Techniques for determining equality of the maximum nullity and the zero forcing number of a graph

by

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

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ABSTRACT

The maximum nullity of a simple graph G over a field $\mathcal{F}$ is the maximum nullity over all symmetric matrices over $\mathcal{F}$ whose ijth entry (where $i \neq j$) is nonzero if and only if ij is an edge in G and is zero otherwise. The zero forcing number of a graph is the minimum cardinality over all zero forcing sets. It is known that the zero forcing number of a graph is an upper bound for the maximum nullity of the graph (see [1]). In this dissertation, we search for characteristics of a graph that guarantee the maximum nullity of the graph and the zero forcing number of the graph are the same by studying a variety of graph parameters which bound the maximum nullity of a graph below. Graph parameters that are considered are a Colin de Verdiére type parameter and the vertex connectivity. We also use matrices, such as a divisor matrix of a graph and an equitable partition of the adjacency matrix of a graph, to establish a lower bound for the nullity of the graph's adjacency matrix. Last, we introduce a new graph parameter and show that it has the same value as the nullity of the graph's adjacency matrix, which is a lower bound for the maximum nullity of a graph.

CHAPTER 1. Introduction

Combinatorial matrix theory uses graphs and other combinatorial techniques to establish properties of matrices. An inverse eigenvalue problem can be described as follows: Given a set of numbers determine if there exists a matrix with certain properties which has eigenvalues corresponding to the given set. A trivial inverse eigenvalue problem is to construct a symmetric matrix having eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. The matrix containing diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_k$ with zeros on the off diagonal is a matrix that satisfies such criteria. Changing the required properties of the matrix changes the difficulty of the problem. Refer to [8] for a survey on inverse eigenvalue problems more generally. The inverse eigenvalue problem of a graph (IEPG) is a well-known combinatorial matrix theory problem. Using tools such as the maximum eigenvalue multiplicity over a set of matrices gives insight into the solution of the IEPG for some graph families. For instance, if we know that the maximum multiplicity over a set of matrices is m, then we know that a matrix having eigenvalues corresponding to a set that contains a number occurring more than m times does not exist.

The *set of symmetric matrices of a graph G over a field $\mathcal{F}$*, denoted by $\mathcal{S}(\mathcal{F}, G)$, is the set of symmetric matrices $A = [a_{ij}]$ having the same off-diagonal nonzero pattern as the adjacency matrix of G (for $i \neq j, a_{ij} \neq 0 \iff ij \in E(G)$) with free diagonal entries ($a_{ii} \in \mathcal{F}$). The *maximum nullity of a graph G over a field $\mathcal{F}$*, denoted by $\mathrm{M}(\mathcal{F}, G)$ is the maximum nullity over $\mathcal{S}(\mathcal{F}, G)$. Because the diagonal of a matrix $A \in \mathcal{S}(\mathcal{F}, G)$ is unrestricted and the nullity of $A - \lambda I$ is the multiplicity of λ as an eigenvalue of A the maximum multiplicity of matrices in $\mathcal{S}(\mathcal{F}, G)$ is the same as the maximum nullity. The *minimum rank of a graph G over a field $\mathcal{F}$*, denoted by $\mathrm{mr}(\mathcal{F}, G)$, is the minimum rank over $\mathcal{S}(\mathcal{F}, G)$. Whenever the field is not specified, the field is understood to be the real numbers $\mathbb{R}$. Observe that $\mathrm{mr}(\mathcal{F}, G) + \mathrm{M}(\mathcal{F}, G) = |G|$, where $|G|$ is defined to be the number of vertices in the graph G. This makes solving for $\mathrm{M}(\mathcal{F}, G)$ equivalent to solving the associated minimum rank problem. See [12] for a discussion on the motivation of the minimum rank problem.

The maximum nullity over a set of matrices that can be described by a graph has been well studied (see [1, 3, 16]). While determining the maximum nullity over a set of matrices described by a graph is not easy to compute, there are graph parameters that allow us to bound the maximum nullity. For some graphs, these bounds are enough to determine the maximum nullity. Unfortunately, the bounds available are not enough to determine the maximum nullity for all graphs.

Let Z be a subset of $V(G)$ such that every vertex in Z is colored blue and all other vertices are colored white. The *color change rule* for zero forcing is: A blue vertex can change a white vertex blue if the white vertex is the only white vertex adjacent to the blue vertex. (Vertices v and u are said to be *adjacent* if and only if $\{v, u\} \in E(G)$.) In this case, we say that the blue vertex forced the white vertex blue. A *zero forcing set* is a subset of $V(G)$ such that after applying the color change rule until no more changes are possible, all vertices in G are colored blue. The *zero forcing number* of a graph G, denoted by $\mathrm{Z}(G)$, is the minimum cardinality over all zero forcing sets. A *chronological list of forces* is a sequence of forces performed in the given order. The term zero forcing refers to forcing entries in the null vector to be zero, which leads to the relationship that the maximum nullity of a graph is bounded above by the zero forcing number of the graph.

Proposition 1.1. [1, Proposition 2.4] Let G be a graph and let $\mathcal{F}$ be a field. Then

$$\mathrm{M}(\mathcal{F}, G) \leq \mathrm{Z}(G).$$

The problem of characterizing graphs that have the property that the maximum nullity of the graph is equal to zero forcing number of the graph was first posed in [1]. While this problem is still open, there are many families of graphs that have their maximum nullity equal to their zero forcing number. A list of families of graphs having this property can be found in [17] including trees, cycles, complete graphs, complete bipartite graphs, completely subdivided graphs, and graphs with less than 8 vertices. The zero forcing number of a graph can be computed by using mathematical software. However, determining the maximum nullity of a graph is a challenging problem.

A notable contribution of this dissertation is the introduction of a combinatorial technique for computing the nullity of the adjacency matrix. We apply this and other techniques to establish $\mathrm{M}(G) = \mathrm{Z}(G)$ for many additional families of graphs G, in some cases, this is established fro arbitrary fields. In Chapter 2, we apply the Colin de Vedière parameter ξ and graph minors to establish a lower bound for the maximum nullity of a graph. Moreover, we use this tool to show that the maximum nullity and zero forcing number of all extended cube graphs and some circulant graphs are equal (extended cube graphs and circulant graphs are defined in Chapter 2). In Chapter 3, we use the vertex connectivity of a graph to bound the maximum nullity and determine the maximum nullity and zero forcing number of some additional circulant graphs. In Chapter 4, we use known results related to the divisor matrix of an equitable partition of a graph to bound the maximum nullity for some graphs. In Chapter 5, we use a decomposition for the adjacency matrix of a graph introduced in [5] by Barrett et al. to determine the nullity of the adjacency matrix. This allows us to establish field independent minimum rank for a specific subfamily of the extended cube graph. In Chapter 6, we use the combinatorial technique for computing the nullity of the adjacency matrix described above to show that the Aztec diamond graphs have field independent minimum rank.

Graph parameters of interest are the vertex connectivity and a Colin Verdière type parameter. Equitable partitions, equitable decompositions of the adjacency matrix, and the new parameter nullity of a graph are use to establish the nullity of the adjacency matrix which serves as a lower bound for the maximum nullity of the graph. In the remainder of this introduction we define these terms and families of graphs after a few additional definitions. We also state some results from the literature that will be used.

A *graph*, denoted by G, consists of a set $V(G)$ called a *vertex set* and an *edge set* $E(G)$ where the edge set contains two element subsets of the vertex set. For convenience, when $\{v, u\} \in E(G)$ we may drop the brackets and write vu. The *order* of a graph, denoted by $|G|$, is the number of vertices in the graph. The spectrum of a symmetric matrix A, denoted by $\operatorname{spec}(A)$, is the multiset of eigenvalues of A. The nullity of a symmetric matrix, denoted by $\operatorname{null}(A)$, is the number of times zero occurs in spec(A). The rank of a symmetric matrix A, denoted by rank(A), is the dimension of the vector space spanned by the rows of A.

A matrix $A \in \mathcal{S}(G)$ is said to have the *Strong Arnold Property* (SAP) if there does not exist a nonzero symmetric matrix X having the following three properties: (1) $AX = 0$, (2) $A \circ X = 0$, (3) $I \circ X = 0$ where $\circ$ is the Hadamard (entrywise) product. The Colin de Verdière type parameter associated with the maximum nullity is

$$\xi(G) = \max\{\text{null}(A) \,|\, A \in \mathcal{S}(G) \text{ and } A \text{ has the SAP}\}.$$

Clearly $\xi(G) \leq \mathrm{M}(G) \leq \mathrm{Z}(G)$ for all graphs G. The parameter ξ was introduced in 2005 in [4] to gain more insight on the minimum rank of a graph.

The *vertex connectivity*, denoted by $\kappa(G)$, of a graph is the smallest number of vertices needed to be deleted to disconnect a noncomplete graph and $\kappa(K_n) = n-1$. In 2007, building on the work of Lovász, Saks, Schrijver [19], [18], Hein van der Holst [20] showed that the vertex connectivity of a graph is a lower bound for the maximum nullity of a graph. Although not published, it is worth noting that in a AIM workshop the minimum degree and vertex connectivity of a graph were used to show that the maximum nullity is equal to the zero forcing number for certain circulant graphs. (In 1962, Frank Harary showed that the vertex connectivity of those graphs is the same as the minimum degree and they are now called *Harary graphs*.)

Note that $A(G) - \lambda I$ with $\lambda \in \mathbb{Z}$ can be viewed as a matrix over any field $\mathcal{F}$ of characteristic p with each integer interpreted as its residue modulo p. Thus $A(G) - \lambda I \in \mathcal{S}(\mathcal{F}, G)$. When we view $A \in \mathcal{F}^{nxn}$ we write rank($\mathcal{F}, A$) for the rank which may depend on $\mathcal{F}$. An *optimal matrix over a field* $\mathcal{F}$ is a matrix $A \in \mathcal{S}(G)$ such that rank($\mathcal{F}, A$) $=$ mr($\mathcal{F}, G$). We say that an integer matrix $A \in \mathcal{S}(\mathcal{F}, G)$ that has entries $-1, 0, 1$ on the off diagonal is *universally optimal* if for all fields $\mathcal{F}$, rank($\mathcal{F}, A$) $=$ mr($\mathcal{F}, G$). The minimum rank of a graph G is said to be *field independent* if for all fields $\mathcal{F}$, mr($\mathcal{F}, G$) $=$ mr(G). The minimum rank problem over fields other than the real numbers was studied as early as 2004 by Wayne Barrett, Hein van der Holst, and Raphael Loewy in [6]. In 2009, Dealba, et. al [11] used universally optimal matrices to establish minimum rank field independence for many graphs listed in [17].

Proposition 1.2. [11, Corollary 2.3] If $A \in \mathbb{Z}^{n\times n}$, then $\operatorname{rank}(\mathbb{Z}_p, A) \leq \operatorname{rank}(A)$ for every prime p.

Corollary 1.3. Let G be a graph having the property that for some $\lambda \in \mathbb{Z}$, $\operatorname{rank}(A(G)-\lambda I) = |G| - \mathrm{Z}(G)$, or equivalently, $\operatorname{null}(A(G) - \lambda I) = \mathrm{Z}(G)$. Then the minimum rank of G is field independent and $A(G) - \lambda I$ is universally optimal, and $\mathrm{M}(\mathcal{F}, G) = \mathrm{Z}(G)$ for all fields $\mathcal{F}$.

Proof. By Proposition 1.1, $|G| - \operatorname{mr}(\mathcal{F}, G) = \mathrm{M}(\mathcal{F}, G) \leq \mathrm{Z}(G)$ and by Proposition 1.2 we have $\operatorname{rank}(\mathcal{F}, A(G) - \lambda I) \leq \operatorname{rank}(A(G) - \lambda I)$, so $\operatorname{null}(A(G) - \lambda I) \leq \operatorname{null}(\mathcal{F}, A(G) - \lambda I)$. It follows that

$$\mathrm{Z}(G) \geq \mathrm{M}(\mathcal{F}, G) \geq \operatorname{null}(\mathcal{F}, A(G) - \lambda I) \geq \operatorname{null}(A(G) - \lambda I) = \mathrm{Z}(G).$$

Therefore, $\operatorname{mr}(\mathcal{F}, G) = \operatorname{rank}(\mathcal{F}, A(G) + \lambda I) = |G| - \mathrm{Z}(G)$ which shows that G has field independent minimum rank and $A(G) + \lambda I$ is universally optimal. □

Observation 1.4. Let G be a graph. If there exists a prime p such that $\operatorname{mr}(\mathbb{Z}_p, G) \neq \operatorname{mr}(G)$ then G does not have field independent minimum rank.

A *generalized Petersen Graph*, denoted by $P(n, k)$, is a graph having a labeled vertex set $\{u_0, u_1, \ldots u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$ and edge set $\big\{\{u_i u_{i+1 \bmod n}\}, \{v_i v_{i+k \bmod n}\}, \{u_i v_i\} : i = 0, 1, 2, \ldots, n-1\big\}$, for $n \geq 3$ and k a positive integer less than $\lfloor \frac{n}{2} \rfloor$. In [2], the adjacency matrix was used to show that the maximum nullity is equal to the zero forcing number for certain generalized Petersen graphs.

Theorem 1.5. [2, Theorem 2.4] *Let r be a positive integer. Then*

$$\mathrm{M}(P(15r, 2)) = \mathrm{Z}(P(15r, 2)) = 6 \quad \textit{and} \quad \mathrm{M}(P(24r, 5)) = \mathrm{Z}(P(24r, 5)) = 12$$

and the maximum nullity is attained by the adjacency matrix.

Corollary 1.6. Let r be a positive integer. Then the two subfamilies $P(15r, 2)$ and $P(24r, 5)$ have field independent minimum rank with universally optimal matrices. Moreover, for all fields $\mathcal{F}$,

$$\mathrm{M}(\mathcal{F}, P(15r, 2)) = \mathrm{Z}(P(15r, 2)) \quad \text{and} \quad \mathrm{M}(\mathcal{F}, P(24r, 5)) = \mathrm{Z}(P(24r, 5)).$$

The *Cartesian product* of the graphs G and H, denoted by $G\Box H$, has vertex set $\{(v,w)|v \in V(G), w \in V(H)\}$ and edge set

$$\{(v_1,w_1)(v_2,w_2)\,|\,(v_1 = v_2 \text{ and } w_1w_2 \in E(H)) \text{ or } (v_1v_2 \in E(G) \text{ and } w_1 = w_2)\}.$$

Theorem 1.7. [1, Theorem 3.8] *Let* $k \geq 3$*. Then* $\mathrm{M}(C_k\Box P_t) = \mathrm{Z}(C_k\Box P_t) = \min\{k, 2t\}$*.*

Example 1.8. By Theorem 1.7, $\mathrm{M}(C_7\Box P_2) = 4$ which implies $\mathrm{mr}(C_7\Box P_2) = 10$. By computation via SageMath (see [22]), there does not exist a matrix in $\mathcal{S}(\mathbb{Z}_2, C_7\Box P_2)$ having rank equal to 10. Therefore by Observation 1.4, $C_7\Box P_2$ does not have field independent minimum rank.

Example 1.8 shows that the generalized Petersen graphs do not have field independent minimum rank field independent since $C_7\Box P_2$ is isomorphic to $P(7,1)$. It is known that $C_n\Box P_t$ does not have field independent minimum rank (see [11, Example 3.5]).

An *equitable partition* of a graph is a partition of the vertex set $V_0, V_1, \ldots, V_k$ such that for all $v \in V_i$ the number b_{ij} of neighbors in V_j is constant for all V_j. Let $V_0, V_1, \ldots, V_k$ be an equitable partition of $V(G)$. We say a *divisor* of G is a weighted directed graph with vertex set $V_0, V_1, \ldots, V_k$ and arc (V_i, V_j) having weight b_{ij} if and only if $b_{ij} \neq 0$. The matrix $[b_{ij}]$ is the *divisor matrix* associated with the equitable partition $V_0, V_1, \ldots, V_k$. It is known that an equitable partition of a graph G can be used to find specific eigenvalues of $A(G)$ (see [9]).

An *automorphism* of a graph G is an isomorphism ϕ from $V(G)$ to $V(G)$ such that $\phi(i)$ is adjacent to $\phi(j)$ if and only if i is adjacent to j. Let G be a graph with $v, u \in V(G)$ and let ϕ be an automorphism of G. Define the relation $\approx$ on the vertices of G by $v \approx u$ if and only if there exists a nonnegative integer j for which $v = \phi^j(u)$. This relation is an equivalence relation on the vertices of G and the equivalence classes are the *orbits* of ϕ. Let ϕ be an uniform automorphism of G with orbit size k where $1 < k$. A *transversal* of ϕ is a subset of $V(G)$ containing exactly one vertex from each orbit of ϕ. The ℓ-power of transversal T is defined to be the following transversal,

$$T_\ell = \{\phi^\ell(v)\,|\,v \in T\}$$

for $\ell \in \{0, 1, 2, \ldots, k-1\}$. It is straightforward to see that T_ℓ is a transversal and $\cup_{\ell=0}^{k-1} T_\ell = V(G)$.

Given an automorphism ϕ, an $n \times n$ matrix $A = [a_{ij}]$ associated with the graph G on n vertices such that

$$a_{\phi(i),\phi(j)} = a_{ij}$$

for all $i, j \in \{1, 2, \ldots, n\}$, is called $\phi-$*compatible*. An $n \times n$ matrix A associated with the graph G is called $\phi-$*automorphism compatible* if it is $\phi-$compatible for every automorphism ϕ of G. Recently in 2017, Barrett et al. used equitable partitions of a graph in [5] to decompose $A(G)$. This decomposition can be used to determine all eigenvalues of $A(G)$. As a result, this decomposition is useful for determining a lower bound for the maximum nullity. Moreover, it can be use to establish a potential candidate for an universally optimal matrix.

A *general graph* is a graph that may contain loops (edges of the form vv) and/or multi-edges (two edges containing the same vertices u and v are called *multi-edges*). Let G be a general graph and let $v, u \in V(G)$. The neighborhood of v in a general graph G, denoted by $N_G(v)$, is a multiset containing vertices of $V(G)$ such that k copies of u are in $N_G(v)$ if and only if there are k copies of uv in $E(G)$. Let X and Y be multisets containing elements of $V(G)$. The general graph G_{v+X} is obtained from G by adding one edge vw for each $w \in N_G(x)$ and for every $x \in X$ (see Figure 1.1). Suppose $N_{G_{v+Y}}(v) \subseteq N_{G_{v+X}}(v)$. Then the general graph G_{v+X-Y} is obtained from G_{v+X} by deleting one edge vw for each $w \in N_G(y)$ and for every $y \in Y$ (see Figure 1.1). In the case that X and Y consists of a single vertex x or y, we write G_{v+x} or G_{v+x-y}.

We define a color change rule as follows: In a graph G, having each vertex colored red or white, a white vertex u can be colored red if there exists a white vertex v and multisets of white vertices X, Y such that

1. $u \notin \{v\} \cup X \cup Y$, and
2. $N_{G_{u+U_k}}(u) = N_{G_{v+X-Y}}(v)$

for some nonnegative integer k and the multiset U_k containing k copies of u, (whenever $k = 0$, U_k is the empty set and $N_{G_{u+U_k}}(u) = N_G(u)$). In this case we say that u can be colored red by (v, X, Y, k).

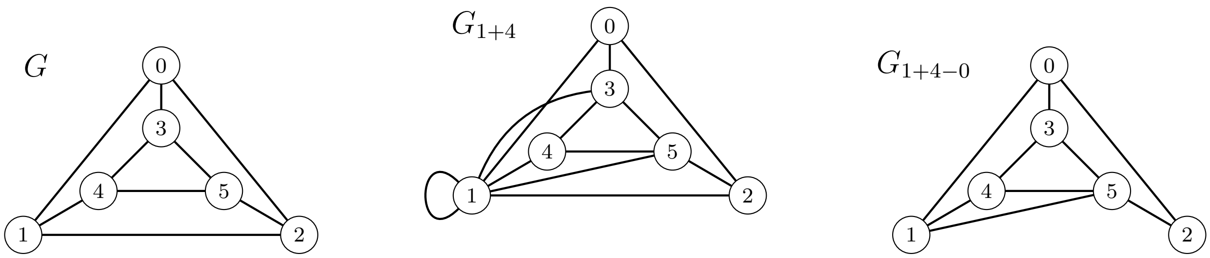


Figure 1.1: This shows the graph G_{1+4-0}.

Example 1.9. Figure 1.1 illustrates the process of creating G_{1+4-0}. Moreover, vertices 1 and 3 have the same neighborhood in G_{1+4-0}, so vertex 3 can be colored red in G by $(1, \{4\}, \{0\}, 0)$. We can also color vertex 5 red. Consider the general graph G_{1+4-2} in which vertices 1 and 5 have the same neighborhood in G_{1+4-2}.

A set of red vertices is called a *red set*, denoted by $\mathcal{R}$, if the vertices $v_1, v_2, \ldots, v_t$ of $\mathcal{R}$ can be sequentially colored red. The *nullity of a graph G*, denoted by $\operatorname{null}(G)$, is the maximum cardinality over the set of all red sets.

CHAPTER 2. An Application of the Strong Arnold Property

In this section, we use the Colin de Verdière type parameter ξ to show that the maximum nullity and zero forcing number of various families of graphs are equal.

Example 2.1. By using SageMath (see [15]), $A(C_8\square P_3)$ has the SAP and $\operatorname{null}(A(C_8\square P_3)) = 6$. By Theorem 1.7, $\mathrm{M}(C_8\square P_3) = \mathrm{Z}(C_8\square P_3) = 6$. Therefore, $\xi(C_8\square P_3) = \mathrm{M}(C_8\square P_3) = \mathrm{Z}(C_8\square P_3) = 6$.

An edge *contraction* of a graph G is defined to be a deletion of two adjacent vertices v_1 and v_2 and an insertion of a vertex u such that $uv \in E(G)$ if and only if $vv_1 \in E(G)$ or $vv_2 \in E(G)$. A graph H is a *minor* of a graph G if H can be constructed from G by performing edge deletions, vertex deletions, and/or contractions. We write $H \preceq G$ when H is a minor of G. Note that $G \preceq G' \preceq G''$ implies $G \preceq G''$.

Observation 2.2. Let $3 \leq k \leq n$ and $1 \leq r \leq t$. Then $C_k\square P_r \preceq C_n\square P_t$.

Theorem 2.3. [4, Corollary 2.5] *If H is a minor of G then $\xi(H) \leq \xi(G)$.*

Definition 2.4. Let H be a minor of G. We say that H is a *zero forcing minor* of G if $\mathrm{Z}(G) \leq \mathrm{Z}(H)$.

Theorem 2.5. *Let H be a zero forcing minor of G such that $\xi(H) = \mathrm{Z}(H)$. Then $\xi(G) = \mathrm{M}(G) = \mathrm{Z}(G) = \mathrm{Z}(H)$.*

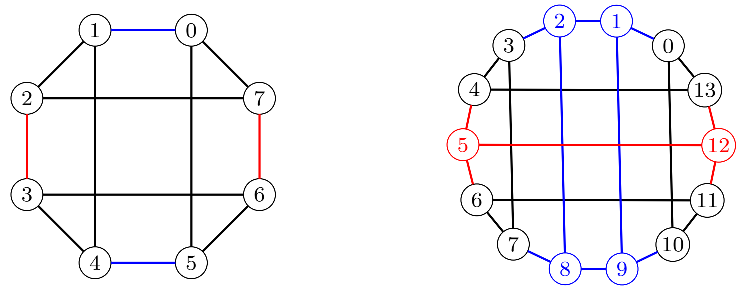
Proof. Given that H is a zero forcing minor, $\mathrm{Z}(G) \leq \mathrm{Z}(H)$. By Theorem 2.3, $\xi(H) \leq \xi(G)$ and it follows that

$$\mathrm{Z}(H) = \xi(H) \leq \xi(G) \leq \mathrm{M}(G) \leq \mathrm{Z}(G) \leq \mathrm{Z}(H).$$

Thus the parameters $\xi(G), \mathrm{M}(G), \mathrm{Z}(G)$ are equal to $\mathrm{Z}(H)$. $\square$

Corollary 2.6. Let $G = C_n\square P_3$ such that $8 \leq n$. Then

$$\xi(G) = \mathrm{M}(\mathcal{F}, G) = \mathrm{Z}(G) = 6.$$



The cube graph $\mathrm{ECG}(1,2)$

Figure 2.1: Applying two vertical and one horizontal subdivision edge insertion on the cube graph gives $\mathrm{ECG}(1,2)$.

A *k-subdivision* of an edge, say uv, is an operation on a graph in which edge uv is deleted, vertices $v_1, v_2, \ldots, v_k$ and edges $uv_1, v_1v_2, v_2v_3, \ldots, v_kv$ are added. We say the edge uv has been k-subdivided. Whenever $k = 1$ we simply say that the edge uv has been subdivided. A *k-subdivision edge insertion* on the edges uv and wx is an operation on a graph in which edges uv and wx are k-subdivided adding vertices $v_1, v_2, \ldots, v_k$ and $x_1, x_2, \ldots, x_k$, respectively, and edges $v_1x_1, v_2x_2, \ldots, v_kx_k$ are added. The *cube graph* Q_3 can be described by an 8 - cycle containing a labeled vertex set $\{0, 1, \ldots, 7\}$ and added edges $\{\{0,5\}, \{1,4\}, \{2,7\}, \{3,6\}\}$ as shown in Figure 2.1.

Proposition 2.7. [20, Lemma 8] For the cube graph, $\xi(Q_3) = 4 = \mathrm{M}(G) = \mathrm{Z}(G)$.

Definition 2.8. (Extended cube graph) A *vertical $k-$subdivision edge insertion* on the cube graph is a $k-$subdivision edge insertion on the edges $\{0,1\}$ and $\{4,5\}$. A *horizontal $k-$subdivision edge insertion* on the cube graph is a $k-$subdivision edge insertion on the edges $\{2,3\}$ and $\{6,7\}$, with the numbering as in Figure 2.1. An *extended cube graph*, denoted by $\mathrm{ECG}(t,k)$, is the cube graph with a horizontal $t-$subdivision edge insertion, a vertical $k-$subdivision edge insertion, and a relabeling around the cycle containing vertex set $\{0, 1, \ldots, 7 + 2(t+k)\}$.

Figure 2.1 shows $\mathrm{ECG}(1,2)$. Notice that $\mathrm{ECG}(t,k)$ isomorphic to the graph $\mathrm{ECG}(k,t)$. For simplicity we consider the extended cube graphs with $t \leq k$. The graph $\mathrm{ECG}(1,1)$ is called the

Bidiakis cube. It was shown in [2, Proposition 5.1] that the maximum nullity and zero forcing number of the Bidiakis cube are the same, motivating the creation of the extended cube graphs.

Observe that in $\mathrm{ECG}(t,k)$, as we draw it, the top endpoints of the vertical edges are $0,\dots,k+1$, the left endpoints of the horizontal edges are $k+2,\dots,t+k+3$, the lower endpoints of the vertical edges are $t+k+4,\dots,t+2k+5=n-t-3$, and the right endpoints of the horizontal edges are $t+2k+6,\dots,2t+2k+7=n-1$.

Observation 2.9. Let G be a graph constructed from the graph H by performing a subdivision edge insertion. Then $H \preceq G$.

Proposition 2.10. Let G be an extended cube graph $\mathrm{ECG}(t,k)$. Then $\mathrm{Z}(G) \le 4$.

Proof. Let n be the number of vertices of G and let $r = n-t-3$. The set $\{0, r, r+1, n-1\}$ is a zero forcing set with simultaneous forces

$$\begin{array}{ccccccccc} 0 & \to & 1 & \to & 2 & \to & \cdots & \to & k+2 \\ r & \to & r-1 & \to & r-2 & \to & \cdots & \to & r-(k+2) = k+t+3. \end{array}$$

These forcing sequences run simultaneously in parallel, i.e., $0 \to 1$ and $r \to r-1$ are simultaneous, etc. After the above forces are completed, the following forces run in parallel.

$$\begin{array}{ccccccccc} (k+2) & \to & (k+2)+1 & \to & (k+2)+3 & \to & \cdots & \to & (k+2)+t \\ (n-1) & \to & (n-1)-1 & \to & (n-1)-2 & \to & \cdots & \to & (n-1)-t = 2k+t+6 \end{array}$$

□

Corollary 2.11. Let G be the extended cube graph $\mathrm{ECG}(t,k)$. Then

$$\xi(G) = \mathrm{M}(G) = \mathrm{Z}(G) = 4.$$

Proof. Let H be the cube graph. By Proposition 2.7, $\xi(H) = \mathrm{Z}(\mathrm{H}) = 4$. By Theorem 2.5, H is a zero forcing minor of G. Thus $\xi(G) = \mathrm{M}(G) = \mathrm{Z}(G) = 4$ by Theorem 2.5. □

A *circulant* graph, denoted by $\mathrm{Circ}[n,S]$, is a graph with vertex set $\{0,1,\dots,n-1\} \subseteq \mathbb{Z}$ and a *connection set* $S \subseteq \{1,2,\dots,\frac{n}{2}\} \subseteq \mathbb{Z}$, where the edge set of $\mathrm{Circ}[n,S]$ is precisely $\big\{\{i, i\pm s\} : s \in S\big\}$

with arithmetic performed modulo n (see Figure 2.2). For any $a \in [n]$, the graphs $\mathrm{Circ}[n, S]$ and $\mathrm{Circ}[n, aS]$ are isomorphic whenever a and n are relatively prime. Thus if there exists $b \in S$ such that $\gcd(b, n) = 1$, then $1 \in b^{-1}S$ and $\mathrm{Circ}[n, S] \cong \mathrm{Circ}[n, b^{-1}S]$. For simplicity, all circulant graphs considered here have 1 in the connection set.

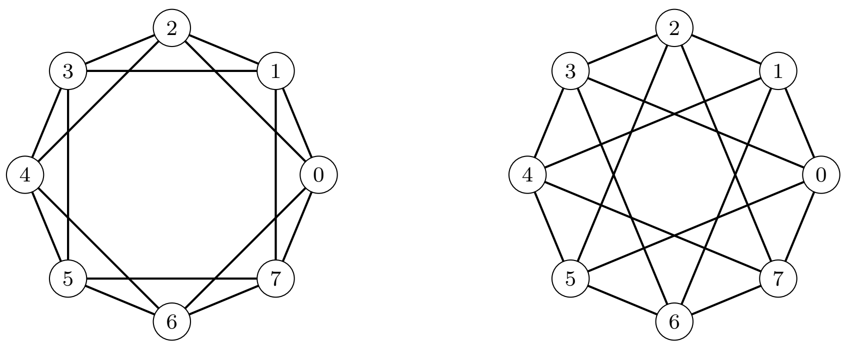


Figure 2.2: The circulants $\mathrm{Circ}[8, \{1, 2\}]$ and $\mathrm{Circ}[8, \{1, 3\}]$.

Observation 2.12. For positive integer k, the circulant $\mathrm{Circ}[4k, \{1, 3, \ldots, 2k-1\}] = K_{2k,2k}$ and the circulant $\mathrm{Circ}[4k+2, \{1, 3, \ldots, 2k+1\}] = K_{2k+1,2k+1}$.

Proposition 2.13 and Theorem 3.4 below were found by several groups in 2009 and 2010 but not published. Some of these results were also published in [10]. We state these results and give formal proofs of the results for clarity.

Proposition 2.13. [14, Proposition 2.1] Let G be a circulant graph $\mathrm{Circ}[n, S]$ and let $m = \max\{i | i \in S\}$. Then $\mathrm{Z}(G) \leq 2m$.

Proof. We will show that $Z = \{0, 1, \ldots, 2m-1\}$ is a zero forcing set. Suppose $s \in S$ and $s \neq m$. Then $1 \leq s < m$ and it follows that $m \pm s \in Z$. If $s = m$, then $m - s = m - m = 0$ which implies $m - s \in Z$. This shows that all neighbors of m except for $2m$ are in Z; clearly $m \in Z$. Hence m can force $2m$. Using a similar argument $m + i$ forces $2m + i$ for $i \in \{1, 2, \ldots, n - 2m - 1\}$. A forcing sequence is listed as

$$m \to 2m,\ m+1 \to 2m+1,\ \ldots,\ m + (n - 2m - 1) = n - m - 1 \to 2m + (n - 2m - 1) = n - 1. \quad \square$$

Observation 2.14. Let n be a multiple of k, $G = C_{n/k} \square P_k$, and $H = \mathrm{Circ}[n, \{1, k\}]$. Then $G \preceq H$. This is illustrated in Figure 2.3.

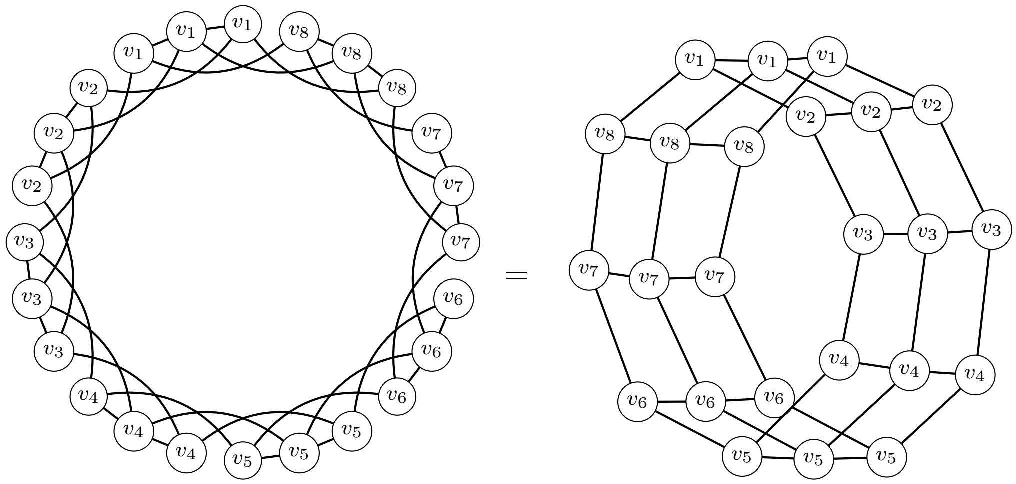


Figure 2.3: By applying edge deletions to Circ[24, {1, 3}] and relabeling the vertices, it is clear that $C_8 \square P_3 \preceq \mathrm{Circ}[24, \{1, 3\}]$.

The next result may also be true for $n < 24$ but our proof needs n to be big enough to use results from $\mathrm{Z}(C_8 \square P_3) = \mathrm{Z}(\mathrm{Circ}[24, \{1, 3\}]) = 6$.

Theorem 2.15. *Let $n \geq 24$ be a multiple of 3 and let $G = \mathrm{Circ}[n, \{1, 3\}]$. Then $\xi(G) = \mathrm{M}(G) = \mathrm{Z}(G) = 6$.*

Proof. In Example 2.1, we showed that $6 = \xi(C_8 \square P_3)$. By Observations 2.2 and 2.14 $C_8 \square P_3 \preceq C_{n/3} \square P_3 \preceq G$, and $\mathrm{Z}(G) \leq 6$ by Proposition 2.13. Therefore, $C_8 \square P_3$ is a zero forcing minor of G, and $\xi(G) = \mathrm{M}(G) = \mathrm{Z}(G) = 6$ by Theorem 2.5. $\square$

Remark 2.16. For every positive integer t, $\mathrm{Circ}[2t, \{1, t\}]$ is the Moebius ladder graph. The edges $\{i, i + t\}$ of $\mathrm{Circ}[2t, \{1, t\}]$ are the rungs in the Moebius ladder. It was shown in [1, Proposition 3.9] that all Moebius ladder graphs have both their maximum nullity and zero forcing number equal to 4.

CHAPTER 3. An Application of Vertex Connectivity

In this section, we use the known results for the vertex connectivity of a graph to show that the maximum nullity and zero forcing number for some circulant graphs are the same.

Theorem 3.1. [20, Theorem 4] *Let G be a graph. Then $\kappa(G) \leq \xi(G)$.*

Corollary 3.2. Let G be a graph. Then $\kappa(G) \leq \xi(G) \leq \mathrm{M}(G) \leq \mathrm{Z}(G)$.

Observation 3.3. Let G be a circulant graph $\mathrm{Circ}[n, S]$ such that S does not contain $\frac{n}{2}$. Then $\delta(G) = 2|S|$.

The circulant graph $\mathrm{Circ}[n, \{1, 2, \ldots, t\}]$ is called a *consecutive circulant*. It is known that a consecutive circulant is a Harary graph (see [21, Example 4.1.4]), and it is shown in [21, Theorem 4.1.5] that the vertex connectivity and the minimum degree of a Harary graph are equal.

Theorem 3.4. [14, Corollary 2.2] *Let $2t + 1 \leq n$ and let $G = \mathrm{Circ}[n, \{1, 2, \ldots, t\}]$. Then*

$$\kappa(G) = \delta(G) = \xi(G) = \mathrm{M}(G) = \mathrm{Z}(G) = 2t.$$

Proof. By Observation 3.3, $\delta(G) = 2t$. Since G is a Harary graph, $\kappa(G) = \delta(G)$. By Corollary 3.2, we have the following inequalities $\kappa(G) = \delta(G) \leq \xi(G) \leq \mathrm{M}(G) \leq \mathrm{Z}(G)$. An upper bound for the zero forcing number of G is $2t$, which is given by Proposition 2.13. Therefore, $\kappa(G) = \delta(G) = \xi(G) = \mathrm{M}(G) = \mathrm{Z}(G) = 2t$. □

When n is odd and $t = \lfloor \frac{n}{2} \rfloor$ the circulant $\mathrm{Circ}[n, \{1, 2, \ldots, t\}] = K_n$. The equality of $\kappa, \delta, \xi,$ and, Z shown for consecutive circulants in Theorem 3.4 is not true for all circulant graphs as shown in the next example.

Example 3.5. Let G be the graph $\mathrm{Circ}[8, \{1, 3\}] = K_{4,4}$. By considering $G = K_{4,4}$, we see that $\kappa(G) = \delta(G) = 4$ and $\mathrm{Z}(G) = 6$, since $\mathrm{Z}(K_{a,b}) = a + b - 2$. It was shown in [4, Corollary 2.8] that $\xi(G) = \min\{4, 4\} + 1 = 5$.

For $n = 2m+1$, if n is prime, $\gcd(m-1, n) = \gcd(m, n) = 1$. So $\mathrm{Circ}[n, \{1, \ldots, m-2, m\}] \cong K_n - C_n \cong \mathrm{Circ}[n, [m-1]]$. However $\mathrm{Circ}[22, \{1, 2, 3, 4, 5, 6, 7, 8, 10\}] \not\cong \mathrm{Circ}[22, \{1, 2, 3, 4, 5, 6, 7, 8, 9\}]$. Thus the discussion below covers graphs that are not consecutive circulants.

Proposition 3.6. Let $H = \mathrm{Circ}[n, [m] \setminus \{m-1\}]$ where $n > 9$ and $m = \lceil n/2 \rceil - 1$. Then $\mathrm{Z}(H) \leq 2(m-1)$.

Proof. Observe first that $2(m-1) = \delta(H) \leq \mathrm{Z}(H)$. We will consider the case when n is odd first. Then $n = 2m+1$. Since $m-1$ is not in the connection set, i is not adjacent to $i + (m-1)$ or $i - (m-1)$. Note that $i - (m-1) \equiv i + n - (m-1) \equiv i + 2m + 1 - (m-1) \equiv i + m + 2 \bmod n$. It follows that 0 is not adjacent to $m-1$ or $m+2$, and 3 is not adjacent to $m+2$ or $m+5$. Consider the set $Z = V(H) \setminus \{m-2, m-1, m+2\}$. Then $0 \to m-2$ and $3 \to m-1$. After these two forces any vertex adjacent to $m+2$ can force $m+2$, which shows that Z is a zero forcing set.

When n is even, $n = 2m+2$. Since $m-1$ and $\frac{n}{2}$ are not in the connection set, i is not adjacent to $i + (m-1)$, $i - (m-1) \equiv i + m + 3$ or $i + n/2 \equiv i + (m+1)$. It follows that 0 is not adjacent to $m-1$, $m+1$, or $m+3$, and 2 is not adjacent $m+1$, $m+3$, or $m+5$. Consider the set $Z = V(H) \setminus \{2, m-1, m+3\}$. Then $0 \to 2$ and $2 \to m-1$. Any vertex adjacent to $m+3$ can force $m+3$, which shows that Z is a zero forcing set. $\square$

Theorem 3.7. [7, Theorem 1] *Let G be a circulant graph $\mathrm{Circ}[n, \{s_1, s_2, \ldots, s_k\}]$. There exists a proper divisor d of n such that the number of distinct positive residues modulo d of $s_1, s_2, \ldots, s_k, n - s_k, n - s_{k-1}, \ldots, n - s_1$ is less than $\min\{d-1, \frac{\delta(G)}{n} d\}$ if and only if $\kappa(G) < \delta(G)$.*

Theorem 3.8. *Let $H = \mathrm{Circ}[n, [m] \setminus \{m-1\}]$ where $n \geq 10$ and $m = \lceil n/2 \rceil - 1$. Then $\kappa(G) = \delta(G) = \xi(G) = \mathrm{M}(G) = \mathrm{Z}(G) = 2(m-1)$.*

Proof. Since $2(m-1) = \delta(G) = \mathrm{Z}(G)$, we need only to show $\kappa(G) = \delta(G)$. Let d be a positive divisor of n and let $S' = \{1, 2, \ldots, m-2, m, n-m, n-(m-2), \ldots, n-1\}$. If $d < m$, then $d - 1 \leq m-2$ and $1, 2, \ldots, d-1$ are $d-1$ distinct residue of S' modulo d. Note that $d = m$ is impossible since m does not divide $2m+1$ or $2m+2$, as $m \geq 3$. If n is even and $d = \frac{n}{2}$, then

$\frac{\delta(G)}{n}d = \frac{\delta(G)}{2} = \frac{2(m-1)}{2} = m - 1 < d - 1$. Furthermore $1, 2, \ldots, m-2, m$ are $m-1$ distinct residue of S' modulo d which is greater than or equal to $\frac{\delta(G)}{n}d$. Therefore, by Theorem 3.7 it must be the case that $\kappa(G) = \delta(G)$. $\square$

CHAPTER 4. An Application of Equitable Partitions

In this section we use an equitable partition of a circulant graph to bound the nullity of the graph. It fact, the lower bound is obtained from the nullity of a circulant graph of small order which possesses the same connection set as the circulant graph of interest.

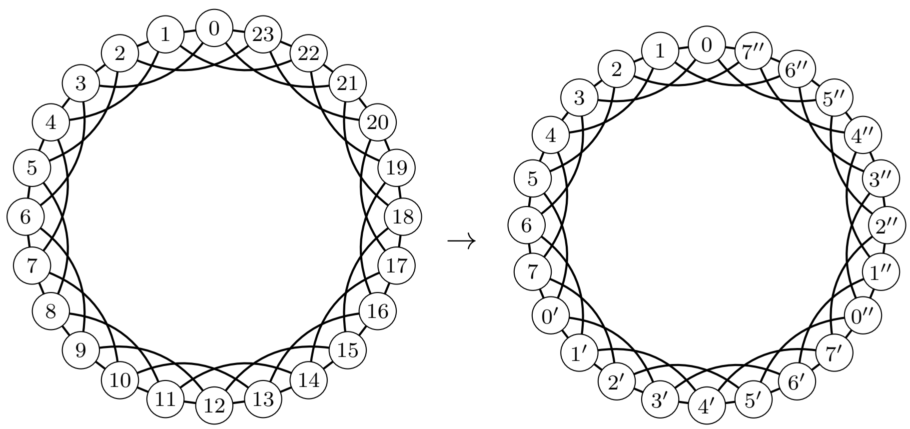


Figure 4.1: Circ[24, $\{1, 3\}$] and a relabeling showing how the vertices can be equitably partitioned.

Example 4.1. Figure 4.1 shows the graph of the circulant Circ[24, $\{1, 3\}$]. By partitioning the vertex set of Circ[24, $\{1, 3\}$] as in Figure 4.1, it is clear that the partition $V_i = \{i, i', i''\}$ for $i = 0, 1, \ldots, 7$ is an equitable partition of Circ[24, $\{1, 3\}$].

Proposition 4.2. [13, Page 196] Let ϕ be an automorphism of G. Then the orbits of ϕ give an equitable partition of $V(G)$.

Note that the equitable partition in Example 4.1 is obtained from the automorphism $\varphi(i) = i+8$.

Theorem 4.3. [9, Theorem 3.9.5] *Let G be a graph and let D be a divisor matrix of some equitable partition of $V(G)$. Then the eigenvalues of D are eigenvalues of $A(G)$ (including multiplicity).*

Theorem 4.4. *Let G be the circulant graph* $\mathrm{Circ}[nk, S]$ *where k is a positive integer and $S \subseteq \left[\left\lceil \frac{n}{2} \right\rceil - 1\right]$. Then the adjacency matrix of the circulant graph* $\mathrm{Circ}[n, S]$ *is a divisor matrix of G.*

Proof. The orbits of the automorphism $\varphi(t) \equiv t + n \bmod nk$ of G are

$$V_i = \{r \in V(\mathrm{Circ}[nk, S]) \,|\, r \equiv i \bmod n\}.$$

Hence the partition $V_0, V_1, \ldots, V_{n-1}$ is an equitable partition of G.

Let $[b_{ij}]$ be the divisor matrix of G with respect to the given equitable partition and let $[a_{ij}]$ be the adjacency matrix of $\mathrm{Circ}[n, S]$. It suffices to show for all i and j, $b_{ij} \leq 1$ and b_{ij} is nonzero if and only if a_{ij} is nonzero. Suppose s_1 and s_2 are distinct elements in S, V_iV_j is an arc, and $i + s_1 \in V_j$. Since $s_1, s_2 \in \left[\left\lceil \frac{n}{2} \right\rceil - 1\right]$, $s_1 \pm s_2 \not\equiv 0 \bmod n$ which implies $i + s_1 \not\equiv i \pm s_2 \bmod n$ and $i \pm s_2 \not\equiv j \bmod n$. Hence $i \pm s_2 \notin V_j$. Also $s_1 \in \left[\left\lceil \frac{n}{2} \right\rceil - 1\right]$, so $2s_1 \not\equiv 0 \bmod n$ which implies $i + s_1 \not\equiv i - s_1 \bmod n$ and $i - s_1 \not\equiv j \bmod n$. This shows that $i - s_1, i + s_2, i - s_2 \notin V_j$. Hence $b_{ij} \leq 1$ for all i and j.

Suppose V_i is adjacent to V_j. Then there exists a vertex $\ell \in V_i$ and $p \in V_j$ such that ℓ is adjacent to p, in G. Thus, $\ell - p \equiv i - j \bmod n$. By definition of adjacency in G, for some $s \in S$, $\ell \equiv p + s \mod nk$ or $\ell \equiv p - s \mod nk$. Hence $\ell - p \equiv s \bmod nk$ or $p - \ell \equiv s \bmod nk$. Thus, $i - j \equiv \ell - p \equiv s \bmod n$ or $i - j \equiv \ell - p \equiv -s \bmod n$. In either case, i is adjacent to j in $\mathrm{Circ}[n, S]$. Now suppose i is adjacent to j in $\mathrm{Circ}[n, S]$ where $0 \leq i, j \leq n - 1$ as integers. Then it must be the case that $j = i + s \bmod n$ or $j = i - s \bmod n$. In $\mathrm{Circ}[nk, S]$, $i \in V_i$ and $i + s \in V_j$ or $i - s \in V_j$. In either case, $b_{ij} \neq 0$ in the divisor matrix of G. $\square$

When n is even in Theorem 4.4 the connection set cannot be extended to include $\frac{n}{2}$.

Example 4.5. Let $G = \mathrm{Circ}[12, \{1, 3\}]$ and $H = \mathrm{Circ}[6, \{1, 3\}] = K_{3,3}$. Furthermore, using the equitable partition described in the proof of Theorem 4.4,

$$V_0 = \{0, 6\}, V_1 = \{1, 7\}, V_2 = \{2, 8\}, V_3 = \{3, 9\},$$

$b_{0,1} = 1, b_{0,2} = 0, b_{0,3} = 2,$

$$[b_{ij}] = \begin{pmatrix} 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \end{pmatrix},$$

and $A(\mathrm{Circ}[6, \{1, 3\}])$ is not the divisor matrix of G. By computation, the eigenvalues of $A(G)$ are $\pm 4, \pm\sqrt{3}, \pm 1, 0$ and H is bipartite and 3 - regular which implies ± 3 are eigenvalues of H. This shows that the adjacency matrix of H is not a divisor matrix of G.

The next corollary is a direct result of Theorem 4.3 and Theorem 4.4.

Corollary 4.6. Consider the circulant graph $\mathrm{Circ}[nk, S]$ where k is a positive integer and $S \subseteq \left[\lceil \frac{n}{2} \rceil - 1\right]$. Then

$$\mathrm{spec}(A(\mathrm{Circ}[n, S])) \subseteq \mathrm{spec}(A(\mathrm{Circ}[nk, S]))$$

and $\mathrm{null}(A(\mathrm{Circ}[n, S])) \leq \mathrm{null}(A(\mathrm{Circ}[nk, S]))$.

It was shown in Theorem 2.15 that $\mathrm{M}(\mathrm{Circ}[3k, \{1, 3\}]) = \mathrm{Z}(\mathrm{Circ}[3k, \{1, 3\}]) = 6$ for $k \geq 8$. The next result establishes field independence, in addition to showing that the maximum nullity equals the zero forcing number for many additional circulants..

Theorem 4.7. *Let k be a positive integer and let ℓ be an odd integer between 3 and 21. Then*

$$\mathrm{M}(\mathrm{Circ}[(\ell^2 - 1)k, \{1, \ell\}]) = \mathrm{Z}(\mathrm{Circ}[(\ell^2 - 1)k, \{1, \ell\}]) = 2\ell,$$

$\mathrm{Circ}[(\ell^2 - 1)k, \{1, \ell\}]$ *has field independent minimum rank, and its adjacency matrix is an universally optimal matrix.*

Proof. Let $n = \ell^2 - 1$, $S = \{1, \ell\}$, and $G = \mathrm{Circ}[nk, S]$ for $k \geq 1$. By Proposition 2.13 and Proposition 1.1, $\mathrm{M}(G) \leq \mathrm{Z}(G) \leq 2\ell$. Thus it suffices to show that $\mathrm{null}(A(\mathrm{Circ}[n, S])) = 2\ell$. This is easily verified using computer software (SageMath offers commands for computing the adjacency matrix of a graphs and its nullity). □

Conjecture 4.8. For all positive values of k and odd ℓ,

$$\mathrm{M}(\mathrm{Circ}[(\ell^2-1)k,\{1,\ell\}]) = \mathrm{Z}(\mathrm{Circ}[(\ell^2-1)k,\{1,\ell\}]) = 2\ell.$$

and field independent minimum rank with universally optimal matrix $A(G)$.

CHAPTER 5. An Application of Equitable Decompositions

In this section, we use the equitable decomposition of the adjacency matrix to establish field independent minimum rank of a graph. The graphs of interest are the extended cube graphs $\mathrm{ECG}(6q+1, 6q+1)$ where q is a nonnegative integer.

Example 5.1. In general, the extended cube graphs do not have field independent minimum rank. Some extended cube graphs are isomorphic to the Cartesian product of a cycle and a path. For instance, $\mathrm{ECG}(0, 3)$ is isomorphic to $C_7 \Box P_2$. It was shown in Example 1.8 that $\mathrm{mr}(\mathbb{Z}_2, C_7 \Box P_2) \neq \mathrm{mr}(C_7 \Box P_2)$.

Observation 5.2. The adjacency matrix of a graph is automorphism compatible.

The next theorem is stated in [5] for automorphism compatible matrices, but as noted there it could be stated for a $\phi-$compatible matrix and we do so.

Theorem 5.3. [5, Theorem 3.8] *Let G be a graph on n vertices, let ϕ be an uniform automorphism of G of orbit size k, let T_0 be a transversal of the orbits of ϕ, and let A be an $\phi-$compatible matrix in $\mathcal{S}(G)$. Set $A_\ell = A[\mathrm{T}_0, \mathrm{T}_\ell]$, $\ell = 0, 1, \ldots, k-1$, let $\omega = e^{2\pi i/k}$, and define*

$$B_j = \sum_{\ell=0}^{k-1} \omega^{j\ell} A_\ell, \quad j = 0, 1, \ldots, k-1.$$

Then for some invertible matrix S

$$S^{-1}AS = B_0 \oplus B_1 \oplus \cdots \oplus B_{k-1} \tag{5.1}$$

and

$$\sigma(A) = \sigma(B_0) \cup \sigma(B_1) \cup \cdots \cup \sigma(B_{k-1}).$$

The decomposition in (5.1) is called an equitable decomposition of A.

Observation 5.4. Let G be a extended cube graph $\mathrm{ECG}(t,t)$ on n vertices and let $r = \frac{n}{4}$. Then the function $\varphi(x) \equiv x + r \mod n$ is a uniform automorphism for G. The function φ can also be written as a permutation,

$$\phi = (0, 0+r, 0+2r, 0+3r)(1, 1+r, 1+2r, 1+3r)\cdots(r-1, r-1+r, r-1+2r, r-1+3r).$$

Furthermore, $T_0 = \{0, 1, \ldots, r-1\}$ is a transversal.

Example 5.5. The following is an example of constructing the eigenvalues of $\mathrm{ECG}(1,1)$ using an equitable decomposition. As in Observation 5.4,

$$\varphi(x) \equiv x + 3 \mod 12,$$

is an automorphism with permutation representation $\phi = (0,3,6,9)(1,4,7,10)(2,5,8,11)$, and the transversals are $T_0 = \{0,1,2\}$, $T_1 = \{3,4,5\}$, $T_2 = \{6,7,8\}$, $T_3 = \{9,10,11\}$. Let

$$A_0 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \phi^0(0)=0 & \phi^0(1)=1 & \phi^0(2)=2 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}, \quad A_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \phi^1(0)=3 & \phi^1(1)=4 & \phi^1(2)=5 \end{array} \\ \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array},$$

$$A_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \phi^2(0)=6 & \phi^2(1)=7 & \phi^2(2)=8 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \end{array}, \quad \text{and} \quad A_3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} \phi^3(0)=9 & \phi^3(1)=10 & \phi^3(2)=11 \end{array} \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{array}.$$

Hence

$$B_0 = A_0 + A_1 + A_2 + A_3 = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$$

and it follows that the spectrum of B_0 is $\{3, 0, -2\}$ with eigenvector $x_0 = [1, -2, 1]^T$ corresponding to the eigenvalue 0. Also,

$$B_1 = A_0 + iA_1 - A_2 - iA_3 = \begin{pmatrix} 0 & 1 & -1-i \\ 1 & -1 & 1 \\ -1+i & 1 & 0 \end{pmatrix}$$

and the spectrum of B_1 is approximately $\{1.561552, 0, -2.561552\}$ with eigenvector $x_1 = [i, 1+i, 1]^T$ corresponding to the eigenvalue 0,

$$B_2 = A_0 - A_1 + A_2 - A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

has spectrum $\{2, 0, -1\}$ with eigenvector $x_2 = [1, 0, -1]^T$ corresponding to the eigenvalue 0, and

$$B_3 = A_0 - iA_1 - A_2 + iA_3 = \begin{pmatrix} 0 & 1 & -1+i \\ 1 & 1 & 1 \\ -1-i & 1 & 0 \end{pmatrix}$$

has spectrum approximately $\{1.561552, 0, -2.561552\}$ with eigenvector $x_3 = [-1, -1-i, i]^T$ corresponding to the eigenvalue 0. Using SageMath (see [22]), we compute the eigenvalues of $\mathrm{ECG}(1, 1)$ to be approximately

$$\{3, 2, 1.561552, 1.561552, 0, 0, 0, 0, -1, -2, -2.561552, -2.561552\}$$

which is the union of the spectra of B_0, B_1, B_2, B_3.

Theorem 5.6. *Let G be a extended cube graph* $\mathrm{ECG}(6q + 1, 6q + 1)$ *for some nonnegative integer q. Then G has field independent minimum rank and $A(G)$ is a universally optimal matrix.*

Proof. First we will show that the adjacency matrix of each such extended cube graph has nullity at least 4. Hence by Corollary 2.11 the adjacency matrix realizes the maximum nullity.

It was shown in Example 5.5 that the nullity of $\mathrm{ECG}(1, 1)$ has nullity equal to 4, so we assume $q > 0$. Let G be a extended cube graph $\mathrm{ECG}(6q + 1, 6q + 1)$ and let n be the number of vertices of G. Consider the uniform automorphism

$$\varphi(x) = x + r \mod n$$

where $r = \frac{n}{4}$ given by Observation 5.4. By Theorem 5.3, G has the following spectrum

$$\operatorname{spec}(A(G)) = \operatorname{spec}(B_0) \cup \operatorname{spec}(B_1) \cup \operatorname{spec}(B_2) \cup \operatorname{spec}(B_3)$$

for the matrices B_i corresponding to φ. We show that B_0, B_1, B_2, B_3 each have nullity at least 1, which implies $A(G)$ has nullity at least 4.

The transversals with respect to φ are $T_0 = \{0, 1, 2, \ldots, r-1\}$, $T_1 = \{r, r+1, r+2, \ldots, 2r-1\}$, $T_2 = \{2r, 2r+1, 2r+2, \ldots, 3r-1\}$, $T_3 = \{3r, 3r+1, 3r+2, \ldots, 4r-1\}$. Hence $k = 4$ and $\omega = e^{2\pi i/4} = i$. For the graph ECG$(1,1)$, let $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ be the corresponding matrices used in Theorem 5.3 to construct $\tilde{B}_0, \tilde{B}_1, \tilde{B}_2, \tilde{B}_3$ such that

$$\operatorname{spec}(A(\mathrm{ECG}(1,1))) = \operatorname{spec}(\tilde{B}_0) \cup \operatorname{spec}(\tilde{B}_1) \cup \operatorname{spec}(\tilde{B}_2) \cup \operatorname{spec}(\tilde{B}_3)$$

and $\tilde{B}_0 = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3$. Also, let $\tilde{x}_0, \tilde{x}_1, \tilde{x}_2$ be the eigenvectors of $\tilde{B}_0, \tilde{B}_1, \tilde{B}_2$ respectively, corresponding to the eigenvalue 0. It follows that A_0, A_1, A_2, A_3 are the matrices

$$A_0 = \begin{pmatrix} \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & \cdots & 0 \\ \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & \cdots & 0 \\ 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 \\ 0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 \\ 0 & \cdots & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 \end{pmatrix},$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \tilde{A}_1 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{A}_2 \\ 0 & 0 & 0 & \cdots & 0 & \tilde{A}_2 & 0 \\ 0 & 0 & 0 & \cdots & \tilde{A}_2 & 0 & 0 \\ \vdots & \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ 0 & 0 & \tilde{A}_2 & \cdots & 0 & 0 & 0 \\ 0 & \tilde{A}_2 & 0 & \cdots & 0 & 0 & 0 \\ \tilde{A}_2 & 0 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad \text{and} \;\; A_3 = A_1^T \, .$$

By definition,

$$B_j = i^{0j} A_0 + i^j A_1 + i^{2j} A_2 + i^{3j} A_3 = A_0 + i^j A_1 + (-1)^j A_2 + i^{3j} A_3 \tag{5.2}$$

for $j = 0, 1, 2, 3$, so the following matrices are constructed

$$B_0 = A_0 + A_1 + A_2 + A_3 \qquad\qquad B_1 = A_0 + iA_1 - A_2 - iA_3$$

$$B_2 = A_0 - A_1 + A_2 - A_3 \qquad\qquad B_3 = A_0 - iA_1 - A_2 + iA_3.$$

Writing B_j in terms of the matrices $\tilde{A}_0, \tilde{A}_1, \tilde{A}_2, \tilde{A}_3$ we get the following matrix

$$\begin{pmatrix}
\tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & (-1)^j\tilde{A}_2+i^{3j}\tilde{A}_3 \\
\tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & \cdots & 0 & 0 & (-1)^j\tilde{A}_2 & 0 \\
0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & \cdots & 0 & (-1)^j\tilde{A}_2 & 0 & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \iddots & & & \vdots \\
0 & \cdots & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & (-1)^j\tilde{A}_2 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0+(-1)^j\tilde{A}_2 & i^{0j}\tilde{A}_1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & (-1)^j\tilde{A}_2 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & \cdots & 0 \\
\vdots & & & \iddots & & & \ddots & \ddots & \ddots & & \vdots \\
0 & 0 & (-1)^j\tilde{A}_2 & 0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 \\
0 & (-1)^j\tilde{A}_2 & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 \\
i^j\tilde{A}_1+(-1)^j\tilde{A}_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0
\end{pmatrix}.$$

For simplicity of notation let $\hat{x}_1 = -\tilde{A}_2\tilde{x}_1$ and $\hat{x}_2 = -\tilde{x}_2$. We show that $x_0 = \bigoplus_{m=1}^{2q+1}\tilde{x}_0$, $x_1 = \bigoplus_{m=1}^{q}(\tilde{x}_1 \oplus \hat{x}_1) \oplus \tilde{x}_1$, and $x_2 = \bigoplus_{m=1}^{q}(\tilde{x}_2 \oplus \hat{x}_2) \oplus \tilde{x}_2$, are eigenvectors corresponding to eigenvalue 0 for B_0, B_1, and B_2 respectively. Since $B_3 = B_1^T$ it follows that B_3 also has a zero eigenvalue so we omit showing that B_3 has an eigenvalue of zero. Observe that

$$B_0 = \begin{pmatrix}
\tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{A}_2+\tilde{A}_3 \\
\tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{A}_2 & 0 \\
0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & \cdots & 0 & \tilde{A}_2 & 0 & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \iddots & & & \vdots \\
0 & \cdots & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0+\tilde{A}_2 & \tilde{A}_1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \tilde{A}_2 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & \cdots & 0 \\
\vdots & & & \iddots & & & \ddots & \ddots & \ddots & & \vdots \\
0 & 0 & \tilde{A}_2 & 0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 \\
0 & \tilde{A}_2 & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 \\
\tilde{A}_1+\tilde{A}_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0
\end{pmatrix}.$$

The product B_0x_0 reduces down to the following vector

$$\begin{pmatrix}
\tilde{A}_0\tilde{x}_0 + \tilde{A}_1\tilde{x}_0 + \tilde{A}_2\tilde{x}_0 + \tilde{A}_3\tilde{x}_0 \\
\tilde{A}_0\tilde{x}_0 + \tilde{A}_1\tilde{x}_0 + \tilde{A}_2\tilde{x}_0 + \tilde{A}_3\tilde{x}_0 \\
\vdots \\
\tilde{A}_0\tilde{x}_0 + \tilde{A}_1\tilde{x}_0 + \tilde{A}_2\tilde{x}_0 + \tilde{A}_3\tilde{x}_0 \\
\tilde{A}_0\tilde{x}_0 + \tilde{A}_1\tilde{x}_0 + \tilde{A}_2\tilde{x}_0 + \tilde{A}_3\tilde{x}_0
\end{pmatrix} = \begin{pmatrix}
\tilde{B}_0\tilde{x}_0 \\
\tilde{B}_0\tilde{x}_0 \\
\vdots \\
\tilde{B}_0\tilde{x}_0 \\
\tilde{B}_0\tilde{x}_0
\end{pmatrix} = 0,$$

since $\tilde{B}_0 = \tilde{A}_0 + \tilde{A}_1 + \tilde{A}_2 + \tilde{A}_3$.

To compute B_1x_1 consider the fact that $\tilde{x}_1 = [i, 1+i, 1]^T$ is an eigenvector for $\tilde{B}_1$. So by definition, $\hat{x}_1 = [-1, -1-i, -i]^T$ and

$$\tilde{A}_0\hat{x}_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1-i \\ -i \end{pmatrix} = -A_0\tilde{x}_1$$

$$\tilde{A}_1\hat{x}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1-i \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1+i \\ 1 \end{pmatrix} = i\tilde{A}_1\tilde{x}_1.$$

Since $\tilde{A}_2^2 = I$ it follows that $\tilde{A}_2\hat{x}_1 = -\tilde{x}_1$. Also,

$$\tilde{A}_3\hat{x}_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ -1-i \\ -i \end{pmatrix} = \begin{pmatrix} -i \\ 0 \\ 0 \end{pmatrix} = -i \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} i \\ 1+i \\ 1 \end{pmatrix} = -i\tilde{A}_3\tilde{x}_1.$$

In other words,

$$\tilde{A}_0\hat{x}_1 = (-1-i)\mathbf{1}^T\,, \tilde{A}_1\hat{x}_1 = i\tilde{A}_1\tilde{x}_1\,, \tilde{A}_2\hat{x}_1 = -\tilde{x}_1\,, \text{ and } \tilde{A}_3\hat{x}_1 = -i\tilde{A}_3\tilde{x}_1$$

and these values are used to reduce the entries of the next product. We have that B_1x_1 is

$$\begin{pmatrix}
\tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & -\tilde{A}_2-i\tilde{A}_3 \\
\tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & \cdots & 0 & 0 & -\tilde{A}_2 & 0 \\
0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & \cdots & 0 & -\tilde{A}_2 & 0 & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \iddots & & & \vdots \\
0 & \cdots & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & -\tilde{A}_2 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0+-\tilde{A}_2 & \tilde{A}_1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & -\tilde{A}_2 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & \cdots & 0 \\
\vdots & & & \iddots & & & \ddots & \ddots & \ddots & & \vdots \\
0 & 0 & -\tilde{A}_2 & 0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 \\
0 & -\tilde{A}_2 & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 \\
i\tilde{A}_1+-\tilde{A}_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0
\end{pmatrix} x_1.$$

We show that the product $B_1 x_1$ is given by the vector,

$$\left(\begin{array}{c}
\tilde{A}_0\tilde{x}_1 + \tilde{A}_1\hat{x}_1 + (-\tilde{A}_1 - i\tilde{A}_3)\tilde{x}_1 \\
-------------- \\
\tilde{A}_3\tilde{x}_1 + \tilde{A}_0\hat{x}_1 + \tilde{A}_1\tilde{x}_1 + (-\tilde{A}_2\hat{x}_1) \\
\tilde{A}_3\hat{x}_1 + \tilde{A}_0\tilde{x}_1 + \tilde{A}_1\hat{x}_1 + (-\tilde{A}_2\tilde{x}_1) \\
\tilde{A}_3\tilde{x}_1 + \tilde{A}_0\hat{x}_1 + \tilde{A}_1\tilde{x}_1 + (-\tilde{A}_2\hat{x}_1) \\
\tilde{A}_3\hat{x}_1 + \tilde{A}_0\tilde{x}_1 + \tilde{A}_1\hat{x}_1 + (-\tilde{A}_2\tilde{x}_1) \\
\vdots \\
\tilde{A}_3\hat{x}_1 + \tilde{A}_0\tilde{x}_1 + \tilde{A}_1\hat{x}_1 + (-\tilde{A}_2\tilde{x}_1) \\
\tilde{A}_3\tilde{x}_1 + \tilde{A}_0\hat{x}_1 + \tilde{A}_1\tilde{x}_1 + (-\tilde{A}_2\hat{x}_1) \\
-------------- \\
\tilde{A}_3\hat{x}_1 + (\tilde{A}_0 - \tilde{A}_2)\tilde{x}_1 + \tilde{A}_1\hat{x}_1 \\
-------------- \\
\tilde{A}_3\hat{x}_1 + \tilde{A}_0\tilde{x}_1 + \tilde{A}_1\hat{x}_1 + (-\tilde{A}_2\tilde{x}_1) \\
\tilde{A}_3\tilde{x}_1 + \tilde{A}_0\hat{x}_1 + \tilde{A}_1\tilde{x}_1 + (-\tilde{A}_2\hat{x}_1) \\
\vdots \\
\tilde{A}_3\tilde{x}_1 + \tilde{A}_0\hat{x}_1 + \tilde{A}_1\tilde{x}_1 + (-\tilde{A}_2\hat{x}_1) \\
\tilde{A}_3\hat{x}_1 + \tilde{A}_0\tilde{x}_1 + \tilde{A}_1\hat{x}_1 + (-\tilde{A}_2\tilde{x}_1) \\
-------------- \\
(i\tilde{A}_1 - \tilde{A}_2)\tilde{x}_1 + \tilde{A}_3\hat{x}_1 + \tilde{A}_0\tilde{x}_1
\end{array}\right) = 0$$

which implies that both matrices B_1 and B_3 has a zero eigenvalue. Note the that each entry in the product takes on one of the following values,

$$
\begin{aligned}
\tilde{A}_0\tilde{x}_1 + \tilde{A}_1\hat{x}_1 + (-\tilde{A}_2 - i\tilde{A}_3)\tilde{x}_1 &= \tilde{A}_0\tilde{x}_1 + i\tilde{A}_1\tilde{x}_1 - \tilde{A}_2\tilde{x}_1 - i\tilde{A}_3\tilde{x}_1 \\
&= (\tilde{A}_0 + i\tilde{A}_1 - \tilde{A}_2 - i\tilde{A}_3)\tilde{x}_1 = \tilde{B}_1\tilde{x}_1 = 0 \\
\tilde{A}_3\tilde{x}_1 + \tilde{A}_0\hat{x}_1 + \tilde{A}_1\tilde{x}_1 + (-\tilde{A}_2\hat{x}_1) &= (1,0,0)^T + (-1-i,-1-i,-1-i)^T \\
&\quad + (0,0,i)^T + (i,1+i,1)^T = 0 \\
\tilde{A}_3\hat{x}_1 + \tilde{A}_0\tilde{x}_1 + \tilde{A}_1\hat{x}_1 + (-\tilde{A}_2\tilde{x}_1) &= -i\tilde{A}_3\tilde{x}_1 + \tilde{A}_0\tilde{x}_1 + i\tilde{A}_1\tilde{x}_1 - \tilde{A}_2\tilde{x}_1 \\
&= (\tilde{A}_0 + i\tilde{A}_1 - \tilde{A}_2 - i\tilde{A}_3)\tilde{x}_1 = \tilde{B}_1\tilde{x}_1 = 0 \\
\tilde{A}_3\hat{x}_1 + (\tilde{A}_0 - \tilde{A}_2)\tilde{x}_1 + \tilde{A}_1\hat{x}_1 &= 0 \quad \text{by (5.2)} \\
(i\tilde{A}_1 - \tilde{A}_2)\tilde{x}_1 + \tilde{A}_3\hat{x}_1 + \tilde{A}_0\tilde{x}_1 &= i\tilde{A}_1\tilde{x}_1 - \tilde{A}_2\tilde{x}_1 - i\tilde{A}_3\tilde{x}_1 + \tilde{A}_0\tilde{x}_1 \\
&= (\tilde{A}_0 + i\tilde{A}_1 - \tilde{A}_2 - i\tilde{A}_3)\tilde{x}_1 = \tilde{B}_1\tilde{x}_1 = 0.
\end{aligned}
$$

Finally, we compute B_2x_2,

$$
\begin{pmatrix}
\tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{A}_2-\tilde{A}_3 \\
\tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & 0 & \cdots & 0 & 0 & \tilde{A}_2 & 0 \\
0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & 0 & \cdots & 0 & \tilde{A}_2 & 0 & 0 \\
\vdots & & \ddots & \ddots & \ddots & & & \iddots & & & \vdots \\
0 & \cdots & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & \tilde{A}_2 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0+\tilde{A}_2 & \tilde{A}_1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \tilde{A}_2 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 & \cdots & 0 \\
\vdots & & & \iddots & & & \ddots & \ddots & \ddots & & \vdots \\
0 & 0 & \tilde{A}_2 & 0 & \cdots & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 & 0 \\
0 & \tilde{A}_2 & 0 & 0 & \cdots & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0 & \tilde{A}_1 \\
-\tilde{A}_1+\tilde{A}_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \tilde{A}_3 & \tilde{A}_0
\end{pmatrix} x_2 =
\left(\begin{array}{c}
\tilde{A}_0\tilde{x}_2+\tilde{A}_1(-\tilde{x}_2)+(\tilde{A}_2-\tilde{A}_3)\tilde{x}_2 \\
\hdashline
\tilde{A}_3\tilde{x}_2+\tilde{A}_0(-\tilde{x}_2)+\tilde{A}_1\tilde{x}_2+\tilde{A}_2(-\tilde{x}_2) \\
\tilde{A}_3(-\tilde{x}_2)+\tilde{A}_0\tilde{x}_2+\tilde{A}_1(-\tilde{x}_2)+\tilde{A}_2\tilde{x}_2 \\
\tilde{A}_3\tilde{x}_2+\tilde{A}_0(-\tilde{x}_2)+\tilde{A}_1\tilde{x}_2+\tilde{A}_2(-\tilde{x}_2) \\
\tilde{A}_3(-\tilde{x}_2)+\tilde{A}_0\tilde{x}_2+\tilde{A}_1(-\tilde{x}_2)+\tilde{A}_2\tilde{x}_2 \\
\vdots \\
\tilde{A}_3\tilde{x}_2+\tilde{A}_0(-\tilde{x}_2)+\tilde{A}_1\tilde{x}_2+\tilde{A}_2(-\tilde{x}_2) \\
\tilde{A}_3(-\tilde{x}_2)+\tilde{A}_0\tilde{x}_2+\tilde{A}_1(-\tilde{x}_2)+\tilde{A}_2\tilde{x}_2 \\
\hdashline
\tilde{A}_3(-\tilde{x}_2)+(\tilde{A}_0+\tilde{A}_2)\tilde{x}_2+\tilde{A}_1(-\tilde{x}_2) \\
\hdashline
\tilde{A}_3(-\tilde{x}_2)+\tilde{A}_0\tilde{x}_2+\tilde{A}_1(-\tilde{x}_2)+\tilde{A}_2\tilde{x}_2 \\
\tilde{A}_3\tilde{x}_2+\tilde{A}_0(-\tilde{x}_2)+\tilde{A}_1\tilde{x}_2+\tilde{A}_2(-\tilde{x}_2) \\
\vdots \\
\tilde{A}_3(-\tilde{x}_2)+\tilde{A}_0\tilde{x}_2+\tilde{A}_1(-\tilde{x}_2)+\tilde{A}_2\tilde{x}_2 \\
\tilde{A}_3\tilde{x}_2+\tilde{A}_0(-\tilde{x}_2)+\tilde{A}_1\tilde{x}_2+\tilde{A}_2(-\tilde{x}_2) \\
\hdashline
(-\tilde{A}_1+\tilde{A}_2)\tilde{x}_2+\tilde{A}_3(-\tilde{x}_2)+\tilde{A}_0\tilde{x}_2
\end{array}\right).
$$

Each entry in the previous vector is $(\tilde{A}_0 - \tilde{A}_1 + \tilde{A}_2 - \tilde{A}_3)\tilde{x}_2$ which is zero. This shows that $B_2x_2 = 0$. Since the adjacency matrix of G was used to establish $\mathrm{M}(G) = \mathrm{Z}(G)$, by Corollary 1.3 the graph G has field independent minimum rank and $A(G)$ is universally optimal matrix. $\square$

Conjecture 5.7. Let $t \equiv 0, 1, 2 \bmod 6$ and r be an integer which is greater than $\lfloor t/6 \rfloor$. Then the extended cube graphs $\mathrm{ECG}(t, 6r - t - 4)$ has field independent minimum rank and their adjacency matrices are universally optimal.

Using computational software, $\mathrm{null}(A(\mathrm{ECG}(t, 6r - t - 4))) = 4 = \mathrm{Z}(\mathrm{ECG}(t, 6r - t - 4))$ for every $r \in \{2, \ldots, 12\}$ when $t \in \{0, 1, 2, 6, 7, 8\}$. Note that in Conjecture 5.7 when $t = 6q + 1$ for some nonnegative integer q it is the case that $t \equiv 1 \bmod 6$. Moreover, when $r = 2q + 1$, $6r - t - 4 = 6q + 1$, and $r > \lfloor t/6 \rfloor = \lfloor \frac{6q+1}{6} \rfloor = q$. In this case, $\mathrm{ECG}(t, 6r - t - 4)$ is the same as the graph $\mathrm{ECG}(6q + 1, 6q + 1)$ as in Theorem 5.6. Also, $t \equiv 1 \bmod 6$ implies $6r - t - 4 \equiv 1 \bmod 6$, $t \equiv 0 \bmod 6$ implies $6r - t - 4 \equiv 2 \bmod 6$, and $t \equiv 2 \bmod 6$ implies $6r - t - 4 \equiv 0 \bmod 6$.

CHAPTER 6. An Application of the Nullity of a Graph

In this section, we use the nullity of the Aztec diamond graphs and to some circulants to compute the maximum nullity and the zero forcing number, and show that they have field independent minimum rank with the adjacency matrix as a universally optimal matrix.

Observation 6.1. Let u, v be white vertices of $V(G)$, X and Y be multisets containing white vertices of $V(G)$, and k be a nonnegative integer. Then u can be colored red by (v, X, Y, k) if and only if

$$(k+1) \cdot \mathrm{row}_{A(G)}(u) = \mathrm{row}_{A(G)}(v) + \sum_{x \in X} \mathrm{row}_{A(G)}(x) - \sum_{y \in Y} \mathrm{row}_{A(G)}(y). \tag{6.1}$$

Theorem 6.2. *Let G be a simple graph. Then* $\mathrm{null}(G) = \mathrm{null}(A(G))$.

Proof. Let G be a graph with all vertices initially colored white. Suppose that at some stage the vertices $u_1, u_2, \ldots, u_{q-1}$ have been sequentially colored red, the remaining vertices colored white, and that each $\mathrm{row}_{A(G)}(u_i)$ can be expressed as a linear combination of rows indexed $W = V(G) \setminus \{u_1, u_2, \ldots, u_{q-1}\}$.

Suppose that v and the vertices of X, Y are white and u_q can be colored red by (v, X, Y, k). We show that $\mathrm{row}_{A(G)}(u_i)$ for $i = 1, 2, \ldots, q$ can each be expressed as a linear combination of rows indexed by $W' = W \setminus \{u_q\}$. Let $W' = \{w_1, w_2, \ldots, w_\ell\}$. By (6.1), $\mathrm{row}_{A(G)}(u_q)$ can be expressed as a linear combination of rows indexed by W'. We know that, $\mathrm{row}_{A(G)}(u_i)$ can be expressed as a linear combination of the rows associated with the vertices in $W = W' \cup \{u_q\}$. By substituting the expression for $\mathrm{row}_{A(G)}(u_q)$ into that for $\mathrm{row}_{A(G)}(u_i)$, we see that $\mathrm{row}_{A(G)}(u_i)$ is a linear combination of rows associated with vertices in W'. At the conclusion of this process $\mathrm{rank}(A(G)) \leq n - \mathrm{null}(G)$, so $\mathrm{null}(G) \leq \mathrm{null}(A(G))$.

Let W be a set of linearly independent rows of $A(G)$ that forms a basis for the row space of $A(G)$. Let $r = |W|$ and let $v_1, v_2, \ldots, v_r$ be the vertices associated with these rows. Then each row not in W, $\text{row}_{A(G)}(v_j)$ with $j > r$, can be written as

$$\frac{c_1}{d_1}\,\text{row}_{A(G)}(v_1) + \frac{c_2}{d_2}\,\text{row}_{A(G)}(v_2) + \cdots + \frac{c_r}{d_r}\,\text{row}_{A(G)}(v_r)$$

where $c_i, d_i \in \mathbb{Z}$ and $d_i > 0$ for $i = 1, \ldots, r$. By letting $d = \text{lcm}(d_1, d_2, \ldots, d_r)$ we can write

$$d \cdot \text{row}_{A(G)}(v_j) = c_1 s_1\,\text{row}_{A(G)}(v_1) + c_2 s_2\,\text{row}_{A(G)}(v_2) + \cdots + c_r s_r\,\text{row}_{A(G)}(v_r) \tag{6.2}$$

where $s_i = d/d_i \in \mathbb{Z}$. Fix v_j corresponding to a row in W. Let $\ell \in \{1, 2, \ldots, r\}$ such that $c_\ell s_\ell > 0$. Let X be the multiset of vertices consisting of $c_\ell s_\ell - 1$ copies of v_ℓ and $c_i s_i$ copies of v_i for $i \neq \ell$ and $c_i s_i > 0$ and let Y be the multiset of vertex consisting of $c_i s_i$ copies of v_i for $c_i s_i < 0$. Then v_j can be colored red by $(v_\ell, X, Y, d-1)$. This implies $\text{null}(G) \geq n - r \geq n - \text{rank}(A(G)) = \text{null}(A(G))$. □

Corollary 6.3. Let G be a bipartite graph with independent sets $\mathbf{B}$ and $\bar{\mathbf{B}}$ such that $|\mathbf{B}| = |\bar{\mathbf{B}}|$. Let $\mathcal{R} \subseteq \mathbf{B}$ be a red set such that every vertex in $\mathcal{R}$ is colored with some (v, X, Y, k) where $\{v\} \cup X \cup Y$ contains only vertices from $\mathbf{B}$. Then $2|\mathcal{R}| \leq \text{null}(A(G))$.

The *Aztec diamond of order* r is a diamond shape configuration of 2r(r+1) unit squares, as illustrated in Figure 6.1. The *Aztec diamond graph of order* r, denoted by AD_r, is the graph such that vertices $v, u \in V(\text{AD}_r)$ are adjacent if and only if squares v and u share an edge in the Aztec diamond of order r. The vertices of AD_r are labeled by ordered pairs (i, j) where $1 \leq i, j \leq 2r$, $r + 1 \leq i + j \leq 3r + 1$, and $0 \leq |j - i| \leq r$.

Proposition 6.4. Let G be a Aztec diamond graph AD_r. Then $\text{Z}(G) \leq 2r$.

Proof. We show that the set $Z = \{(1, r), (2, r-1), (3, r-2), \ldots, (r, 1)\} \cup \{(1, r+1), (2, r+2), (3, r+3), \ldots, (r, 2r)\}$ is a zero forcing set. For $i \in \{1, 2, \ldots, r\}$ in order (i, j) can force $(i+1, j)$ as long as (i, j) and $(i+1, j)$ exist. □

Theorem 6.5. *Let* AD_r *be a Aztec diamond graph of order* r *and* $\mathcal{F}$ *be an arbitrary field. Then*

$$\text{M}(\mathcal{F}, \text{AD}_r) = \text{Z}(\text{AD}_r) = 2r$$

and field independent minimum rank is established with the universally optimal matrix $A(G)$.

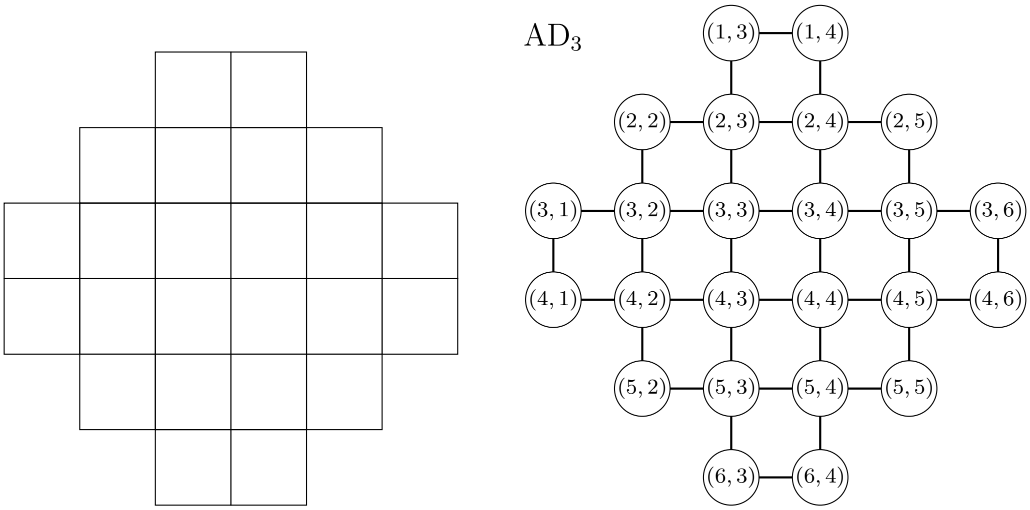


Figure 6.1: The Aztec diamond of order 3 and the Aztec diamond graph AD_3.

Proof. Let $D_\ell = \{(i+\ell, r+2+\ell-i) | 1 \le i \le r+1\}$ for $0 \le \ell \le r-1$. Note that the D_ℓ are independent sets and disjoint. Let $\mathbf{B} = D_0 \cup D_1 \cup D_2 \cup \cdots \cup D_{(r-1)}$. We show that r vertices of $\mathbf{B}$ can be colored red by other vertices of $\mathbf{B}$. The vertex $(r+1, 1)$ in the set D_0 can be colored red by $\big((r,2), \{(i,j) \in D_0 | i < r, j \text{ is even}\}, \{(i,j) \in D_0 | i < r, j \text{ is odd}\}, 0\big)$. See Figure 6.2 for an example. Using a similar argument each D_ℓ has a vertex that can be colored red using only vertices from D_ℓ. Since $\mathbf{B}$ is partitioned into r sets D_ℓ, a total of r vertices that can be colored red. By Corollary 6.3, $2r \le \mathrm{null}(\mathrm{AD}_r)$. By Theorem 6.2 and Proposition 6.4, $2r \le \mathrm{null}(A(\mathrm{AD}_r)) \le \mathrm{M}(\mathrm{AD}_r) \le \mathrm{Z}(\mathrm{AD}_r) \le 2r$. □

Proposition 6.6. Let n be a multiple of 8. Then,

$$\mathrm{Z}(\mathrm{Circ}[n, \{1, \tfrac{n}{2}-1\}]) \le \tfrac{n}{2} + 2.$$

Proof. Let $G = \mathrm{Circ}[n, \{1, \frac{n}{2}-1\}]$. Then $Z = \{0, 1, 2, \dots, \frac{n}{2}, n-1\}$ is a zero forcing set with forces $0 \to n/2+1, 1 \to n/2+2, \cdots, n/2-3 \to n-2$. This shows that $\mathrm{Z}(G) \le \frac{n}{2} + 2$ □

Theorem 6.7. *Let n be a multiple of* 8. *Then,*

$$\mathrm{M}(\mathrm{Circ}[n, \{1, \tfrac{n}{2}-1\}]) = \mathrm{Z}(\mathrm{Circ}[n, \{1, \tfrac{n}{2}-1\}]) = \tfrac{n}{2} + 2$$

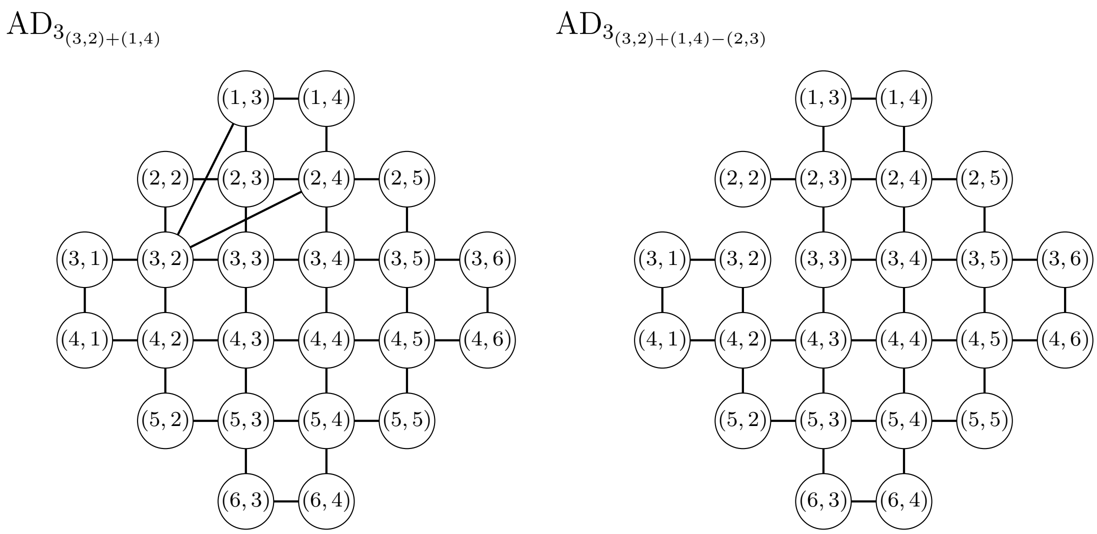


Figure 6.2: Coloring $(4,1)$ red with $((3,2),\{(1,4)\},\{(2,3)\},0)$ in the Aztec diamond graph AD_3.

and field independent minimum rank is established with the universally optimal matrix $A(G)$.

Proof. First note that G is bipartite with partite set $\mathbf{B} = \{2k \,|\, 0 \le k \le \frac{n}{2}-1\}$ and $\bar{\mathbf{B}} = \{2k+1 \,|\, 0 \le k \le \frac{n}{2}-1\}$. We show that $\frac{n}{4}+1$ vertices from $\mathbf{B}$ can be colored red using only white vertices of $\mathbf{B}$. Note that for every vertex v in $\{0,1,2,\dots,\frac{n}{2}-1\}$, v is adjacent to $v+1, v-1, v+\frac{n}{2}-1, v+\frac{n}{2}+1$, and $v+\frac{n}{2}$ is adjacent $v+\frac{n}{2}+1, v+\frac{n}{2}-1, v+\frac{n}{2}+\frac{n}{2}-1 \equiv v-1 \bmod n, v+\frac{n}{2}+\frac{n}{2}+1 \equiv v+1 \bmod n$. Hence, $N_G(v) = N_G(v+\frac{n}{2})$ and v can be colored red by $(v+\frac{n}{2},\emptyset,\emptyset,0)$ where $v \in \{0,2,4,\dots,\frac{n}{2}-2\}$. This shows that $\frac{n}{4}$ vertices from $\mathbf{B}$ can be colored red. The vertex $\frac{n}{2}$ can be colored red by $(\frac{n}{2}+2, \{2i \,:\, 2|i \text{ and } \frac{n}{2}+2 < 2i \le n-1\}, \{2i \,:\, 2\nmid i \text{ and } \frac{n}{2}+2 < 2i \le n-1\}, 0)$. Hence, the vertices of $\{0,2,4,\dots,\frac{n}{2}\}$ can be colored red with the vertices $\mathbf{B} \setminus \{0,2,4,\dots,\frac{n}{2}\}$. By Corollary 6.3, $2(\frac{n}{4}+1) = \frac{n}{2}+2 \le \mathrm{null}(G)$. So by Theorem 6.2 and by Proposition 6.6

$$\frac{n}{2}+2 \le \mathrm{null}(A(G)) \le \mathrm{M}(G) \le \mathrm{Z}(G) \le \frac{n}{2}+2.$$

$\square$

APPENDIX A. SageMath Code

```
def extended_cube_graph(t,k):
    """
    Returns an extended cube graph
    :t: a nonnegative integer
    :k: a nonnegative integer
    :returns: extended cube graph
    """
    n = 2*(t+k)+8
    g = graphs.CycleGraph(n)
    v1=[i+0 for i in range(k+2)]
    v2=[i+k+2 for i in range(t+2)]
    v3=[i+k+t+4 for i in range(k+2)]
    v4=[i+2*k+t+6 for i in range(t+2)]
    horizontal_edges = [(v2[i],v4[-i-1]) for i in range(t+2)]
    vertical_edges = [(v1[i],v3[-i-1]) for i in range(k+2)]
    g.add_edges(horizontal_edges)
    g.add_edges(vertical_edges)
    return g

def list_matrices_Z_2(g,Z,p=2):
    """
    :g: A graph.
```

```
:Z: The zero forcing number of the given graph.
:returns: A set of tuples with entries corresponding to nonzero diagonal
entries of a potential universally optimal matrices.
:Example:
sage: g=graphs.CycleGraph(7)
....:
sage: list_matrices_Z_2(g,2)
....:
The graph has 21 matrices that achieve minimum rank of 5 over Z_2
{(0, 1, 2),
 (0, 1, 2, 3, 5),
 (0, 1, 2, 4, 6),
 (0, 1, 3, 5, 6),
 (0, 1, 4),
 (0, 1, 6),
 (0, 2, 3, 4, 5),
 (0, 2, 4, 5, 6),
 (0, 3, 4),
 (0, 3, 6),
 (0, 5, 6),
 (1, 2, 3),
 (1, 2, 3, 4, 6),
 (1, 2, 5),
 (1, 3, 4, 5, 6),
 (1, 4, 5),
 (2, 3, 4),
 (2, 3, 6),
```

```
 (2, 5, 6),
 (3, 4, 5),
 (4, 5, 6)}
"""
import itertools
n = g.order()
integer_set = [i for i in range(n)]
nonzero_set = set([ ])
for L in range(n+1):
    for subset in itertools.combinations(integer_set, L):
           A=g.am()
           for i in range(L):
               j = subset[i]
               A[j,j]=1
           A=matrix(GF(p),A)
           if A.rank() == n-Z:
               nonzero_set.add(subset)
print("The graph has {} matrices that achieve minimum rank"
" of {} over Z_{}".format(len(nonzero_set),n-Z,p))
return nonzero_set
```

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