and let v_2 and u_2 belong to the forcing chain $\mathfrak{C}_2$ (where v_1 comes before u_1 and u_2 comes before v_2 in the consecutive forcing order). Then v_1 cannot force until v_2 has been forced, but v_2 will not get forced before u_1 is forced. This contradicts B being a zero forcing set of G with forcing chains $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$. So, G is planar.

□

Proof of Theorem 87. We prove the contrapositive. If $\mathrm{ppt}(G) \geq n - 3$, then for any minimum dominating set S of G, $|S| \leq 3$, $|N[S]| \leq 4$, and $N[S] \setminus S$ is a zero forcing set of $G - S$ with size at most 3. By Lemma 88, $G - S$ is planar. Since $N[S] \leq 4$, $G[N[S]]$ is planar, and it follows that G is planar. The graph $G = K_5$ demonstrates that this bound is tight.

We now show the second statement of the theorem. Suppose that $\mathrm{ppt}(G) = n - 4$. For an efficient power dominating set S of G, $|N[S]| \leq 5$. If $|N[S]| \leq 4$, then $N[S] \setminus S$ is a zero forcing set for $G - S$ of size at most 3, so $G - S$ is planar (by Lemma 88) and it follows that G is planar. So, $|N[S]| = 5$. If $|N[S] \setminus S| = 4$, then $\gamma_p(G) = 1$. If $|N[S] \setminus S| \leq 3$, it follows again that $G - S$ is planar, and since G is non planar, we must have that $N[S] = K_5$ and $\gamma_p(G) = 1$. $\square$

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CHAPTER 4. GENERAL CONCLUSIONS

In Chapter 2, bounds on the tree cover number of a graph were given in terms of the number of vertices, and some relationships between the tree cover number and other graph parameters were established. Although not presented this way, the tree cover problem can be thought of as a graph coloring problem: Given a graph G, what is the least number of colors needed to color the vertices such that each color class induces a tree? This formulation connects the idea of tree covers to several other graph coloring parameters such as the vertex-aboricity, which is the least number of colors needed to color the vertices of a graph such that each color class induces a forest. Since a tree cover is a forest cover, the vertex-aboricity is a lower bound on the tree cover number of a graph. Vertex-aboricty is a well explored area, so perhaps similar techniques to those used to study vertex-aboricty may be used to study the tree cover number. To be best of my knowledge, the complexity of the tree cover number has not been determined, although I have reason to believe that it is NP-complete.

In Chapter 3, the power propagation time of a graph is studied. I conjecture that the sum of the power propagation time of a graph and that of its complement is at most the number of vertices, and to date, this conjecture has not been proved or disproved. Furthermore, while general bounds on the power propagation time of a graph were given, it may be interesting to prove nicer bounds for specific graph families and to determine relationships between the power propagation time and other graph parameters.

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