On exponential domination of graphs

by

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The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

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DEDICATION

Graduate school is an emotional roller coaster of highs and lows. I would like to take this opportunity to dedicate this work to the people who supported me during my graduate experience: To my parents, Greg and Ellen, for their constant support of my endeavors and for being the best role models I could ever ask for. To my siblings, Gregory and Jenny, for setting the standard of Dairyko excellence high. To all my friends, old and new, who have shaped me to be the person that I am, have given me a shoulder to lean on, an ear to listen with, and a smile to laugh to. To the Posse Foundation and Pomona Posse 5, who started my way down this path and gave me the drive to become an agent of change.

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ABSTRACT

Exponential domination in graphs evaluates the influence that a particular vertex exerts on the remaining vertices within a graph. The amount of influence a vertex exerts is measured through an exponential decay formula with a growth factor of one-half. An exponential dominating set consists of vertices who have significant influence on other vertices. In non-porous exponential domination, vertices in an exponential domination set block the influence of each other. Whereas in porous exponential domination, the influence of exponential dominating vertices are not blocked. For a graph G, the non-porous and porous exponential domination numbers, denoted $\gamma_e(G)$ and $\gamma_e^*(G)$, are defined to be the cardinality of the minimum non-porous exponential dominating set and cardinality of the minimum porous exponential dominating set, respectively. This dissertation focuses on determining lower and upper bounds of the non-porous and porous exponential domination number of the King grid $\mathcal{K}_n$, Slant grid $\mathcal{S}_n$, n-dimensional hypercube Q_n, and the general consecutive circulant graph $C_{n,[\ell]}$.

A method to determine the lower bound of the non-porous exponential domination number for any graph is derived. In particular, a lower bound for $\gamma_e^*(Q_n)$ is found. An upper bound for $\gamma_e^*(Q_n)$ is established through exploiting distance properties of Q_n. For any grid graph G, linear programming can be incorporated with the lower bound method to determine a general lower bound for $\gamma_e^*(G)$. Applying this technique to the grid graphs $\mathcal{K}_n$ and $\mathcal{S}_n$ yields lower bounds for $\gamma_e^*(\mathcal{K}_n)$ and $\gamma_e^*(\mathcal{S}_n)$. Upper bound constructions for $\gamma_e^*(\mathcal{K}_n)$ and $\gamma_e^*(\mathcal{S}_n)$ are also derived. Finally, it is shown that $\gamma_e(C_{n,[\ell]}) = \gamma_e^*(C_{n,[\ell]})$.

CHAPTER 1. INTRODUCTION

A graph is a representation of a collection of interconnected objects. We denote these objects as *vertices*, and their corresponding relations as *edges.* For instance, a family tree is an example of a graph. The family members are the vertices, while the child-to-parent relationship corresponds to the edges. Another example of a graph is a social network, where an individual online represents a vertex and a connection, such as friendship, is represented by an edge.

1.1 Notation and Definitions

In this section, we provide basic notation and definitions used throughout the dissertation. The majority of the definitions and notations stated are based on Diestel [11]. All graphs are simple and undirected. A graph is an ordered pair $G = (V(G), E(G))$ that consists of a set $V(G)$ of vertices and a set $E(G)$ of edges, where an edge $e = \{u, v\}$ is the two element subset of vertices. The edge $\{u, v\}$ is often denoted by uv. The *order* of the graph G is the cardinality of $V(G)$, and is customarily denoted by n. Two vertices u, v are adjacent, denoted $u \sim v$, if $uv \in E(G)$. The *closed neighborhood* of a vertex v, denoted $N[v]$, is the set of vertices that are adjacent to v, together with v itself. An edge e is *incident* to a vertex v if $v \in e$. The *degree* of a vertex v is the number of edges that are incident to v. The *maximum* degree is denoted as $\Delta(G)$ and the *minimum* degree is denoted as $\delta(G)$. A graph G is *regular* if $\delta(G) = \Delta(G)$ and *subcubic* if $\Delta(G) \leq 3$. The graph H is considered to be a *subgraph* of the graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph H of G is an *induced* subgraph of G if for $u, v \in V(H)$, $u \sim v$ in H if and only if $u \sim v$ in G. Two graphs G and H are *isomorphic*, denoted $G \cong H$, if there exists a bijection $f : V(G) \to V(H)$ such that $u \sim v$ in G if and only if $f(u) \sim f(v)$ in H.

A *complete* graph on n vertices, denoted K_n, is a graph for which every two vertices are adjacent. If a graph G can be decomposed into two disjoint sets $A, B \subset V(G)$ such that no two vertices within

A or B are adjacent, then G is a *bipartite* graph. A *path* of length n, denoted P_n, is a graph whose vertices can be listed as $v_1, v_2, \ldots, v_n$ for which $v_i v_{i+1}$ is an edge for $1 \leq i \leq n-1$. A *cycle* of length n, denoted C_n, is a path graph on n vertices with the additional edge $v_1 v_n$.

A *connected* graph has the property that there exists a path between any two vertices. An *endvertex* is a vertex with degree at most 1. A *tree* is a graph such that any two vertices are connected by exactly one path. A *spanning* tree for the graph G is a tree containing all the vertices of G. A *rooted* tree has a unique vertex called the root. Let T be a rooted tree and consider $v \in V(T)$. A *descendant* of v is any vertex whose path from the root contains v. The *subtree* of T that is rooted in v, denoted T_v, is the subgraph of T that contains the descendants of v and v itself. A *parent* of v is the vertex adjacent to v in the path to the root. A *child* of v is any vertex that has v as a parent. The *depth* of T is the length of the longest path from the root to any vertex. Let $d_0, d_1, \ldots, d_n$ be nonnegative integers. Then $T(d_0, d_1, \ldots, d_n)$ is the rooted tree of depth $n+1$ for which every vertex that is distance k from the root has exactly d_k children for every $0 \leq k \leq n$.

The *n-dimensional hypercube* graph, denoted Q_n, is constructed by creating a vertex for each n-digit binary word. Edges are formed if two vertices differ by one digit in their binary representation. Let $[n] = \{1, 2, \ldots, n\}$. The *consecutive circulant graph*, denoted $C_{n,[\ell]}$, has the set of $[n]$ vertices and vertex v is adjacent to vertex $v \pm i \mod n$ for each $i \in [\ell]$.

For the two sets A and B, the Cartesian product of A and B is defined to be $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. The Cartesian product of two graphs G and H, denoted $G \square H$, is a graph such that $V(G \square H) = V(G) \times V(H)$ and two vertices $(g, h) \sim (g', h')$ in $G \square H$ if and only if either $g = g'$ and $h \sim h'$ in H, or $h = h'$ and $g \sim g'$ in G. Let $G_{m,n} = P_m \square P_n$ be the *standard grid.* A *grid graph* is the standard grid with possibly additional edges added in a regular pattern. The *torus* is defined to be the graph $C_m \square C_n$, where $m \leq n$. The *strong product* of two graphs G and H is the graph $G \boxtimes H$ for which $V(G \boxtimes H) = V(G) \times V(H)$ and two distinct vertices are adjacent whenever in both coordinate places the vertices are adjacent or equal in the corresponding graph. The *King grid* is defined as $\mathcal{K}_n = P_n \boxtimes P_n$. An alternate definition for K_n is in terms of chess, where vertices are represented by the squares on the chessboard and edges are the potential movements of a king chess piece in a single turn. Consider the paths P_n and P_m with vertex sets $[n]$ and $[m]$, respectively. Then the *Slant* grid is defined to be $\mathcal{S}_n = P_n \square P_m$ with the additional edges $\{i, j\} \sim \{i+1, j+1\}$, for $i \in [n-1]$ and $j \in [m-1]$. Notice that $C_m \square C_n$, $\mathcal{K}_n$, and $\mathcal{S}_n$ are all instances of grid graphs.

Let $\operatorname{dist}(u, v)$ denote the length of the shortest path from vertex u to vertex v. Consider $u \in D \subseteq V(G)$ and $v \in V(G)$. The *diameter* of G is defined as $\operatorname{diam}(G) = \max_{u,v \in V(G)} \operatorname{dist}(u, v)$. Let $S_k(v) = \{u \in V(G) : \operatorname{dist}(u, v) = k\}$ denote the sphere of radius k and let $D_k(u) = \{d \in D : \operatorname{dist}(u, d) \leq k\}$ denote the ball of radius k. The annulus with radii r and R is defined to be $A_{r,R}(u) = \{v \in V(G) : r \leq \operatorname{dist}(v, u) \leq R\}$. Let $\overline{\operatorname{dist}}(u, v)$ be the length of the shortest path from vertex u to vertex v that contains no internal vertices of D.

Define $w : V(G) \times V(G) \to \mathbb{R}$ to be a *weight function* of G. For $u, v \in V(G)$, we say that u assigns weight $w(u, v)$ to v. Denote the weight assigned $D \subseteq V(G)$ to v as $w(D, v) := \sum_{u \in D} w(u, v)$, and similarly, the weight assigned by $u \in D$ to $H \subseteq V(G)$ as $w(u, H) := \sum_{h \in H} w(u, h)$.

1.2 Background of the Problem

Domination in graphs is used to study situations that arise when a particular vertex exerts influence on its neighboring vertices. The original domination problem is known as *classical domination*. Consider a graph G and a set $D \subseteq V(G)$. With respect to classical domination, D is a *classical dominating set* if every vertex contained in $V(G) \setminus D$ is adjacent to at least one vertex of D. For $d \in D$ and $v \in V(G)$, the corresponding weight function to classical domination is

$$w(d, v) = \begin{cases} 1 & \text{if } v \in N[d] \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.2.1. Consider the graph $P_2 \square P_5$ with weight function w. Let $V(P_2 \square P_5) = \{a, b, c, d, e, f, g, h, i, j\}$, as shown in Figure 1.1. Notice that $N[c] = \{b, c, d, h\}$, $N[f] = \{a, f, g\}$, and $N[j] = \{e, i, j\}$. This shows that $w(\{c, f, j\}, v) = 1$ for every $v \in V(P_2 \square P_5)$. Therefore the set of filled vertices, $\{c, f, j\}$, form a classical domination set for $P_2 \square P_5$. Notice that $\{c, f, j\}$ is minimum.

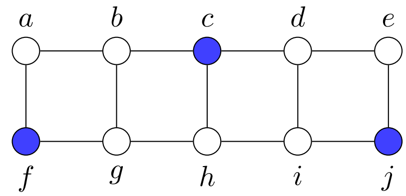

Figure 1.1 Filled vertices form a classical domination set for the graph $P_2 \Box P_5$

It was mentioned in Hedetniemi and Laskar [15] that the roots of domination in graphs can be traced back hundreds of years ago to when chess was developed in India. Chess is a two player game of strategy that is played on an 8×8 board consisting of alternating colored squares. At the start of each game, each player has 16 pieces; one king, one queen, two rooks, two bishops, two knights, and eight pawns. The queen is considered to be the most powerful piece in chess and, excluding any obstructions, can move any number of squares in a row in the diagonal, vertical, or horizontal direction. See Figure 1.2 for an illustration of the potential moves of the marked queen on a chessboard, signified by 'X'. According to Haynes et al. [13], the origin of domination occurred in the 1850's in Europe. Here chess enthusiasts studied problems related to how sets of various chess pieces could *dominate*, or cover, the squares of a chessboard.

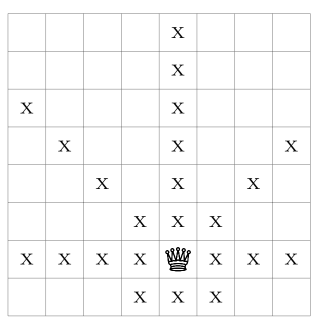

Figure 1.2 Potential moves of a queen chess piece

We briefly discuss on problems and questions related to the queen chess piece. Note that related problems for other chess pieces on a chessboard and other board games are discussed in Rouse [2]. In particular consider the following problem, named the *Five Queens Problem:* what is the smallest number of queen chess pieces needed to ensure that every square on a standard chessboard can be reached by a queen? In [2], it was mentioned that the solution to the Five Queens Problem was five, and Figure 1.3 shows one such solution.

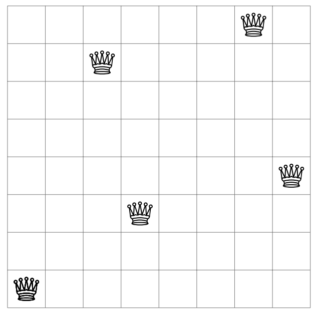

Figure 1.3 Solution to the Five Queens Problem

In the mid 1900's the books, *Theory of Graphs and its Applications* [3] and *Theory of Graphs* [19], were published. These two books clearly defined domination in graphs and created a solid foundation of theory to build upon. In 1975 a survey paper on domination titled, *Towards a theory of domination in graphs* [7], was published. A comprehensive bibliography [14] of over 300 citations was created in 1988 to track all the results in the area. The authors of [7] were credited in [14] for the desire to 'get the ball rolling' within the area of domination and inciting a large interest amongst mathematicians to study domination problems. It was highlighted in [14] that the numerous real world problems that domination modeled and the various parameters that were constructed from domination helped to the increase the popularity of this area.

1.3 Review of Literature

1.3.1 Classical Domination

In this section, selected results from classical domination are discussed. One of the first basic results within the area of domination came from Ore [19] in the following theorem.

Theorem 1.3.1. [19] *Any graph G with $\delta(G) \geq 1$ has a dominating set D such that its complement $\overline{D}$ is also a dominating set.*

Notice that the following is an immediate corollary to Theorem 1.3.1.

Corollary 1.3.2. [19] *Let G be a graph of order n with $\delta(G) \geq 1$, then $\gamma(G) \leq \frac{n}{2}$.*

With an additional restriction on the minimum degree of a graph, Reed [20] presented another upper bound to $\gamma(G)$.

Theorem 1.3.3. [20] *Every graph on n vertices with $\delta(G) = 3$ has a dominating set of size at most $\frac{3n}{8}$.*

The following conjecture was then posed.

Conjecture 1.3.4. [20] *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma(G) \leq \left\lceil \frac{n}{3} \right\rceil$.*

However, Kostochka and Stodolsky constructed a family of graphs showing Conjecture 1.3.4 is false in [18].

Theorem 1.3.5. [18] *There is a sequence $\{G_k\}_{k=1}^{\infty}$ of connected graphs such that for every k, $\delta(G_k) = 3$, $|V(G_k)| = 46k$, $\gamma(G_k) \geq 16k$, and thus*

$$\lim_{k\to\infty} \frac{\gamma(G_k)}{|V(G_k)|} \geq \frac{16}{46} = \frac{1}{3} + \frac{1}{69}.$$

1.3.2 Variants of Domination

There are many variants of classical domination for a graph G. For instance, consider k-*domination.* As described in [13], a set $D \subseteq V(G)$ is a k*-dominating set* if every vertex in $V(G) \setminus D$ is adjacent to at least k members of D. Then for $d \in D$ and $v \in V(G)$, the corresponding weight function for k-domination is as follows:

$$w_k(d, v) = \begin{cases} \frac{1}{k} & \text{if } v \sim d \\ 1 & \text{if } d = v \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.3.6. Consider the graph $P_2 \square P_5$ with the weight function w_2. Let $V(P_2 \square P_5) = \{a, b, \ldots, j\}$, as shown in Figure 1.4. Observe that $S_1(a) = \{b, f\}, S_1(c) = \{b, d, h\}, S_1(e) =$

$\{d,j\}, S_1(g) = \{b,f,h\}$, and $S_1(i) = \{d,h,j\}$. Therefore $w_2(\{b,d,f,h,j\},v) \geq 1$ for all $v \in V(P_2 \Box P_5)$. It follows that the set of filled vertices, $\{b,d,f,h,j\}$, form a 2-dominating set for $P_2 \Box P_5$. Note that $\{b,d,f,h,j\}$, is minimum.

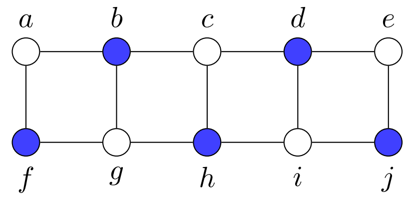

Figure 1.4 Filled vertices form a 2-dominating set for the graph $P_2 \Box P_5$

A particular variant of classical domination, called distance domination, has a framework that models real world situations in which the influence of a vertex extends beyond its immediate neighborhood. As described in [13], a set $D \subseteq V(G)$ is a *distance-k dominating set* if the distance between each vertex $v \in V(G) \setminus D$ and at least one member of D is at most k. Then for $d \in D$ and $v \in V(G)$, the corresponding weight function for distance domination is as follows:

$$w_{\leq k}(d,v) = \begin{cases} 1 & \text{if } \operatorname{dist}(d,v) \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Example 1.3.7. Consider the graph $P_3 \Box P_4$ with weight function $w_{\leq 2}$. Let $V(P_3 \Box P_4) = \{a,b,\ldots,l\}$, as shown in Figure 1.5. Notice that $S_1(i) = \{e,j\}$, $S_2(i) = \{a,f,k\}$, $S_1(d) = \{c,h\}$, and $S_2(d) = \{b,g,l\}$. Therefore $w_{\leq 2}(\{d,i\},v) = 1$ for all $v \in V(P_3 \Box P_4)$. This shows that the set of filled vertices, $\{d,i\}$, form a distance-2 dominating set for $P_3 \Box P_4$. Note that $\{d,i\}$ is minimum.

However, the instance in which influence decays as the distance increases had no such study prior to 2009. This motivated Dankelmann et al. [10] to create a new variant of classical domination, called *exponential domination.* Generally speaking, consider the set of vertices D that exerts influence on its surrounding vertices that decays exponentially by a factor of one-half. Any vertex that is influenced substantially, meaning receives weight at least one, by members of D is considered

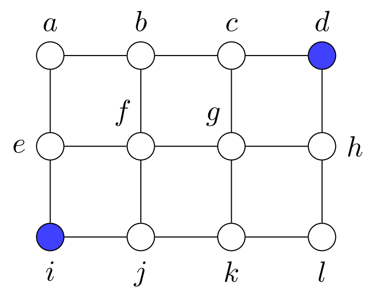

Figure 1.5 Filled vertices form a distance-2 dominating set for the graph $P_3 \Box P_4$

to be exponentially dominated. In the discussion of [10], it was said that exponential domination has real world applications. In particular, it models the dissemination of information in social networks where the information's influence decays exponentially with each share.

There are two types of exponential domination; porous and non-porous. A *porous exponential dominating set* is a set $D \subseteq V(G)$ such that $w_e^*(D,v) \geq 1$ for every $v \in V(G)$, where the weight function w_e^* is given by

$$w_e^*(u,v) = \left(\frac{1}{2}\right)^{\mathrm{dist}(u,v)-1}.$$

The *porous exponential domination number,* $\gamma_e^*(G)$, equals the cardinality of the smallest porous exponential dominating set.

Example 1.3.8. Consider the graph $P_3 \Box P_4$ with weight function w_e^*. Let $V(P_3 \Box P_4) = \{a, b, \ldots, l\}$, as shown in Figure 1.6. Notice that

$$\begin{aligned}
&w_e^*(\{a,g,l\},b) = \tfrac{13}{8}, \quad w_e^*(\{a,g,l\},c) = \tfrac{7}{4}, \quad w_e^*(\{a,g,l\},d) = \tfrac{5}{4}, \\
&w_e^*(\{a,g,l\},e) = \tfrac{13}{8}, \quad w_e^*(\{a,g,l\},f) = \tfrac{7}{4}, \quad w_e^*(\{a,g,l\},h) = \tfrac{17}{8}, \\
&w_e^*(\{a,g,l\},i) = 1, \quad w_e^*(\{a,g,l\},j) = \tfrac{5}{4}, \quad w_e^*(\{a,g,l\},k) = \tfrac{17}{8}.
\end{aligned}$$

As $w_e^*(\{a,g,l\},v) \geq 1$ for every $v \in V(P_3 \Box P_4)$, the set of filled vertices, $\{a,g,l\}$, form a porous exponential dominating set for $P_3 \Box P_4$. Note that $\{a,g,l\}$ is minimum.

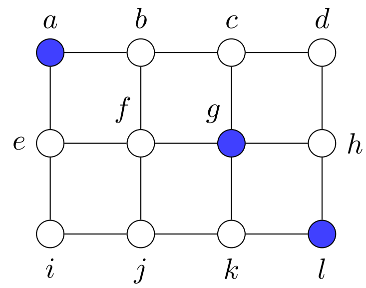

Figure 1.6 Filled vertices form a porous exponential dominating set for the graph $P_3 \Box P_4$

A *non-porous exponential dominating set* is a set $D \subseteq V(G)$ such that $w_e(D, v) \geq 1$ for every $v \in V(G)$, where the weight function

$$w_e(u, v) = \left(\frac{1}{2}\right)^{\overline{\text{dist}}(u,v)-1}.$$

The *non-porous exponential domination number,* $\gamma_e(G)$, represents the cardinality of the smallest non-porous exponential dominating set. Observe that $\{a, g, l\}$ from Example 1.3.8 also forms a non-porous exponential domination set.

Example 1.3.9. *Consider the graph* $T(3, 2, 2)$ *with weight function* w_e. *Let* $V(T(3, 2, 2)) = \{a, b, \ldots, v\}$, *as shown in Figure 1.7. Observe that*

$$\begin{array}{ccc}
w_e(\{e,f,g,h,i,j\},a) = 3 & w_e(\{e,f,g,h,i,j\},b) = 3 & w_e(\{e,f,g,h,i,j\},c) = 3 \\
w_e(\{e,f,g,h,i,j\},d) = 3 & w_e(\{e,f,g,h,i,j\},k) = 1 & w_e(\{e,f,g,h,i,j\},l) = 1 \\
w_e(\{e,f,g,h,i,j\},m) = 1 & w_e(\{e,f,g,h,i,j\},n) = 1 & w_e(\{e,f,g,h,i,j\},o) = 1 \\
w_e(\{e,f,g,h,i,j\},p) = 1 & w_e(\{e,f,g,h,i,j\},q) = 1 & w_e(\{e,f,g,h,i,j\},r) = 1 \\
w_e(\{e,f,g,h,i,j\},s) = 1 & w_e(\{e,f,g,h,i,j\},t) = 1 & w_e(\{e,f,g,h,i,j\},u) = 1
\end{array}$$

$$w_e(\{e,f,g,h,i,j\},v) = 1.$$

We have now shown that $w_e(\{e, f, g, h, i, j\}, \alpha) \geq 1$ *for every* $\alpha \in V(T(3, 2, 2))$. *Thus the set of filled vertices,* $\{e, f, g, h, i, j\}$, *form a non-porous exponential dominating set for* $T(3, 2, 2)$.

Note that $\{a, b, c, d\}$ from Example 1.3.9 forms a porous exponential domination set for for $T(3, 2, 2)$.

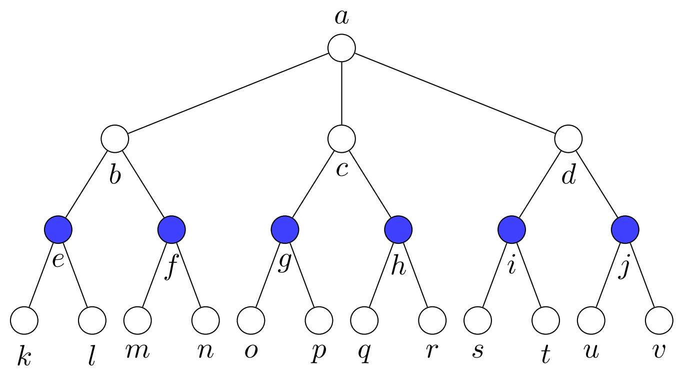

Figure 1.7 Filled vertices form a non-porous exponential dominating set for the graph $T(3, 2, 2)$.

1.3.3 Exponential Domination

The other variants of classical domination rely solely on the local influence of their respective dominating vertices to dominate neighboring vertices. Exponential domination is the only such framework in which the influence of an exponential dominating vertex is global with respect to other vertices. There are limited results on non-porous exponential domination and even less for porous exponential domination. As mentioned in Henning et al. [17], the fact that exponential domination is the only global variant of domination, and the difficulty in studying such a concept, may be a possible explanation to the lack of results in the area.

We now discuss notable findings in exponential domination. Immediate results from the definition of exponential domination noted in [10] are that for a graph G, $\gamma_e(G) = 1$ if and only if $\gamma(G) = 1$ and

$$\gamma_e^*(G) \leq \gamma_e(G) \leq \gamma(G). \tag{1.3.1}$$

For the remainder of the dissertation, we focus only on porous and non-porous exponential domination. For the sake of simplicity, we use the weight function w^* to represent w_e^*, the weight function for porous exponential domination and w to represent w_e, the weight function for non-porous exponential domination. Elementary results for P_n, the path on n vertices and C_n, the cycle on n vertices, are stated in the following lemma and proposition.

Lemma 1.3.10. [10] *For every integer* n,

$$\gamma_e^*(P_n) = \gamma_e(P_n) = \left\lceil \frac{n+1}{4} \right\rceil.$$

Proposition 1.3.11. [10] *For every integer* $n \geq 3$,

$$\gamma_e(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ \lceil \frac{n}{4} \rceil & \text{if } n \neq 4. \end{cases}$$

The results discussed in [10] focused mainly on non-porous exponential domination. Here porous exponential domination was used to determine a lower bound on the non-porous exponential domination number. The general upper and lower bounds for the non-porous exponential domination number are given in the following theorem.

Theorem 1.3.12. [10] *If G is a connected graph of order n and diameter* $\operatorname{diam}(G)$, *then*

$$\left\lceil \frac{\operatorname{diam}(G)+2}{4} \right\rceil \leq \gamma_e(G) \leq \frac{2}{5}(n+2)$$

The proof of Theorem 1.3.12 is split into two parts, one determining the lower bound and the other establishing the upper bound. Note that in each part, the global nature of exponential domination is localized. The lower bound is shown via contradiction. Through the application of several minor lemmas, it is shown that $\gamma_e(G) < \gamma_e^*(G)$, which contradicts (1.3.1). The proof of the upper bound utilizes the fact that T, the spanning tree of G, has the property that $\gamma_e(G) \leq \gamma_e(T)$. Through a detailed case analysis, it is shown that the upper bound holds. There is additional discussion on the sharpness of the bounds determined in Theorem 1.3.12. Observe that Lemma 1.3.10 shows that the lower bound is sharp. Through brute force, it was verified that there is no tree T of order $n \leq 10$ with the property that $\gamma_e(T) = \frac{2(n+2)}{5}$. However, brute force could not be applied for trees of order $n > 10$. Through the use of computers, [10] searched to find trees so that the value $\frac{\gamma_e(T)}{n+2}$ is maximized. The best result for the upper bound of the non-porous exponential domination number of trees occurred via the construction of an infinite family of trees $\mathcal{T}$ such that $\lim_{n\to\infty} \frac{\gamma_e(T)}{n+2} = \frac{144}{379} \approx 0.380$, for $T \in \mathcal{T}$, with n denoting the order of T. Outside of infinite families, it was shown that the tree $T_0 = T(2,3,3,4,3,4,2,1)$ of order $n = 375$ has $\gamma_e(T_0) = 144$, so $\frac{\gamma_e(T_0)}{n+2} = \frac{144}{377} \approx 0.382$. Therefore the upper bound for Theorem 1.3.12 is not known to be sharp. In the concluding remarks, [10] posed the following two open questions:

1. Let T be a tree. Is there a polynomial-time algorithm to determine $\gamma_e(T)$.

2. Under what conditions is $\gamma_e(G) = \gamma(G)$?

The papers Bessy et al. [4], Henning et al. [16] and [17] address the open questions posed in [10]. We now summarize the main results from these papers. First, non-porous exponential domination in subcubic graphs was studied [4]. Here the authors were able to manipulate properties of subcubic graphs that somewhat localized exponential domination. This resulted in an upper bound for the weight that a vertex receives from the exponential dominating set, which simplified determining the non-porous exponential domination number of subcubic graphs. The following theorem is the best result for the lower and upper bounds of the non-porous exponential domination number of any subcubic graph G, and it is shown that the upper bound is tight.

Theorem 1.3.13. [4] *If G is a connected subcubic graph of order n, then*

$$\frac{n}{6\log_2(n+2)+4} \leq \gamma_e(G) \leq \frac{n+2}{3}.$$

Similarly as for the upper bound in Theorem 1.3.12, [4] used that the spanning tree H of G has the property that $\gamma_e(G) \leq \gamma_e(H)$. The upper bound was shown for all trees through the use of induction on the order the tree T. Putting it all together gave that the upper bound held for all graphs. The lower bound in Theorem 1.3.13 was shown to be true through the use of a clever counting argument that manipulated the weight a particular vertex received from the non-porous exponential dominating set, along with the use of an auxiliary theorem.

In the remainder of [4], there is a focus on the complexity of the non-porous exponential domination number of subcubic trees.

Theorem 1.3.14. [4] *Given a subcubic tree T, $\gamma_e(T)$ can be determined in polynomial time.*

Notice that Theorem 1.3.14 shows that there exists a polynomial time algorithm that computes the non-porous exponential domination number of a subcubic tree, and answers the first open question posed in [10] for subcubic trees. Further, [4] establishes that determining a minimum non-porous exponential dominating set of a given subcubic graph is APX-hard. However no such algorithm is known for general trees, nor for the porous exponential domination number of subcubic trees.

In [16], the second open problem from [10] is addressed. The hereditary class $\mathcal{G}$ is the set of graphs G for which $\gamma_e(H) = \gamma(H)$ for every induced subgraph H of G [16]. Through the use of minimal forbidden induced subgraphs, the authors characterize a large subclass of $\mathcal{G}$. Theorem 1.3.15, Corollary 1.3.16, and Corollary 1.3.17 use the graphs depicted in Figure 1.8.

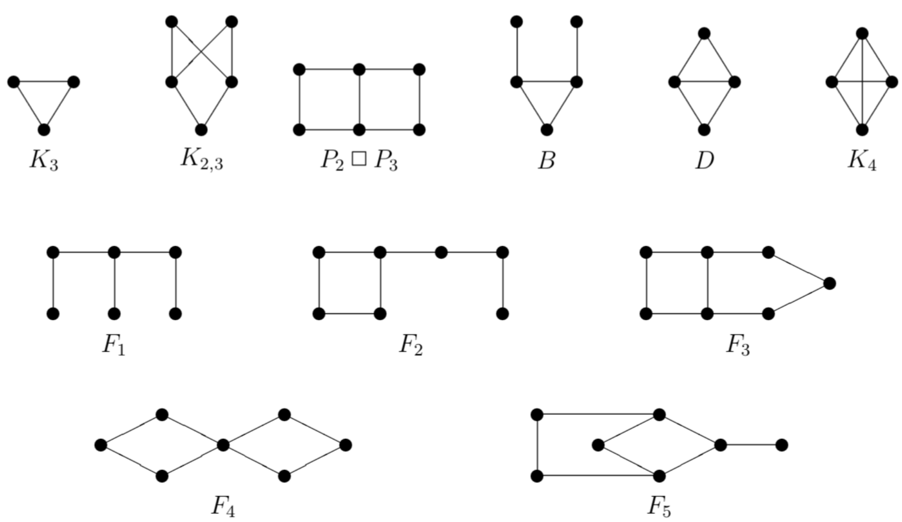

Figure 1.8 The graphs K_3, $K_{2,3}$, $P_2\square P_3$, B, D, K_4, and $F_1, \ldots, F_5$ from [16]

Theorem 1.3.15. [16] *If G is a $\{B, D, K_4, K_{2,3}, P_2\square P_3\}$-free graph, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\{P_7, C_7, F_1, \ldots, F_5\}$-free.*

The challenging proof details of Theorem 1.3.15 are omitted as they are not directly related to the focus of this dissertation. As K_3 is an induced subgraph of the graphs B, D, and K_4, consider the following corollary.

Corollary 1.3.16. [16] *If G is a $\{K_3, K_{2,3}, P_2 \square P_3\}$-free graph, then $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\{P_7, C_7, F_1, \ldots, F_5\}$-free.*

A complete characterization for trees contained in $\mathcal{G}$ is then given in the following corollary.

Corollary 1.3.17. [16] *If T is a tree, then $\gamma(F) = \gamma_e(F)$ for every induced subgraph F of T if and only if T is $\{P_7, F_1\}$-free.*

For a general graph G, there still is no efficient algorithm to determine if $\gamma(G) = \gamma_e(G)$. Two conjectures are posed in [16] that give further insight to this open problem.

Conjecture 1.3.18. [16] *There is a finite set $\mathcal{F}$ of graphs such that a graph G satisfies $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if G is $\mathcal{F}$-free.*

Conjecture 1.3.19. [16] *A graph G satisfies $\gamma(H) = \gamma_e(H)$ for every induced subgraph H of G if and only if $\gamma(H) = \gamma_e^*(H)$ for every induced subgraph H of G.*

The focus of [17] is to relate the parameters of classical domination with exponential domination. In particular, they give results that fill the gaps in (1.3.1), with an emphasis on subcubic graphs and extend results from [4]. The technique of *linear programming* was used to help determine lower bounds on the exponential domination number of subcubic trees.

Linear programing is a tool used in optimization that takes a set of linear inequalities, or constraints, and outputs the optimal solution of the linear objective function. Notice that a linear inequality creates a half space, and the finite intersection of half spaces forms a convex polytope. Therefore a linear program searches for a point within the polytope that optimizes the linear objective function. If such a point exists, then there is a solution, otherwise there is no feasible solution. An integer program is a linear program, with the restriction that the variables can only be assigned integer values. In essence, finding the minimum exponential dominating number is an optimization problem because the aim is to find the fewest exponential dominating vertices that exponentially dominate the remaining vertices.

The following integer program is formulated for a graph so that the optimum value is $\gamma_e^*(G)$.

Integer Program 1.3.20. [17]

$$\begin{aligned} \min \sum_{u \in V(G)} x(u) & & & \\ s.t. \sum_{u \in V(G)} \left(\frac{1}{2}\right)^{\mathrm{dist}(u,v)-1} x(u) & \geq & 1 & \quad \forall v \in V(G) \\ x(u) & \in & \{0,1\} & \quad \forall u \in V(G). \end{aligned}$$

A new exponential domination parameter called the *fractional porous exponential domination number,* denoted $\gamma_{e,f}^*(G)$, is introduced in [17]. The following linear program is a relaxation of Integer Program 1.3.20, and the corresponding optimum value is equivalent to $\gamma_{e,f}^*(G)$.

Linear Program 1.3.21. [17]

$$\begin{aligned} \min \sum_{u \in V(G)} x(u) & & & \\ s.t. \sum_{u \in V(G)} \left(\frac{1}{2}\right)^{\mathrm{dist}(u,v)-1} x(u) & \geq & 1 & \quad \forall v \in V(G) \\ x(u) & \geq & 0 & \quad \forall u \in V(G). \end{aligned}$$

A notable result using the concept of the fractional porous exponential domination number of a graph is summarized in the following theorem.

Theorem 1.3.22. [17] *If T is a subcubic tree of order n, then $\gamma_{e,f}^*(T) = \frac{n+2}{6}$.*

The value of $\gamma_{e,f}^*(T)$ determined in Theorem 1.3.22 is a consequence of applying Linear Program 1.3.21 to subcubic trees. The corresponding objective function is $\sum_{u \in V(T)} x(u)$, where

$$x(u) = \begin{cases} \frac{1}{3}, & \text{if } u \text{ is an endvertex of } T \\ \frac{1}{6}, & \text{if } u \text{ has degree 2 in } T \\ 0, & \text{if } u \text{ has degree 3 in } T. \end{cases}$$

The bounds of the non-porous exponential domination number of a connected graph from Theorem 1.3.12 were improved in Bessy et al. [5]. In particular, the upper bound was strengthened with the following theorem.

Theorem 1.3.23. [5] *If G is a connected graph of order n, then $\gamma_e(G) \leq \frac{43}{108}(n+2)$.*

The proof of Theorem 1.3.23 is similar to the proof of Theorem 1.3.12. Since $\frac{43}{108}$ is approximately $\frac{2}{5}$, we see that [5] did not drastically improve the bound. To sharpen the bound any further, a complex case analysis would be needed. It was also acknowledged in [5] that the process of localizing the global influence of exponential dominating vertices does not necessarily produce the best upper bound.

Although most results related to exponential domination have had a focus on subcubic graphs, there has been study on other graphs. The porous exponential domination number of $C_m \Box C_n$ was determined in Anderson et al. [1]. Figure 1.9 shows the tile T such that when the infinite torus $C_\infty \Box C_\infty$ is tiled with T, a porous exponential dominating set D is formed. For the sake of simplicity, T is depicted as a 13×13 chessboard, where the vertices are represented as squares; 'X' denotes the location of a member of D. Notice that there is exactly one member of D in every row and column of T. The construction led the authors to the following theorem, which determines the upper bound on the asymptotic density of $\gamma_e^*(C_m \Box C_n)$.

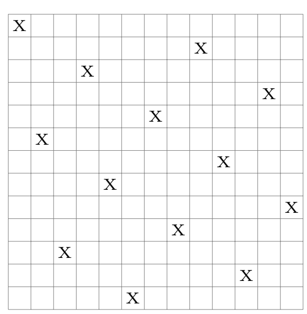

Figure 1.9 13×13 exponential dominating set tile for $C_\infty \Box C_\infty$

Theorem 1.3.24. [1] $\lim_{m,n\to\infty} \frac{\gamma_e^*(C_m \Box C_n)}{mn} \leq \frac{1}{13}$.

Through a naive counting argument, a lower bound on $\gamma_e^*(C_m \Box C_n)$ was established:

Theorem 1.3.25. [1] *For all* $m, n > 3$,

$$\frac{mn}{15.875} < \gamma_e^*(C_m \square C_n).$$

The results of Theorems 1.3.24 and 1.3.25 lead [1] to Conjecture 1.3.26. Observe that if this conjecture is proven to be true, then the exact value of $\gamma_e^*(C_m \square C_n)$ will be known.

Conjecture 1.3.26. *For all* m *and* n, $\frac{\gamma_e(C_m \square C_n)}{mn} \geq \frac{1}{13}$ *and this bound is sharp (take* $m = n = 13$*).*

In the unpublished work of Bozeman et al. [6], the lower bound for $\gamma_e^*(C_m \square C_n)$ determined in Theorem 1.3.25 was improved through the use of linear programming. Notice that Theorem 1.3.27 further supports Conjecture 1.3.26.

Theorem 1.3.27. [6] *For all* $m, n \geq 11$,

$$\frac{mn}{13.761891939197298} \leq \gamma_e^*(C_m \square C_n).$$

In conclusion, there are a number of results already known within the area of exponential domination. Table 1.1 contains a summary of the main results from articles on exponential domination. The entries in the table are given in ascending order with respect to the year published or posted online.

1.4 Organization of the Dissertation

This dissertation is written under the format of a collection of papers submitted to journals.

Chapter 1 presents definitions, discusses the history of exponential domination, and summarizes the literature within this area.

Chapter 2 contains the paper "A linear programming method for exponential domination" [9]. In this paper we first describe a technique to find a lower bound for the porous exponential domination number of any graph. A linear programming method is given to determine the lower bound of the porous exponential domination number of the King grid, denoted $\mathcal{K}_n$, and the Slant grid, denoted $\mathcal{S}_n$. Another method is applied to construct a lower bound to the porous exponential domination

Table 1.1 Highlighted results from literature on exponential domination

Reference	Result Number	Graph & Conditions	Result
[10]	Theorem 1.3.12	G connected	$\frac{1}{4}(\operatorname{diam}(G)+2) \leq \gamma_e(G) \leq \frac{2}{5}(n+2)$
[1]	Theorem 1.3.25	$C_m \square C_n$ $m, n \geq 3$	$\frac{mn}{15.875} < \gamma_e^*(C_m \square C_n)$
[4]	Theorem 1.3.13	G, connected, $\Delta(G) \leq 3$	$\frac{n}{6\log_2(n+2)+4} \leq \gamma_e(G) \leq \frac{1}{3}(n+2)$
[6]	Theorem 1.3.27	$C_m \square C_n$, $m, n \geq 11$	$\frac{mn}{13.761891939197298} \leq \gamma_e^*(C_m \square C_n)$
[16]	Corollary 1.3.17	tree T, P_7, F_1	$\gamma(F) = \gamma_e(F)$ for every induced subgraph F of T if and only if T is $\{P_7, F_1\}$-free
[5]	Theorem 1.3.23	G, connected	$\gamma_e(G) \leq \frac{43}{108}(n+2)$
[17]	Theorem 1.3.22	tree T, $\Delta(T) \leq 3$	$\gamma_{e,f}^*(T) = \frac{n+2}{6}$.

number of the n-dimensional hypercube, denoted Q_n. Furthermore, upper bound constructions for $\gamma_e^*(\mathcal{K}_n)$, $\gamma_e^*(\mathcal{S}_n)$, and $\gamma_e^*(Q_n)$ are given.

Chapter 3 contains the paper titled "On exponential domination of the consecutive circulant graph" [8]. In this paper we determine the lower bound of the porous exponential domination number of $C_{n,[\ell]}$, the general consecutive circulant graph. We also find an upper bound to the non-porous exponential domination number of $C_{n,[\ell]}$. With the aid of a result from [10], we show that

$$\gamma_e(C_{n,[\ell]}) = \gamma_e^*(C_{n,[\ell]}) = \left\lceil \frac{n}{3\ell+1} \right\rceil.$$

Chapter 4 gives final remarks and observations on exponential domination and discusses future directions.

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CHAPTER 2. A LINEAR PROGRAMMING METHOD FOR EXPONENTIAL DOMINATION

Modified form of a submitted paper

Michael Dairyko[1] and Michael Young

Abstract

For a graph G, the set $D \subseteq V(G)$ is a porous exponential dominating set if $1 \leq \sum_{d \in D} (2)^{1-\text{dist}(d,v)}$ for every $v \in V(G)$, where $\text{dist}(d, v)$ denotes the length of the shortest dv path. The porous exponential dominating number of G, denoted $\gamma_e^*(G)$, is the minimum cardinality of a porous exponential dominating set. For any graph G, a technique is derived to determine a lower bound for $\gamma_e^*(G)$. Specifically for a grid graph H, linear programing is used to sharpen bound found through the lower bound technique. Lower and upper bounds are determined for the porous exponential domination number of the King Grid $\mathcal{K}_n$, the Slant Grid $\mathcal{S}_n$, and the n-dimensional hypercube Q_n.

AMS 2010 Subject Classification: Primary 05C69; Secondary 90C05

Keywords: porous exponential domination, linear programming, grid graphs, n-dimensional hypercube

2.1 Introduction

Domination in graphs is a tool used to model situations in which a vertex exerts influence on its neighboring vertices. For a graph G, a set $D \subseteq V(G)$ is a *dominating set* if every vertex contained in $V(G) \setminus D$ is adjacent to at least one vertex of D. The *domination number*, denoted $\gamma(G)$, is the cardinality of a minimum domination set.

[1]Primary Researcher and Author.

Exponential domination was first introduced in [6] and is a variant of domination that models situations in which the influence an object exerts decreases exponentially as the distance increases. In particular exponential domination models the dissemination of information in social networks where the information's influence decays exponentially with each share [6]. Therefore, exponential domination analyzes objects with a global influence. Other variants of domination investigate objects with local influence. There are two parameters within exponential domination; porous and non-porous. This paper focuses on porous exponential domination. A *porous exponential dominating set* is a set $D \subseteq V(G)$ such that $w^*(D, v) \geq 1$ for every $v \in V(G)$, where the weight function w^* is given by $w^*(u, v) = 2^{1-\text{dist}(u,v)}$ and $\text{dist}(u, v)$ represents the length of the shortest uv path. The *porous exponential domination number* of G, denoted by $\gamma_e^*(G)$, is the cardinality of a minimum porous exponential dominating set. For the sake of simplicity, we will refer to porous exponential domination as exponential domination. See Section 2.1.1 for technical definitions.

Section 2.2 develops a technique to determine the lower bound of the exponential domination number of any graph. Furthermore, with respect to grid graphs, a method using linear programing sharpens the lower bound. Section 2.3 applies the lower bound technique described in Section 2.2, to find lower bounds for the exponential domination number of the King grid $\mathcal{K}_n$, the Slant grid $\mathcal{S}_n$, and the n-dimensional hypercube Q_n. Upper bound constructions are then found for $\gamma_e^*(\mathcal{K}_n)$, $\gamma_e^*(\mathcal{S}_n)$ and $\gamma_e^*(Q_n)$.

2.1.1 Preliminaries

All graphs are simple and undirected. A graph $G = (V(G), E(G))$ is an ordered pair that is formed by a set of *vertices* $V(G)$ and a set of *edges* $E(G)$, where an edge is the two element subset of vertices. For the two sets A and B, the Cartesian product of A and B is defined to be $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$. Consider the graph G and the set $D \subseteq V(G)$. Let $w : V(G) \times V(G) \to \mathbb{R}$ be a *weight function*. For $u, v \in V(G)$, we say that u assigns weight $w(u, v)$ to v. Denote the weight assigned by D to v as $w(D, v) := \sum_{d \in D} w(d, v)$, and similarly, the weight assigned by $d \in D$ to $H \subseteq V(G)$ as $w(d, H) := \sum_{h \in H} w(d, h)$. Let $\text{m}(G) = \max_{d \in D} w(d, V(G))$.

The pair (D, w) dominates G if $w(D, v) \geq 1$ for all $v \in V(G)$. The *excess weight* that the vertex v receives from D is defined as $\mathrm{exc}(D, v) = w(D, v) - 1$. We denote $\mathrm{exc}(D) = \sum_{v \in V(G)} \mathrm{exc}(D, v)$ to be the total excess weight that D sends out. Let $S_k(v) = \{u \in V(G) : \mathrm{dist}(u, v) = k\}$ denote the sphere of radius k.

Linear programing is an optimization technique that takes a set of linear inequalities, or constraints, and finds the best solution of a linear objective function. An integer program is a linear program, with the restriction the variables can only be assigned integer values. Observe that $\gamma_e^*(G)$ is equivalent to finding the optimal value of the following integer program introduced by Henning et al.:

Integer Program 2.1.1. *[9]*

$$\begin{array}{rcll} \min \displaystyle\sum_{u \in V(G)} x(u) & & & \\ s.t. \displaystyle\sum_{u \in V(G)} \left(\frac{1}{2}\right)^{\mathrm{dist}(u,v)-1} x(u) & \geq & 1 & \forall v \in V(G) \\ x(u) & \in & \{0, 1\} & \forall u \in V(G). \end{array}$$

Notice that it is only feasible to run the program for graphs of small size, as the computation time for this integer program greatly increases as the size of the graph increases. To be able to run the program on graphs with larger sizes, the constraints in Integer Program 2.1.1 can be relaxed as shown in the following linear program.

Linear Program 2.1.2. *[9]*

$$\begin{array}{rcll} \min \displaystyle\sum_{u \in V(G)} x(u) & & & \\ s.t. \displaystyle\sum_{u \in V(G)} \left(\frac{1}{2}\right)^{\mathrm{dist}(u,v)-1} x(u) & \geq & 1 & \forall v \in V(G) \\ x(u) & \geq & 0 & \forall u \in V(G). \end{array}$$

The Cartesian product of two graphs G and H, denoted $G \square H$, is a graph such that $V(G \square H) = V(G) \times V(H)$ and two vertices $(g, h) \sim (g', h')$ in $G \square H$ if and only if either $g = g'$ and $h \sim h'$ in H, or $h = h'$ and $g \sim g'$ in G. Let $G_{m,n} = P_m \Box P_n$ be the *standard grid.* A *grid graph* is the standard grid with possibly additional edges added in a regular pattern. Notice that linear programming is a natural technique to apply to grid graphs. Observe that asymptotically, $G_{m,n}$ is equivalent to the torus $C_m \Box C_n$, which yields the same lower bound for the corresponding exponential domination number.

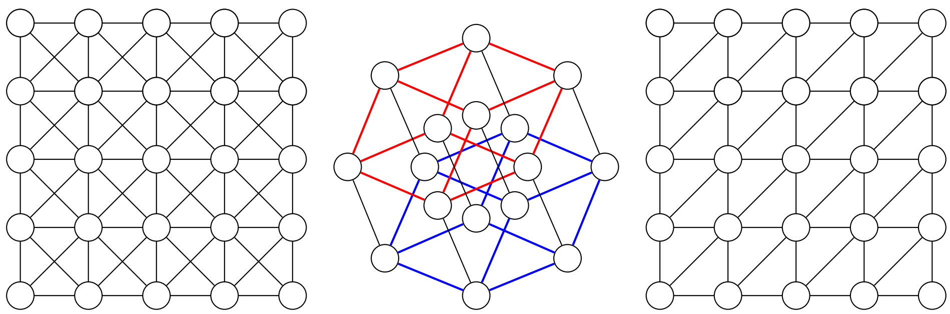

Figure 2.1 An illustration of $\mathcal{K}_5$, Q_4, and $\mathcal{S}_5$

The *strong product* of two graphs G and H is the graph $G \boxtimes H$ for which $V(G \boxtimes H) = V(G) \times V(H)$ and two distinct vertices are adjacent whenever in both coordinate places the vertices are adjacent or equal in the corresponding graph. The *King grid* is defined as $\mathcal{K}_n = P_n \boxtimes P_n$. Let $[n] = \{1, 2, \ldots, n\}$. Consider the paths P_n and P_m with vertex sets $[n]$ and $[m]$, respectively. Then the *Slant* grid is defined to be $\mathcal{S}_n = P_n \Box P_m$ with the additional edges $\{i, j\} \sim \{i + 1, j + 1\}$, for $i \in [n - 1]$ and $j \in [m - 1]$. Notice that $\mathcal{K}_n$ and $\mathcal{S}_n$ are both instances of grid graphs. The *n-dimensional hypercube* graph, denoted Q_n, is constructed by creating a vertex for each n-digit binary word. Edges are formed if two vertices differ by one digit in their binary representation. See Figure 2.1 for an illustration of $\mathcal{K}_5$, Q_4, and $\mathcal{S}_5$.

2.1.2 Motivation

For $m \leq n$ consider $C_m \Box C_n$, the *torus* graph. Exponential domination of $C_m \Box C_n$ was first studied in [1]. Figure 2.2 is a visual representation of $C_{13} \Box C_{13}$, where X denotes a member of D, an exponential domination set. Observe that there is one member of D in every row and column, therefore giving an upper bound construction for $\gamma_e(C_m \square C_n)$ when m and n are multiples of 13. The following theorem extends this idea to large graphs.

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Figure 2.2 13×13 exponential dominating set tile for $C_\infty \square C_\infty$

Theorem 2.1.3. *[1]* $\lim_{n\to\infty} \frac{\gamma_e^*(C_m \square C_n)}{mn} \leq \frac{1}{13}$.

Notice that Theorem 2.1.3 directly implies that for $m, n \geq 13$, $\gamma_e^*(C_m \square C_n) \leq \left\lceil \frac{mn}{13} \right\rceil + o(n^2)$. Through a naive counting argument, it was shown that for $m, n \geq 3$, $\left\lceil \frac{mn}{15.875} \right\rceil \leq \gamma_e^*(C_m \square C_n)$ [1]. These results lead to the following conjecture.

Conjecture 2.1.4. *For all m and n, $\left\lceil \frac{mn}{13} \right\rceil \leq \gamma_e^*(C_m \square C_n)$.*

The lower bound for $\gamma_e^*(C_m \square C_n)$ was improved in [5] by taking the counting argument from [1] and applying it to linear programming.

Theorem 2.1.5. *[5] For all $m, n \geq 11$, $\left\lceil \frac{mn}{13.761891939197298} \right\rceil \leq \gamma_e^*(C_m \square C_n)$.*

This paper was motivated by the work on determining $\gamma_e^*(C_m \square C_n)$ from [1] and [5]. The case specific lower bound technique from [5] is generalized to all graphs and the linear programming method detailed in [5] is generalized to all grid graphs.

2.2 A Lower Bound Technique

In this section, a technique for determining the lower bound of the exponential domination number of any graph is derived. Through the use of linear programing, this technique is improved specifically for grid graphs. Note that the bound in Lemma 2.2.1 is sharp if $w^*(v, V(G)) = \mathrm{m}(G)$ for every $v \in V(G)$.

Lemma 2.2.1. *Let D be an exponential dominating set for the graph G. If $k|D| \leq \mathrm{exc}(D)$, then*

$$\left\lceil \frac{|V(G)|}{\mathrm{m}(G) - k} \right\rceil \leq |D|.$$

Proof. Observe that

$$\begin{aligned} |V(G)| &\leq \sum_{v \in V(G)} w(D, v) = \sum_{d \in D} \sum_{v \in V(G)} w(d, v) \leq |D| \, \mathrm{m}(G) - \mathrm{exc}(G) \\ &\leq |D| \left(\mathrm{m}(G) - \frac{\mathrm{exc}(D)}{|D|} \right) \leq |D| \left(\mathrm{m}(G) - k \right). \end{aligned}$$

□

Remark 2.2.2. In Lemma 2.2.1, the value k is needed to compute the lower bound. For grid graphs, linear programming can be used to determine such a value of k. Mixed Integer Linear Program 2.2.3 is derived through the use of Linear Program 2.1.2 with two additional constraints. See Section 2.2.1 for the construction details. Let $x_{\min}$ be the optimal solution found from Mixed Integer Linear Program 2.2.3. As $w^*(D, v) \geq 1$ for all $v \in V(G)$, it follows that $|I| < x_{\min}$. Therefore $k = x_{\min} - |I|$.

Mixed Integer Linear Program 2.2.3.

$$\begin{aligned} \min \quad & \sum_{i \in I} [A\mathbf{x}]_i \\ \text{s.t.} \quad & A\mathbf{x} \geq \mathbf{1} \\ & A\mathbf{x} \leq b \\ & \mathbf{x} \geq \mathbf{0} \\ & x_i \leq 2, \ i \in I \\ & x_1 = 2. \end{aligned}$$

Remark 2.2.4. Observe that Remark 2.2.2 localizes the global nature of exponential domination. Recall that exponential domination has a growth factor of $\frac{1}{2}$. Therefore this method can be applied to the variant of exponential domination with the growth factor of $\frac{1}{p}$ for $p \geq 3$. Furthermore, the method can be applied to other variants of domination to obtain a lower bound for the corresponding domination number. However, it is unclear whether the lower bound derived will be significant.

2.2.1 Mixed Integer Linear Program Setup

The setup for Mixed Integer Linear Program 2.2.3 is now discussed. Consider the $m \times n$ grid graph G and let D be a corresponding exponential dominating set. For a fixed $d_0 \in D$ and given an odd positive integer $r \leq \min\{m, n\}$, define H to be the $r \times r$ subgrid of G centered at d_0. Label the set of vertices $V(H)$ as $\{v_1, v_2, \ldots, v_{r^2}\}$ and let the indices of the interior vertices of H be defined as

$$I = \left\{ i : v_i \in V(H) \text{ and } \operatorname{dist}(d_0, v_i) < \left\lfloor \frac{r}{2} \right\rfloor \right\}.$$

Then for $1 \leq k \leq r^2$, define $S_k = v_k \cup \{u \in V(G \setminus H) : \operatorname{dist}(u, v_k) \leq \operatorname{dist}(u, h)\ \forall h \in V(H)\}$ and $x_k = w^*(S_k \cap D, v_k)$. Notice that $S_i = v_i$ for every $i \in I$. Therefore for $1 \leq k, j \leq r^2$, it follows that $w^*(S_k \cap D, v_j) \leq x_k \left(\frac{1}{2}\right)^{\operatorname{dist}(v_k, v_j)}$. Thus, by the construction of S_k,

$$w^*(D, v_j) \leq \sum_{k=1}^{r^2} w^*(S_k \cap D, v_j) \leq \sum_{k=1}^{r^2} x_k \left(\frac{1}{2}\right)^{\operatorname{dist}(v_k, v_j)}.$$

Let A be the $r^2 \times r^2$ matrix such that $[A]_{kj} = \left(\frac{1}{2}\right)^{\operatorname{dist}(v_k, v_j)}$. Furthermore, let $\vec{x} = [x_1, x_2, \ldots, x_{r^2}]^\intercal$, where x_1 corresponds to d_0, and $\vec{w} = [w^*(D, v_1), w^*(D, v_2), \ldots, w^*(D, v_{r^2})]^\intercal$. Then observe that $\vec{w} \leq A\vec{x}$. The aim is to minimize $w^*(d_0, v_i)$ for all $i \in I$, while still satisfying that $w^*(D, v_i) \geq 1$. Therefore the objective function is to minimize $\sum_{i \in I} [A\mathbf{x}]_i$, where $\mathbf{x}$ is a vector of r^2 nonnegative variables.

Let $\mathbf{0}$ and $\mathbf{1}$ denote the $0s$ and $1s$ vectors of length r. Then the two constraints of Linear Program 2.1.2 with respect to the grid graph construction are that $A\mathbf{x} \geq \mathbf{1}$ and $\mathbf{x} \geq \mathbf{0}$. The remaining two constraints of Mixed Integer Linear Program are now discussed. By construction, any member of D assigns itself weight 2, and the remaining vertices do not have any initial weight. This gives the first integer constraint that $x_1 = 2$ and $x_i \leq 2$, for $i \in I$. Observe that it is necessary to determine an upper bound for $w^*(D, v_i)$ for each $v_i \in V(H)$ so that $w^*(d_0, v_i)$ can be decreased by the appropriate amount. To ensure this, we want

$$0 \leq w^*(d_0, v_i) - \text{exc}(D, v_i) = w^*(d_0, v_i) - (w^*(D, v_i) - 1).$$

This implies that $w^*(D, v_i) \leq 1 + w^*(d_0, v_i)$. Let b be the real valued vector such that $b_i = 1 + \left(\frac{1}{2}\right)^{\text{dist}(d_0, v_i)-1}$ for $1 \leq i \leq r^2$. Therefore, the second constraint is $A\mathbf{x} \leq b$.

2.3 Main Results

In this section the lower bound technique discussed in Section 2.2 is applied and upper bound constructions are found to bound the exponential domination number of the the King grid $\mathcal{K}_n$, Slant grid $\mathcal{S}_n$, and n-dimensional hypercube Q_n.

2.3.1 The King Grid $\mathcal{K}_n$

For small values of n, the exact value of $\gamma_e^*(\mathcal{K}_n)$ can be determined using Integer Program 2.1.1. Figure 2.3 visualizes the location of the corresponding exponential dominating vertices for $\gamma_e^*(\mathcal{K}_n)$, denoted by 'X'. See Code 2.6.1 for the corresponding SAGE code.

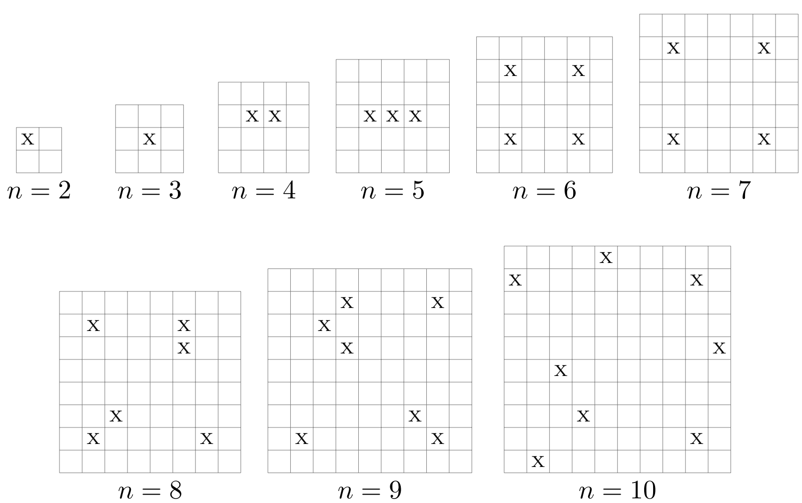

Figure 2.3 Minimum exponential dominating sets of $\mathcal{K}_n$, $2 \leq n \leq 10$

Let D be an exponential dominating set for $\mathcal{K}_n$. Notice that for $d \in D$, it follows that $|S_k(v)| = 8k$ for $k \geq 1$. Then,

$$w^*(d, V(\mathcal{K}_n)) < 2 + \sum_{k=1}^{\infty} 8k \left(\frac{1}{2}\right)^{k-1} = 2 + \left(\frac{8}{(1-\frac{1}{2})^2}\right) = 34.$$

This shows that $\mathrm{m}(\mathcal{K}_n) < 34$. This fact, along with the optimal values of k determined by Mixed Integer Linear Program 2.2.3 can be applied with Lemma 2.2.1 to determine a lower bound for $\gamma_e^*(\mathcal{K}_n)$. See Table 2.1 for a summary of these results. Observe that for $n \geq 11$, there is no feasible solution with Mixed Integer Linear Program 2.2.3. This is caused by the constraint $Ax \leq b$, since it puts a bound on the reduction of how much weight the center vertex can send out to the remaining interior vertices. Thus the best use of Mixed Integer Linear Program 2.2.3 will occur at $n = 7$.

Table 2.1 Lower Bounds for $\gamma_e^*(\mathcal{K}_n)$ for small values of n

n	3	5	7	9	11
k	1	5.7806	10.6905	10.4103	$\emptyset$
$\gamma_e^*(\mathcal{K}_n) \geq$	$\frac{n^2}{33}$	$\frac{n^2}{28.2194}$	$\frac{n^2}{23.3095}$	$\frac{n^2}{23.5897}$	$\emptyset$

Theorem 2.3.1. *For all $n \geq 7$, $\left\lceil \frac{n^2}{23.3095033018} \right\rceil \leq \gamma_e^*(\mathcal{K}_n)$.*

Proof. Let D be a minimum exponential dominating set for $\mathcal{K}_n$. For each $d \in D$, let H be the 7×7 grid centered at d. The corresponding solution to Mixed Integer Linear Program 2.2.3 gives $x_{\min} = 35.6904966982$. Therefore let $k = 35.6904966982 - 25 = 10.6904966982$ and recall that $\mathrm{m}(\mathcal{K}_n) < 34$. Therefore result follows from Lemma 2.2.1. □

Figure 2.4 shows a construction of a 23×23 tile $T_{\mathcal{K}}$, where 'X' denotes the location of an exponential dominating vertex. In particular, when $\mathcal{K}_\infty$ is tiled with $T_{\mathcal{K}}$, the exponential dominating set $D_{\mathcal{K}}$ is formed. The following theorem uses $T_{\mathcal{K}}$ to determines an upper bound for the asymptotic density of $\gamma_e^*(\mathcal{K}_n)$.

Theorem 2.3.2. $\lim_{n\to\infty} \dfrac{\gamma_e^*(\mathcal{K}_n)}{n^2} \leq \dfrac{1}{23}$

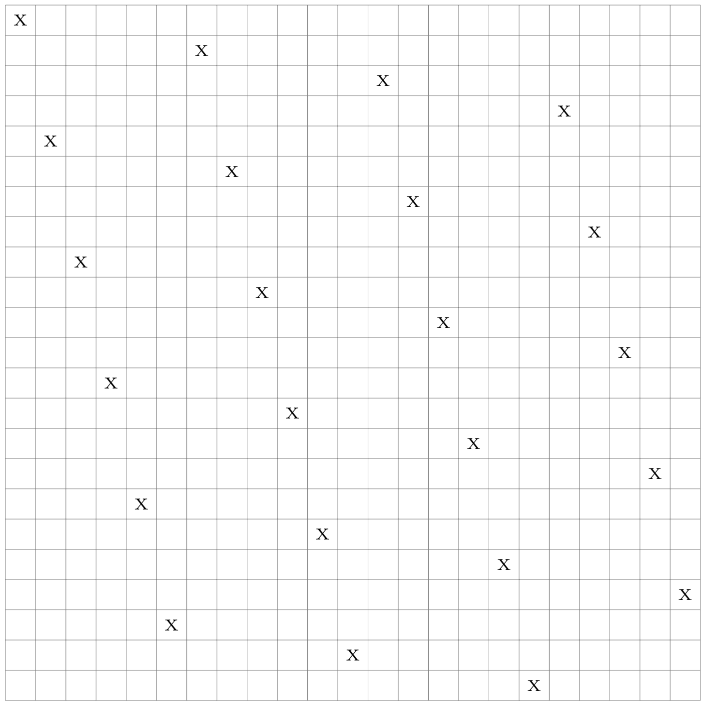

Figure 2.4 $T_{\mathcal{K}}$, the 23×23 exponential dominating set tile for $\mathcal{K}_\infty$

Proof. Let $n = 23q + r$, for some $q, r \in \mathbb{Z}$ and $0 \le r < 23$. Let H denote the $23q \times 23q$ subgrid of $\mathcal{K}_n$. Notice that we may tile H with the tiling scheme $T_{\mathcal{K}}$, as shown in Figure 2.4. Let $D_{\mathcal{K}}$ be the exponential dominating set that contains the $23q^2$ vertices used to tile H, as well as $V(\mathcal{K}_n \setminus H)$. Therefore $\gamma_e^*(\mathcal{K}_n) \le 23q^2 + 46qr + r^2$, and we obtain the following asymptotic density:

$$\lim_{n\to\infty} \frac{\gamma_e^*(\mathcal{K}_n)}{n^2} \le \lim_{q\to\infty} \frac{23q^2 + 46qr + r^2}{(23q + r)^2} \le \frac{1}{23} + \lim_{q\to\infty} \frac{46qr + r^2}{(23q + r)^2} \le \frac{1}{23},$$

as the limit equals zero. $\square$

Theorem 2.3.3. *For all $n \ge 23$, $\gamma_e^*(\mathcal{K}_n) \le \left\lceil \frac{n^2}{23} \right\rceil + o(n^2)$.*

Proof. This result follows directly from Theorem 2.3.2. $\square$

Similarly to Conjecture 2.1.4, we make the following conjecture.

Conjecture 2.3.4. *For all n, $\left\lceil \frac{n^2}{23} \right\rceil \le \gamma_e^*(\mathcal{K}_n)$.*

2.3.2 The Slant Grid $\mathcal{S}_n$

Integer Program 2.1.1 can be utilized in terms of $\mathcal{S}_n$ to determine the exact value of $\gamma_e^*(\mathcal{S}_n)$ for small values of n. These values, as well as the locations of the exponential dominating vertices, are illustrated in Figure 2.5. Notice that 'X' denotes a member of $\gamma_e^*(\mathcal{S}_n)$. See Code 2.6.3 for the corresponding SAGE code.

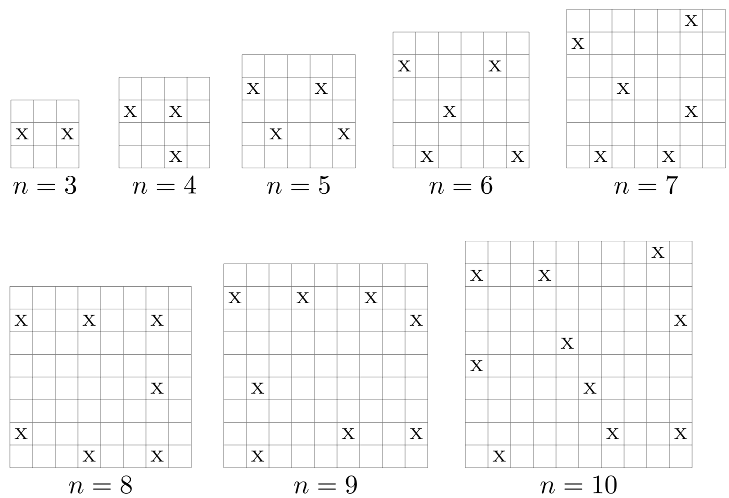

Figure 2.5 Minimum exponential dominating sets of $\mathcal{S}_n$, $3 \leq n \leq 10$.

Let D be an exponential dominating set for $\mathcal{S}_n$. Notice that for $d \in D$, we have that $|S_k(d)| \leq 6k$ for $k \geq 1$. Then we can bound the total weight that d sends to $V(\mathcal{S}_n)$ with

$$w^*(d, V(H_n)) < 2 + \sum_{k=1}^{\infty} 6k \left(\frac{1}{2}\right)^{k-1} = 2 + \left(\frac{6}{(1-\frac{1}{2})^2}\right) = 26.$$

Therefore it follows that $\mathrm{m}(\mathcal{S}_n) < 26$.

This fact, along with the optimal values of k determined by Mixed Integer Linear Program 2.2.3 can be applied with Lemma 2.2.1 to determine a lower bound for $\gamma_e^*(\mathcal{S}_n)$. See Table 2.2 for a summary of these results. Observe that for $n \geq 9$, there is no feasible solution with Mixed Integer Linear Program 2.2.3. This is caused by the constraint $Ax \leq b$, since it puts a bound on the reduction of how much weight the center vertex can send out to the remaining interior vertices. Thus the best use of Mixed Integer Linear Program 2.2.3 will occur at $n = 7$.

Table 2.2 Lower Bounds for $\gamma_e^*(\mathcal{S}_n)$ for small values of n

n	3	5	7	9
k	1.2353	3.9774	6.2655	$\emptyset$
$\gamma_e^*(\mathcal{S}_n) \geq$	$\frac{n^2}{24.7647}$	$\frac{n^2}{22.0226}$	$\frac{n^2}{19.7345}$	$\emptyset$

Theorem 2.3.5. *For all* $n \geq 7$, $\left\lceil \frac{n^2}{19.7344975348} \right\rceil \leq \gamma_e^*(\mathcal{S}_n)$.

Figure 2.6 shows a construction of a 19×19 tile $T_{\mathcal{S}}$, such that when $\mathcal{S}_\infty$ is tiled with $T_{\mathcal{S}}$, exponential dominating set $D_{\mathcal{S}}$ is formed. Notice that 'X' denotes the location of an exponential dominating vertex. The following theorem uses $T_{\mathcal{S}}$ to determine an upper bound for the asymptotic density of $\gamma_e^*(\mathcal{S}_n)$.

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Figure 2.6 $T_{\mathcal{S}}$, the 19×19 exponential dominating set tile for $\mathcal{S}_\infty$

Theorem 2.3.6.

$$\lim_{n \to \infty} \frac{\gamma_e^*(\mathcal{S}_n)}{n^2} \leq \frac{1}{19}$$

Proof. Let $n = 19q + r$, for some $q, r \in \mathbb{Z}$ and $0 \leq r < 19$. Let H denote the $19q \times 19q$ subgrid of $\mathcal{S}_n$. Notice that we may tile H with the tiling scheme $T_{\mathcal{S}}$, as shown in Figure 2.6. Let $D_{\mathcal{S}}$ be the exponential dominating set that contains the $19q^2$ vertices used to tile H, as well as $V(\mathcal{S}_n \setminus H)$. Therefore $\gamma_e^*(\mathcal{S}_n) \leq 19q^2 + 38qr + r^2$, and we obtain the following asymptotic density:

$$\lim_{n\to\infty} \frac{\gamma_e^*(\mathcal{S}_n)}{n^2} \leq \lim_{q\to\infty} \frac{19q^2 + 38qr + r^2}{(19q + r)^2} \leq \frac{1}{19} + \lim_{q\to\infty} \frac{38qr + r^2}{(19q + r)^2} \leq \frac{1}{19},$$

as the limit equals zero. □

Theorem 2.3.7. *For $n \geq 19$, $\gamma_e^*(\mathcal{S}_n) \leq \left\lceil \frac{n^2}{19} \right\rceil + o(n^2)$.*

Proof. This result follows directly from Theorem 2.3.6. □

Similarly to Conjecture 2.1.4, we make the following conjecture.

Conjecture 2.3.8. *For all n, $\left\lceil \frac{n^2}{23} \right\rceil \leq \gamma_e^*(\mathcal{K}_n)$.*

2.3.3 The n-dimensional hypercube

As Q_n is not a grid graph, the method used to determine a value of k for Lemma 2.2.1 in Remark 2.2.2 cannot be used to find the lower bound $\gamma_e^*(Q_n)$. In order to determine such a lower bound, a new method is used where distance properties of Q_n are exploited.

Let D be a minimum exponential dominating set for Q_n and let $d \in D$. Observe that for $u, v \in V(Q_n)$, the length of the shortest uv path in Q_n can be determined by the minimum number of digits that must be changed to get from u to v. Then for all $v \in V(Q_n)$, we have that:

$$w^*(v, V(Q_n)) = \sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^{i-1} = 2\sum_{i=0}^{n} \binom{n}{i} \left(\frac{1}{2}\right)^{i} \cdot 1^{n-i} = 2\left(\frac{1}{2} + 1\right)^{n} = 2\left(\frac{3}{2}\right)^{n}.$$

Thus it follows that $\mathrm{m}(Q_n) = 2\left(\frac{3}{2}\right)^n$.

In the following theorem, the decomposition in Figure 2.7 and value of $\mathrm{m}(Q_n)$ are used to show that $\left(\frac{4}{3}\right)^n \leq \gamma_e^*(Q_n) \leq (\sqrt{2})^n$ for large n.

Theorem 2.3.9. *For all $n \geq 1$, $\left\lceil \dfrac{2^{n+3}}{2^{4-n} \cdot 3^n - 2n - 9} \right\rceil \leq \gamma_e^*(Q_n) \leq (\sqrt{2})^n$*

$$Q_n = \begin{array}{ccc} Q^{(1)}_{n-2} & \cdots & Q^{(2)}_{n-2} \\ \vdots & & \vdots \\ Q^{(3)}_{n-2} & \cdots & Q^{(4)}_{n-2} \end{array}$$

Figure 2.7 A decomposition of Q_n, where $Q_n = Q_{n-2}\Box K_2\Box K_2$.

Proof. We begin with the lower bound. Let D be an exponential dominating set for Q_n and suppose that $d = \{0, 0, \ldots, 0\} \in D$. Let $A = \{a \in V(Q_n) :\ \text{a has an odd number of } 1's\}$ and $B = V(Q_n) \setminus (A \cup d)$. Let $X \subseteq A, Y \subseteq B$ such that

$$\begin{aligned} X &= \{x \in V(Q_n) :\ dx \in E(Q_n)\}, \\ Y &= \{y \in V(Q_n) :\ xy \in E(Q_n) \text{ for some } x \in X\}. \end{aligned}$$

Then $w^*(d, X) = |X| = n$ and $w^*(d, Y) = \frac{n}{2}$. As (D, w^*) dominates Q_n, $w^*(D \setminus d, Y) \geq \frac{n}{2}$. This implies that $w^*(D \setminus d, X) \geq \frac{n}{4}$, and $w^*(D \setminus d, d) \geq \frac{1}{8}$. Therefore $\text{exc}(D, X) \geq \frac{n}{4}$ and $\text{exc}(D, d) = \frac{9}{8}$, which holds for all $d \in D$. This gives that

$$\text{exc}(D) \geq \left(\frac{9}{8} + \frac{n}{4}\right)|D| = \frac{2n+9}{8}|D|.$$

Then using $\text{m}(Q_n) = 2\left(\frac{3}{2}\right)^n$ and $k = \frac{2n+9}{8}$, the lower bound follows from Lemma 2.2.1.

Now we show the upper bound. From Figure 2.7, $Q_n = Q_{n-2}\Box K_2\Box K_2$. Without loss of generality, let D and D' be two minimum exponential dominating sets for $Q^{(1)}_{n-2}$ and $Q^{(4)}_{n-2}$, respectively, with labeling as in Figure 2.7. Therefore it follows by definition that $w^*(D, v) \geq 1$ for every $v \in V(Q^{(1)}_{n-2})$ and $w^*(D', u) \geq 1$ for every $u \in V(Q^{(4)}_{n-2})$. As neighboring vertices also receive weight, every $s \in V(Q^{(2)}_{n-2})$ and $t \in V(Q^{(3)}_{n-2})$ has $w^*(D, s)$, $w^*(D, t) \geq \frac{1}{2}$ and $w^*(D', s)$, $w^*(D', t) \geq \frac{1}{2}$. This implies that $D \cup D'$ forms an exponential dominating set for Q_n. Let $a_n = \gamma^*_e(Q_n)$ and $a_{n-2} = |D| = |D'|$, so $a_n \leq 2a_{n-2}$. We now show that $a_n \leq 2^{\frac{n}{2}}$ by induction. Observe that when $n = 1$ and $n = 2$, we have that $a_1 = 1 \leq 2^{\frac{1}{2}}$ and $a_2 = 2 \leq 2^1$, respectively. Now suppose that $a_n \leq 2^{\frac{n}{2}}$ holds for all $n < k$. Now consider the case when $n = k$. Then using the inductive hypothesis,

$$a_k \leq 2a_{k-2} \leq 2(2^{\frac{1}{2}(k-2)}) = 2^{\frac{k}{2}}.$$

Therefore by induction, $\gamma_e^*(Q_n) \leq (\sqrt{2})^n$. $\square$

2.4 Acknowledgements

This research was supported in part by the National Science Foundation Award # 1719841. We would like to thanks Dr. Leslie Hogben for her input on this paper.

2.5 Additional work

This section includes work that was not included in the paper *A linear programming method for exponential domination.* For the sake of simplicity, we refer to porous exponential domination as exponential domination. A potential way to increase the lower bound for $\gamma_e^*(\mathcal{K}_n)$ and $\gamma_e^*(\mathcal{S}_n)$ described in Theorem 2.3.1 and Theorem 2.3.5, respectively, would be to add additional constraints to Mixed Integer Linear Program 2.2.3. One such constraint would be to determine a global α for all $u \in V(G)$ such that $w(D_\alpha(u), u) \geq 1$.

Lemma 2.5.1. *Let G be an infinite grid graph for which (D, w) is a dominating pair and suppose that for every $u \in V(G)$, there exists α such that $w^*(D_\alpha(u), u) \geq 1$. Then $\operatorname{exc}(D, u) > 0$ for every $u \in V(G)$.*

Proof. Suppose that for every $u \in V(G)$, there exist an α such that $w^*(D_\alpha(u), u) \geq 1$. Consider $v \in V(G)$, and by assumption there exist α such that $w^*(D_\alpha(v), v) \geq 1$. As G is infinite, there exists $v' \in V(G)$ and α' such that $w^*(D_{\alpha'}(v'), v') \geq 1$ and $\operatorname{dist}(v, v') \geq \alpha + \alpha'$. There then must be at least one member $d \in D$ such that $d \notin D_\alpha(v)$. Therefore it follows that

$$1 \leq w^*(D_\alpha(v), v) < w^*(D_\alpha(v), v) + w^*(d, v).$$

Thus for all $u \in V(G)$, $\operatorname{exc}(D, u) > 0$. $\square$

Observe that the reverse of Lemma 2.5.1 does follow, if the additional assumptions that G is connected and has infinite diameter. However without the added conditions, the result is unknown. This leads to the following conjecture:

Conjecture 2.5.2. *Let G be an infinite grid graph for which (D, w) is a dominating pair. If $\operatorname{exc}(D, u) > 0$ for every $u \in V(G)$, then for every $u \in V(G)$, there exists α such that $w(D_\alpha(u), u) \geq 1$.*

Results with a minimum distance amongst members of the exponential dominating set with respect to the infinite grid graph $G_{\infty,\infty}$ are examined in [1]. Let D be a set of vertices in $G_{\infty,\infty}$, then define the function $I_n(v) = \sum_{w \in D \setminus B_{n-1}(v)} \left(\frac{1}{2}\right)^{\operatorname{dist}(v,w)-1}$ for any vertex v of $G_{\infty,\infty}$ [1].

Lemma 2.5.3. *[1] Given a set D of vertices in the infinite grid graph $G_{\infty,\infty}$, if the minimum distance between vertices in D is at least 5, then for any vertex v, $I_n(v) \leq \frac{35n+36}{147 \cdot 2^{n-4}}$.*

Essentially Lemma 2.5.3 determined an upper bound on the potential weight a vertex receives from members of the exponential dominating set whose distance from the vertex is at least n. With restraints on the minimal distance between exponential dominating vertices, the smallest ball of radius r that must contain an exponential dominating vertex was determined. These findings give rise to another potential constraint to add to Mixed Integer Linear Program 2.2.3. In particular, we focus on the infinite King Grid $\mathcal{K}_\infty$.

Remark 2.5.4. Let $v \in V(\mathcal{K}_\infty)$ and let D be an exponential dominating set. If the minimum distance between vertices in D is at least α, then there are at most $\frac{|S_n(v)|}{\alpha}$ members of $D \cap S_n(v)$.

In the next lemma, we derive a formula to determine an upper bound for the weight that $v \in V(\mathcal{K}_\infty)$ receives from $D \setminus B_{n-1}(v)$ using $A_{n,n+2}(v)$, with the restraint that the minimum distance between members of the exponential dominating set is at least 4.

Lemma 2.5.5. *Consider the infinite King grid $\mathcal{K}_\infty$ and let D be an exponential domination set for which the minimum distance between members of D is at least 4. Then for any vertex v,*

$$w(D \setminus B_{n-1}(v), v) \leq \frac{7n+17}{49 \cdot 2^{n-5}}.$$

Proof. Consider $\mathcal{K}_\infty$ in the chessboard representation. Let D be an exponential domination set for $\mathcal{K}_\infty$ and let $v \in V(\mathcal{K}_\infty)$ such that $v = (0,0)$. Consider $A_{n,n+2}(v)$ and partition $V(A_{n,n+2}(v))$ in the

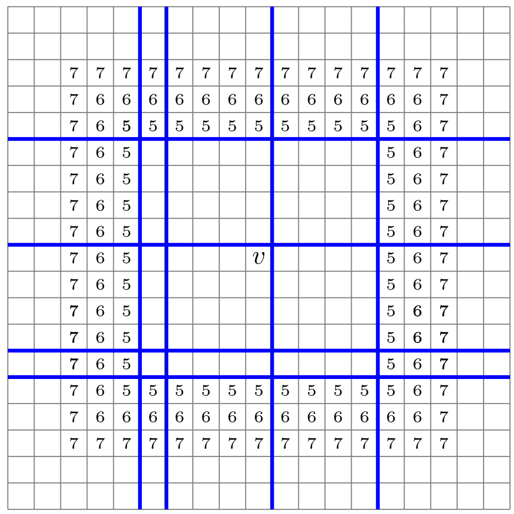

Figure 2.8 Example of $A_{n,n+2}(v)$ for $n = 5$ with Lemma 2.5.5 partition

following manner. For the horizontal lines, let $y = 0.5 - n$ and $y = n - 0.5 - 4k$ for $0 \leq k < \lceil \frac{n}{2} \rceil$. For the vertical lines we use $x = 0.5 - n$ and $x = n - 0.5 - 4k$ for $0 \leq k < \lceil \frac{n}{2} \rceil$. Notice that this partition ensures that the maximum distance between any two vertices within the same part is 3. Therefore by the initial assumption, at most one vertex in each part can be a member of D. See Figure 2.8 for an example of the partition. Observe that there are $4(\lceil \frac{n}{2} \rceil + 1)$ parts in each partition. Using the property that $\lceil x \rceil \leq x + 1$, it follows that

$$|D \cap A_{n,n+2}(v)| \leq 4\left(\left\lceil \frac{n}{2} \right\rceil + 1\right) \leq 2n + 8.$$

As $|S_n(v)| = 8n$, Remark 2.5.4 shows that $|D \cap S_n(v)| \leq 2n$. If the remaining exponential dominating vertices in $A_{n,n+2}(v)$ are contained in $S_{n+1}(v)$, then

$$w^*(D \cap A_{n,n+2}(v), v) \leq 2n\left(\frac{1}{2}\right)^{n-1} + 8\left(\frac{1}{2}\right)^{n} = \frac{n+2}{2^{n-2}}. \tag{2.5.1}$$

Observe that by construction

$$V(G) \setminus B_{n-1}(v) = \bigcup_{k=0}^{\infty} A_{n+3k,n+3k+2}(v). \tag{2.5.2}$$

Putting (2.5.1) and (2.5.2) together gives that

$$w^*(D \setminus B_{n-1}(v), v) \leq \sum_{k=0}^{\infty} \frac{(n+3k)+2}{2^{n+3k-2}} = \frac{7n+17}{49 \cdot 2^{n-5}}.$$

$\square$

Table 2.3 shows the formula derived in Lemma 2.5.5 for small values of n. Notice that when $n = 6$, the vertex v does not receive sufficient weight from D. This implies that there must be an exponential dominating vertex contained in every ball of radius 5.

Table 2.3 Small values of n applied to Lemma 2.5.5

n	5	6	7
$w^*(D \setminus B_{n-1}(v), v)$	1.06122	0.602041	0.336735

The following lemma mimics Lemma 2.5.5, with the added constraint that the minimum distance between members of the exponential dominating set is at least 2.

Lemma 2.5.6. *Consider the infinite King grid $\mathcal{K}_\infty$ and let D be an exponential domination set for which the minimum distance between vertices in D is at least 2. Then for any vertex v,*

$$w(D \setminus B_{n-1}(v), v) \leq \frac{28n + 33}{49 \cdot 2^{n-5}}.$$

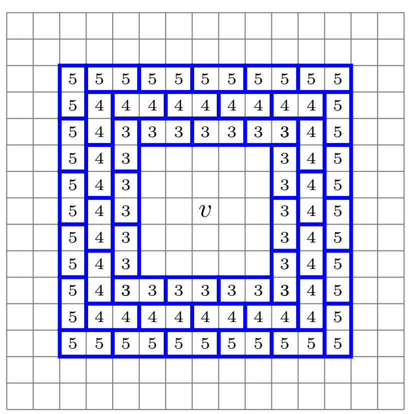

Figure 2.9 Example of $A_{n,n+2}(v)$ for $n = 3$ with Lemma 2.5.6 partition

Proof. Consider $\mathcal{K}_\infty$ in the chessboard representation. Let D be an exponential domination set for $\mathcal{K}_\infty$ and let $v \in V(\mathcal{K}_\infty)$ such that $v = (0, 0)$. Consider $A_{n,n+2}(v)$ and partition $V(A_{n,n+2}(v))$ into horizontal and vertical cells. For $n \leq i \leq n+2$ and $0 \leq k < i$, the horizontal cells consist of the two vertices $(\mp i, \pm(i-2k)), (\mp i, \pm(i-2k-1))$ and the vertical cells consist of the two vertices $(\pm(2k+1-i), \pm i), (\pm(2k+2-i), \pm i)$. By construction, this partition has the property the maximum distance between any two vertices in a cell is 1. See Figure 2.9 for an example of the partition. Notice that there are exactly $12n+12$ cells and at most one vertex of each cell can be a member of D. As $|S_n(v)| = 8n$, Remark 2.5.4 shows that $|D \cap S_n(v)| \leq 4n$. If the remaining exponential dominating vertices in $A_{n,n+2}(v)$ are contained in $S_{n+1}(v)$, then

$$w^*(D \cap A_{n,n+2}(v)) \leq 4n\left(\frac{1}{2}\right)^{n-1} + (8n+12)\left(\frac{1}{2}\right)^{n} = \frac{4n+3}{2^{n-2}}. \tag{2.5.3}$$

Observe that by construction

$$V(G) \setminus B_{n-1}(v) = \bigcup_{k=0}^{\infty} A_{n+3k,n+3k+2}(v). \tag{2.5.4}$$

Putting (2.5.3) and (2.5.4) together gives

$$w^*(D \setminus B_{n-1}(v), v) \leq \sum_{k=0}^{\infty} \frac{4(n+3k)+3}{2^{n+3k-2}} = \frac{28n+33}{49 \cdot 2^{n-5}}.$$

□

Table 2.4 Small values of n applied to Lemma 2.5.6

n	6	7	8
$w^*(D \setminus B_{n-1}(v), v)$	2.05102	1.16837	0.655612

Table 2.4 shows the formula derived in Lemma 2.5.6 for small values of n. Notice that when $n = 8$, the vertex v does not receive sufficient weight from D. This implies that there must be an exponential dominating vertex contained in every ball of radius 7. Therefore if it can be shown that no two exponential dominating vertices are adjacent then the constraint that there must be at least one exponential dominating vertex within every ball of radius 7 can be added to Mixed Integer Linear Program 2.2.3. This motivates the following conjecture

Conjecture 2.5.7. *There exist a minimum exponential dominating set such that no dominating vertices are adjacent.*

2.6 Appendix

This sections consists of the SAGE code referenced throughout *A linear programming method for exponential domination.*

Code 2.6.1. *Integer program that computes $\gamma_e^*(\mathcal{K}_n)$ for small values of n*

```
for n in range(1,11):
    G = graphs.KingGraph( [n,n] ) # Sets up King grid
    KingDist = G.distance_matrix() # King grid distance matrix
    m = KingDist.nrows()

    # This matrix represents the weight that
    # vertex i sends to vertex j in the King grid
    K = matrix(QQ, m,m, lambda i, j: (1/2)^(KingDist[i][j]-1))

    # Sets up the MILP
    p = MixedIntegerLinearProgram(maximization=False, solver="GLPK");
    x = p.new_variable(integer=True, nonnegative=True)

    # creates the objective function
    s = 0
    for i in range(m):
        p.add_constraint(x[i] <= 1)
        s = s + x[i]

    p.set_objective(s)
    p.add_constraint(K*x >= 1)
    print n, p.solve()
```

Code 2.6.2. *Mixed integer linear program to find upper bound of $\gamma_e^*(\mathcal{K}_n)$.*

```
# This function computes the distance between two vertices in the King grid
def distance(a,b):
    ans = max( abs(a[0] - b[0]), abs(a[1] - b[1]) )
    return ans
```

```

# The Linear Program for the King grid
for n in range (3, 13, 2):
    p = MixedIntegerLinearProgram(maximization=False, solver="GLPK");
    x = p.new_variable(real=True, nonnegative=True)
    King = graphs.KingGraph([n,n]) # Generated the nxn king grid
    K = King.vertices()

    A = zero_matrix(RR,n^2,n^2) # Creates the A matrix for LP
    for i in range(n^2):
        for j in range(n^2):
            A[i, j] = (1/2)^ distance(K[i], K[j] )

    KingDist =  zero_matrix(RR,n^2,n^2) # Creates the distance matrix for King grid
    for i in range(n^2):
        for j in range(n^2):
            KingDist[i,j] = distance(K[i],K[j])

    b = [34]*n^2 # creates the b vector for King grid
    for i in range(0,n^2):
        if divmod(i,n)[1] != 0 and divmod(i,n)[1]!= n-1:
            b[i] = 1+ (1/2)^(KingDist[(n^2-1)/2, i] - 1)

    # Finds the inner vertices of King grid
    list =[]
    for i in range(n,n^2-n):
        if divmod(i,n)[1] != 0 and divmod(i,n)[1]!= n-1:
            list = list + [i]
    #make the matrix using only the rows that are inner vertices
    c =A.matrix_from_rows(list)
    sum=0

    one_vec = [1]*n^2
    # Adds in the constraint that 1≤ Ax
    p.add_constraint(A*x ≥ one_vec)
    # Adds the constraint that the weight of the middle vertex is 2
    p.add_constraint(x[(n^2-1)/2] == 2)
    # Adds in the constraint that Ax ≤ b
    p.add_constraint(A*x ≤ b)
```

```
    for i in range(len(list)):
        p.set_integer(x[list[i]])
      # Adds in constraint that inner vertices have weight 0 or 2
        p.add_constraint(x[list[i]] ≤ 2)

    # Sets the objective function
    for i in range(c.nrows()):
        for j in range(c.ncols()):
            sum = sum + c[i][j]*x[j]

    # Computes the minimum exponential dominating number
    p.set_objective(sum)
    ans = 34 + (n-2)^2 - p.solve();
    print n, ans
```

Code 2.6.3. *Integer program that computes $\gamma_e^*(\mathcal{S}_n)$ for small values of n*

```
for n in range(2,11):
    Slant = SlantGrid(n)
    Sverts = Slant.vertices()

# This matrix represents the weight that
# vertex i sends to vertex j in the Slant grid
    A = zero_matrix(QQ,n^2,n^2)
    for i in range(n^2):
        for j in range(n^2):
            A[i, j] = (1/2)^(Slant.distance(Sverts[i], Sverts[j] ) -1)
# Constructs integer program
    p = MixedIntegerLinearProgram(maximization=False, solver="GLPK");
    x = p.new_variable(integer=True, nonnegative=True)
    s = 0
    for i in range(n^2):
        p.add_constraint(x[i] ≤ 1)
        s = s + x[i]

    p.set_objective(s)
```

```
    p.add_constraint(A*x ≥ 1)
    print n, p.solve()
```

Code 2.6.4. *Mixed integer linear program to find a lower bound of* $\gamma_e^*(\mathcal{S}_n)$.

```
def SlantGrid(n):
    S = graphs.Grid2dGraph(n,n)
    for i in range(1, n ):
        for j in range(n-1):
            S.add_edge( (i,j) , (i-1, j+1)  )
    return S

# The Mixed Integer Linear Program for the Slant grid

for n in range(3, 20, 2):
    p = MixedIntegerLinearProgram(maximization=False, solver="GLPK");
    x = p.new_variable(real=True, nonnegative=True)

    Slant = SlantGrid(n) # Generated the nxn Slant grid
    Sverts = Slant.vertices()

    A = zero_matrix(RR,n^2,n^2) # Creates the A matrix for LP
    for i in range(n^2):
        for j in range(n^2):
            A[i, j] = (1/2)^Slant.distance(Sverts[i], Sverts[j] )
    one_vec = [1]*n^2
     # Creates the distance matrix for Slant grid
    SlantDist =  zero_matrix(RR,n^2,n^2)
    for i in range(n^2):
        for j in range(n^2):
            SlantDist[i,j] = Slant.distance(Sverts[i],Sverts[j])

    b = [26]*n^2
    for i in range(0,n^2):
        if divmod(i,n)[1] != 0 and divmod(i,n)[1]!= n-1:
            b[i] = 1+ (1/2)^(SlantDist[(n^2-1)/2, i] - 1)

```

```
    # Finds the inner vertices of Slant grid
    list=[]
    for i in range(n,n^2-n):
        if divmod(i,n)[1] != 0 and divmod(i,n)[1]!= n-1:
            list = list + [i]
    #make the matrix using only the rows that are inner vertices
    c =A.matrix_from_rows(list)
    sum=0

    # Adds in the constraint that 1≤ Ax
    p.add_constraint(A*x ≥ one_vec)
   # Adds the constraint that the weight of the center vertex is 2
    p.add_constraint(x[(n^2-1)/2] == 2)
   # Adds in the constraint that Ax ≤ b
    p.add_constraint(A*x ≤ b)

    for i in range(len(list)):
        p.set_integer(x[list[i]])
        # Adds in constraint that inner vertices have weight 0,1, or 2
        p.add_constraint(x[list[i]] ≤ 2)

    # Sets the objective function
    for i in range(c.nrows()):
        for j in range(c.ncols()):
            sum = sum + c[i][j]*x[j]

    # Computes the minimum exponential dominating number
    p.set_objective(sum)
    ans = 26 + (n-2)^2 - p.solve();
    print n, ans, p.solve() - (n-2)^2
```

Code 2.6.5. *Integer program that computes $\gamma_e^*(Q_n)$ for small values of n*

```
for m in range (1, 8):
    g = graphs.CubeGraph(m)
    M = g.distance_matrix()

```

```
    # sets up the integer program
    p = MixedIntegerLinearProgram(maximization=False, solver="GLPK");
    n = M.nrows()

    # This matrix represents the weight that
    # vertex i sends to vertex j in the hypercube
    K = matrix(QQ, n,n, lambda i, j: (1/2)^(M[i][j]-1))
    x = p.new_variable(integer=True, nonnegative=True)
    s = 0
    for i in range(n):
        p.add_constraint(x[i] <= 1)
        s = s + x[i]

    # sets the objective function and constraints
    p.set_objective(s)
    p.add_constraint(K*x >= 1)
    print m, p.solve()
```

2.7 Bibliography

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CHAPTER 3. ON EXPONENTIAL DOMINATION OF THE GENERALIZED CIRCULANT GRAPH

Modified from of a submitted paper

Michael Dairyko[1] and Michael Young

Abstract

For a graph G, we consider $D \subset V(G)$ to be a porous exponential dominating set if $1 \leq \sum_{d\in D} (2)^{1-\text{dist}(d,v)}$ for every $v \in V(G)$, where $\text{dist}(d,v)$ denotes the length of the smallest dv path. Similarly, $D \subset V(G)$ is a non-porous exponential dominating set is $1 \leq \sum_{d\in D} (2)^{1-\overline{\text{dist}}(d,v)}$ for every $v \in V(G)$, where $\overline{\text{dist}}(d,v)$ represents the length of the shortest dv path with no internal vertices in D. The porous and non-porous exponential dominating number of G, denoted $\gamma_e^*(G)$ and $\gamma_e(G)$, are the minimum cardinality of a porous and non-porous exponential dominating set, respectively. The consecutive circulant graph, $C_{n,[\ell]}$, is the set of n vertices such that vertex v is adjacent to $v \pm i \mod n$ for each $i \in [\ell]$. In this paper we show $\gamma_e(C_{n,[\ell]}) = \gamma_e^*(C_{n,[\ell]}) = \left\lceil \frac{n}{3\ell+1} \right\rceil$.

AMS 2010 Subject Classification: Primary 05C69; Secondary 05C12

Keywords: domination, generalized circulant graph, exponential domination, porous exponential domination

3.1 Introduction

Domination in graphs is a well studied area within graph theory. For a graph G, we consider $D \subset V(G)$ to be a *dominating set* if every member of $V(G) \setminus D$ is adjacent to at least one member of D. The *domination number* of G, denoted $\gamma(G)$, is the minimum cardinality of a dominating set. Define $w : V(G) \times V(G) \to \mathbb{R}$ to be a *weight function* of G. For $u, v \in V(G)$, we say that u assigns weight

[1]Primary Researcher and Author.

$w(u,v)$ to v. Denote the weight assigned by a set of vertices D to v as $w(D,v) := \sum_{d\in D} w(d,v)$, and similarly, the weight assigned by $d \in D$ to $H \subset V(G)$ as $w(d,H) := \sum_{h\in H} w(d,h)$. The pair (D,w) dominates G if $w(D,v) \geq 1$ for every $v \in V(G)$. In the context of domination, the pair (D,w) dominates G where D is a dominating set and w is the following function:

$$w(u,v) = \begin{cases} 1 & \text{if } u \in D \text{ and } uv \in E(G) \\ 0 & \text{otherwise.} \end{cases}$$

A variant of domination, called *exponential domination*, was first introduced in [5]. Their motivation was to create a framework for a particular type of distance domination, one that would better model real world situations in which the influence of a selected vertex on other vertices decreases exponentially as their distance increases. There are two types of exponential domination; *non-porous* and *porous.* In *non-porous exponential domination,* exponential dominating vertices obstruct the influence of each other, whereas there is no such obstruction in *porous exponential domination.* More formally, the weight function for non-porous exponential domination is

$$w(u,v) = \left(\frac{1}{2}\right)^{\overline{\mathrm{dist}}(u,v)-1},$$

where $\overline{\mathrm{dist}}(u,v)$ represents the length of the shortest uv path that does not contain any internal vertices that are in the non-porous exponential dominating set. The *non-porous exponential domination number* of G, denoted by $\gamma_e(G)$, is the cardinality of a minimum non-porous exponential dominating set. The weight function for porous exponential domination is

$$w^*(u,v) = \left(\frac{1}{2}\right)^{\mathrm{dist}(u,v)-1},$$

where $\mathrm{dist}(u,v)$ represents the length of the shortest uv path. The *porous exponential domination number* of G, denoted by $\gamma_e^*(G)$, is the cardinality of a minimum porous exponential dominating set.

Notice that exponential domination differs from the other variants of domination discussed in [6] due to the global influence exponential dominating vertices have on $V(G)$, whereas the dominating vertices of the variants of domination have a more local influence. The relatively few results

[1, 2, 3, 4, 5, 7, 8] in this area has been attributed to the difficulty of working within the global nature of exponential domination. On relating exponential domination to domination, it is known that [5]

$$\gamma_e^*(G) \leq \gamma_e(G) \leq \gamma(G). \tag{3.1.1}$$

Let $[n] = \{1, 2, \ldots, n\}$. The *consecutive circulant graph*, $C_{n,[\ell]}$, is the set of n vertices such that vertex v is adjacent to vertex $v \pm i \mod n$ for each $i \in [\ell]$. Notice that $C_{n,[1]}$ is equivalent to C_n and $C_{n,[\lfloor \frac{n}{2} \rfloor]}$ is equivalent to the complete graph K_n. The following proposition gives an explicit formula for $\gamma_e(C_n)$.

Proposition 3.1.1. *[5]* For every integer $n \geq 3$,

$$\gamma_e(C_n) = \begin{cases} 2 & \text{if } n = 4 \\ \lceil \frac{n}{4} \rceil & \text{if } n \neq 4. \end{cases}$$

No such formula has been determined for $\gamma_e^*(C_n)$. In this paper, we show that that the porous and non-porous exponential domination number of $C_{n,[\ell]}$ are equivalent. Furthermore, in Theorem 3.1.2 when $\ell = 1$ and $m \geq 2$, our results align with Proposition 3.1.1 and fills the gap to $\gamma_e^*(C_n)$. For the sake of simplicity, we will now refer to porous exponential domination as exponential domination, unless stated otherwise.

We still need a few more definitions and notation. Let H be the Hamiltonian cycle of $C_{n,[\ell]}$, where the vertices $v, v+1 \mod n$ form an edge. Label the vertices of $C_{n,[\ell]}$ in the order of H as $V_H = \{0, 1, \ldots, n-1\}$. For $0 \leq i, j \leq n-1$, we denote $\text{dist}_H(i,j)$ to be the length of the shortest path from i to j on H. See Figure 3.1 for an illustration of $C_{8,[2]}$, with the defined labeling. With respect to $V(C_{n,[\ell]})$, denote the interval $[i,j]$ as the set of increasing consecutive integers modulo n from i to j. Let $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$ be the consecutive partition around H. For any exponential dominating set D, let $f_k(D,\mathcal{I}) := |I_k \cap D|$. Also define $z(D,\mathcal{I}) := \{i : f_i(D,\mathcal{I}) = 0\}$, $Z(D,\mathcal{I}) := |z(D,\mathcal{I})|$, and $f^*(D,\mathcal{I}) := \max_{0 \leq i \leq m-1} f_i(D,\mathcal{I})$.

Our main result is the following theorem, whose proof is shown in Section 3.2.2.

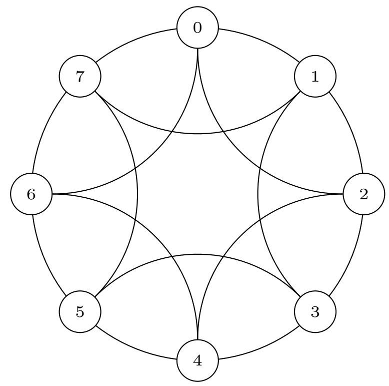

Figure 3.1 An illustration of $C_{8,[2]}$

Theorem 3.1.2. *Let* $n = (3\ell + 1)m + r$*, for* $0 \leq r \leq 3\ell$*. Then*

$$\gamma_e^*(C_{n,[\ell]}) = \gamma_e(C_{n,[\ell]}) = \left\lceil \frac{n}{3\ell + 1} \right\rceil .$$

We now give a brief outline of the proof for Theorem 3.1.2. Through the use of the remarks and lemmas in Section 3.2.1, we show that the above equality holds when $(3\ell + 1)$ divides n. Additionally, the structure of the exponential domination set in this case is shown to be unique up to isomorphism. The main result is proven by exploiting the uniqueness of the exponential domination set when $(3\ell + 1)$ divides n, and applying (3.1.1).

3.2 Exponential domination of consecutive circulants

In this section we prove Theorem 3.1.2, which determines the explicit non-porous and porous exponential domination number of $C_{n,[\ell]}$, and shows that these numbers are equivalent. In the first subsection, we remark on minor results and provide lemmas used to prove the main results. The main results and their proofs are given in the second subsection.

3.2.1 Minor Results and Lemmas

The following remarks and lemmas appear in the order they are referenced in the proofs of Theorems 3.2.12 and 3.1.2.

Remark 3.2.1. Consider $u, v \in V(C_{n,[\ell]})$. Throughout the paper, there will be a need to refer to $\text{dist}(u, v)$. Notice that,

$$\text{dist}(u, v) = \left\lceil \frac{\text{dist}_H(u, v)}{\ell} \right\rceil.$$

Remark 3.2.2. Suppose that $[a, b]$ is an interval on H such that $a < b$ and $\text{dist}_H(a, b) \leq 3\ell + 1$. For notational simplicity, consider the interval $[0, 3\ell + 1]$. Then 0 and $3\ell + 1$ dominates $[1, \ell]$ and $[2\ell + 1, 3\ell]$, respectively, and both 0 and $3\ell + 1$ contribute weight $\frac{1}{2}$ to $[\ell + 1, 2\ell]$. Therefore $[0, 3\ell + 1]$ is exponentially dominated by 0 and $3\ell + 1$. This shows that a, b exponentially dominates $[a, b]$.

Remark 3.2.3. For $n = (3\ell + 1)m$, with $m \geq 2$, let D be a minimum exponential dominating set for $C_{n,[\ell]}$. Fix vertex $i \in V_H$ and construct set $D^* = \{i + (3\ell + 1)t \mod n : 0 \leq t \leq m - 1\}$. Through the application of Remark 3.2.2, D^* forms an exponential dominating set for which $|D| \leq |D^*| = m$.

Remark 3.2.4. Let D be a minimum exponential dominating set for $C_{n,[\ell]}$. From Remark 3.2.3, we have that $|D| \leq m$. Therefore, for every interval I_k with $f_k(D) > 1$, there must exist $f_k(d) - 1$ distinct intervals that contain no members of D. This shows that $\sum_{k=0}^{m-1} f_k(D) \leq m$.

In many of the cases for the proof of Theorem 3.1.2, an exponential dominating set D can be reduced to having at least one interval with no members of D and the remaining intervals with exactly one member of D. The following lemma gives results on D in this situation.

Lemma 3.2.5. *Let $D \subset V(C_{n,[\ell]})$ and $\mathcal{I}$ be a partition such that $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$, where $I_i = [(3\ell + 1)i, (3\ell + 1)i + 3\ell]$ such that $f^*(D, \mathcal{I}) = 1$ and $Z(D, \mathcal{I}) \geq 1$. Let $d_i := I_i \cap D$ for $0 \leq i \leq m - 1$, and consider $z \in z(D, \mathcal{I})$. Then,*

(i) $w^*(D, (3\ell + 1)z + \ell) < \frac{6}{7}$ *and* $w^*(D, (3\ell + 1)z + 2\ell) < \frac{6}{7}$,

(ii) $w^*(D \setminus d_k, (3\ell + 1)z + \ell) < \frac{17}{28}$ *and* $w^*(D \setminus d_k, (3\ell + 1)z + 2\ell) < \frac{5}{14}$, *for* $k \equiv z + 1 \mod m$,

(iii) $w^*(D \setminus d_k, (3\ell + 1)z + \ell) < \frac{5}{14}$ *and* $w^*(D \setminus d_k, (3\ell + 1)z + 2\ell) < \frac{17}{28}$, *for* $k \equiv z - 1 \mod m$.

(iv) $w^*(D \setminus d_k, (3\ell + 1)z + \ell) < \frac{377}{448}$ *and* $w^*(D \setminus d_k, (3\ell + 1)z + 2\ell) < \frac{185}{224}$, *for* $k \equiv z + 2 \mod m$,

(v) $w^*(D \setminus d_k, (3\ell + 1)z + \ell) < \frac{185}{224}$ *and* $w^*(D \setminus d_k, (3\ell + 1)z + 2\ell) < \frac{377}{448}$, *for* $k \equiv z - 2 \mod m$.

Proof. Let $\mathcal{I}$ be the partition such that $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$, where $I_i = [(3\ell+1)i, (3\ell+1)i+3\ell]$. For the sake of simplicity, let $f(D) = f(D,\mathcal{I})$, $f^*(D) = f^*(D,\mathcal{I})$, $z(D) = z(D,\mathcal{I})$, and $Z(D) = Z(D,\mathcal{I})$. Without loss of generality, suppose that $0 \in z(D)$. Then the interval I_0 has $f_0(D) = 0$. Among such D, choose D' to maximize $w^*(D', 2\ell)$ and let $k_0 = \left\lfloor \frac{m}{2} \right\rfloor$. Then the choice of D' implies that

$$I_k \cap D' = \begin{cases} (3\ell+1)k & \text{for } k \leq k_0 \\ (3\ell+1)k + 3\ell & \text{for } k > k_0. \end{cases}$$

Consider $k \leq k_0$ and notice that $\text{dist}_H(2\ell, (3\ell+1)k) = (3\ell+1)k - 2\ell = (3k-2)\ell + k$. By Remark 3.2.1,

$$\text{dist}(2\ell, (3\ell+1)k) = \left\lceil \frac{(3k-2)\ell + k}{\ell} \right\rceil = 3k - 2 + \left\lceil \frac{k}{\ell} \right\rceil. \tag{3.2.1}$$

Using the fact that $1 \leq \left\lceil \frac{k}{\ell} \right\rceil$, it follows that

$$\begin{aligned} \sum_{k=1}^{k_0} w^*(I_k \cap D', 2\ell) &= \sum_{k=1}^{k_0} \left(\frac{1}{2}\right)^{\text{dist}(2\ell,(3\ell+1)k)-1} < \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{\text{dist}(2\ell,(3\ell+1)k)-1} &(3.2.2)\\ &= \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{3k-3+\left\lceil \frac{k}{\ell} \right\rceil} \leq \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{3k-2} &(3.2.3)\\ &= \frac{1}{2}\sum_{t=0}^{\infty} \left(\frac{1}{2}\right)^{3t} = \frac{4}{7}. \end{aligned}$$

Now consider $k > k_0$ and let $k' = m - k$. Then with respect to V_H,

$$\begin{aligned} \text{dist}_H(2\ell, (3\ell+1)k + 3\ell) &= \text{dist}_H((3\ell+1)m + 2\ell, (3\ell+1)k + 3\ell) \\ &= (3\ell+1)m + 2\ell - (3\ell+1)k - 3\ell \\ &= (3k'-1)\ell + k'. \end{aligned}$$

Again, applying Remark 3.2.1 gives,

$$\text{dist}(2\ell, (3\ell+1)k + 3\ell) = \left\lceil \frac{(3k'-1)\ell + k'}{\ell} \right\rceil = 3k' - 1 + \left\lceil \frac{k'}{\ell} \right\rceil. \tag{3.2.4}$$

Notice since k and k' are counters, (3.2.1) and (3.2.4) only differ by ℓ. Furthermore, summing from $k = k_0 + 1$ to $m - 1$ is the same as summing k' from 1 to $m - k_0 - 1$, which equals k_0 or $k_0 - 1$. Therefore we have shown that

$$w^*(D, 2\ell) \quad \leq \quad w^*(D', 2\ell) \quad < \quad \frac{3}{2}\sum_{k=1}^{k_0} w^*(I_k \cap D', 2\ell) \quad < \quad \frac{6}{7}.$$

Applying a symmetric argument gives that $w^*(D, \ell) < \frac{6}{7}$, and (i) has been established. Let $d'_k := I_k \cap D'$ and observe that $w^*(D \setminus d_k, \ell) \leq w^*(D' \setminus d'_k, \ell)$ and $w^*(D \setminus d_k, 2\ell) \leq w^*(D' \setminus d'_k, 2\ell)$. Notice that by construction, $d'_1 = 3\ell + 1$ and $d'_2 = 6\ell + 2$. Then applying

$$\begin{aligned}
\operatorname{dist}_H(d'_1, \ell) &= 3\ell + 1 - \ell &&= 2\ell + 1,\\
\operatorname{dist}_H(d'_1, 2\ell) &= 3\ell + 1 - 2\ell &&= \ell + 1,\\
\operatorname{dist}_H(d'_2, \ell) &= 6\ell + 2 - \ell &&= 5\ell + 2,\\
\operatorname{dist}_H(d'_2, 2\ell) &= 6\ell + 2 - 2\ell &&= 4\ell + 2,
\end{aligned}$$

with Remark 3.2.1, and the fact that $1 = \left\lceil \frac{1}{\ell} \right\rceil$ and $1 \leq \left\lceil \frac{2}{\ell} \right\rceil \leq 2$ yields

$$\begin{array}{rcccccccl}
 & & w^*(d'_1, \ell) & = & \left(\frac{1}{2}\right)^{\operatorname{dist}(3\ell+1,\ell)-1} & = & \left(\frac{1}{2}\right)^{1+\left\lceil \frac{1}{\ell} \right\rceil} & = & \frac{1}{4},\\
 & & w^*(d'_1, 2\ell) & = & \left(\frac{1}{2}\right)^{\operatorname{dist}(3\ell+1,2\ell)-1} & = & \left(\frac{1}{2}\right)^{\left\lceil \frac{1}{\ell} \right\rceil} & = & \frac{1}{2}\\
\frac{1}{64} & \leq & w^*(d'_2, \ell) & = & \left(\frac{1}{2}\right)^{\operatorname{dist}(6\ell+2,\ell)-1} & = & \left(\frac{1}{2}\right)^{4+\left\lceil \frac{2}{\ell} \right\rceil} & \leq & \frac{1}{32},\\
\frac{1}{32} & \leq & w^*(d'_2, 2\ell) & = & \left(\frac{1}{2}\right)^{\operatorname{dist}(6\ell+2,2\ell)-1} & = & \left(\frac{1}{2}\right)^{3+\left\lceil \frac{2}{\ell} \right\rceil} & \leq & \frac{1}{16}.
\end{array}$$

Therefore,

$$w^*(D \setminus d_1, \ell) \quad \leq \quad w^*(D' \setminus d'_1, \ell) \quad < \quad \tfrac{6}{7} \quad - \quad w^*(d'_1, \ell) \quad = \quad \tfrac{17}{28},$$

$$w^*(D \setminus d_1, 2\ell) \quad \leq \quad w^*(D' \setminus d'_1, 2\ell) \quad < \quad \tfrac{6}{7} \quad - \quad w^*(d'_1, 2\ell) \quad = \quad \tfrac{5}{14},$$

$$w^*(D \setminus d_2, \ell) \quad \leq \quad w^*(D' \setminus d'_2, \ell) \quad < \quad \tfrac{6}{7} \quad - \quad w^*(d'_2, \ell) \quad = \quad \tfrac{377}{448},$$

$$w^*(D \setminus d_2, 2\ell) \quad \leq \quad w^*(D' \setminus d'_2, 2\ell) \quad < \quad \tfrac{6}{7} \quad - \quad w^*(d'_2, 2\ell) \quad = \quad \tfrac{185}{224}.$$

Therefore (ii) and (iv) have been verified. Observe that (iii) and (v) are a symmetric argument to (ii) and (iv), respectively. □

Given an exponential dominating set D, the following algorithm details the process in how to construct a new exponential dominating set of the same size. With respect to D, this new exponential dominating set has less intervals that contain no exponential dominating vertices, or has less exponential dominating vertices contained in each interval.

Algorithm 3.2.6. Consider D, an exponential dominating set for $C_{n,[\ell]}$, and the partition $\mathcal{I}$ such that $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$, where $I_i = [(3\ell+1)i, (3\ell+1)i+3\ell]$. For the sake of simplicity, let $f(D) = f(D,\mathcal{I})$, $f^*(D) = f^*(D,\mathcal{I})$ and $Z(D) = Z(D,\mathcal{I})$. Suppose that $3 \leq f^*(D)$. Observe that by Remark 3.2.4, $Z(D) \geq 2$. Without loss of generality, assume that for $0 < b \leq m$, the intervals I_0 and I_b have that $f_0(D) = f_b(D) = 0$. Find the interval I_a such that $a = \min\{1, 2, \ldots, b-1\}$ and $f_a(D) \geq 3$. Furthermore, assume that the remaining $0 < i < b$ have $f_i(D) \geq 1$. Without loss of generality suppose that $a \leq b - a$ (use a reflection if necessary). Observe that there are at least $f^*(D) + b - 2$ exponential dominating vertices contained in $[0, (3\ell+1)b+3\ell]$. We identify the three closest members of $I_a \cap D$ to $(3\ell+1)a + 2\ell$, and the closest member to $(3\ell+1)i + 2\ell$ within $I_i \cap D$ for $1 \leq i \neq a \leq b$, to be defined as $P = \{d_0, d_1, \ldots, d_b\}$ for which

$$d_i \in \begin{cases} I_{i+1} \cap D & \text{if } 0 \leq i \leq a-2 \\ I_a \cap D & \text{if } a-1 \leq i \leq a+1 \\ I_{i-1} \cap D & \text{if } a+2 \leq i \leq b. \end{cases}$$

Then define $S = \{s_0, s_1, \ldots, s_b\}$, such that $s_t = (3\ell+1)t + 2\ell$, and output the set $D' = (D \setminus P) \cup S$.

Lemma 3.2.7. *Given an exponential dominating set $D \subset V(C_{n,[\ell]})$, and the partition $\mathcal{I}$ so that $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$, where $I_i = [(3\ell+1)i, (3\ell+1)i+3\ell]$ and $3 \leq f^*(D,\mathcal{I}) \leq 3\ell+1$, Algorithm 3.2.6 outputs the exponential dominating set D' such that $|D| = |D'|$, $Z(D',\mathcal{I}) = Z(D,\mathcal{I}) - 2$, and $f^*(D,\mathcal{I}) - 2 \leq f^*(D',\mathcal{I}) \leq f^*(D,\mathcal{I})$.*

Proof. For the sake of simplicity, let $f(D) = f(D,\mathcal{I})$, $f^*(D) = f^*(D,\mathcal{I})$ and $Z(D) = Z(D,\mathcal{I})$. Notice that by construction, $|D| = |D'|$ and $\mathrm{dist}_H(s_i, s_{i+1}) = 3\ell + 1$ for each consecutive pair

$s_i, s_{i+1} \in S$. Through applications of Remark 3.2.2, all the vertices in $[s_0, s_b]$ are exponentially dominated by vertices of S. Let $V' = V(C_{n,[\ell]}) \setminus [s_0, s_b]$. We define $V_L = \{v \in V' : \text{dist}_H(v, d_{a+1}) \leq \text{dist}_H(v, d_{a+1} + 1)\}$ and $V_R = V' \setminus V_L$. There are four cases:

1. Consider the case when $v \in V_L$. Then $w^*(d_i, v) \geq w^*(d_{i+1}, v)$ for $0 \leq i \leq b-1$. By construction, $w^*(d_{a-2}, v) + w^*(d_{a-1}, v) \leq 2w^*(d_{a-2}, v) \leq w^*(s_{a-2}, v)$ and $w^*(d_a, v) + w^*(d_{a+1}, v) \leq 2w^*(d_a, v) \leq w^*(s_{a-1}, v)$. Additionally, $w^*(d_i, v) \leq w^*(s_i, v)$ for $0 \leq i \leq a-3$ and $w^*(d_i, v) \leq w^*(s_{i-2}, v)$, for $a+2 \leq i \leq b$. Figure 3.2 visually shows that $w^*(P \cap D, v) < w^*(S, v)$. Then putting it together gives that

$$\begin{aligned} w^*(P \cap D, v) &= \sum_{k=0}^{a-3} w^*(d_k, v) + \sum_{k=a-2}^{a-1} w^*(d_k, v) + \sum_{k=a}^{a+1} w^*(d_k, v) + \sum_{k=a+2}^{b} w^*(d_k, v) \\ &< \sum_{k=0}^{a-3} w^*(s_k, v) + w^*(s_{a-2}, v) + w^*(s_{a-1}, v) + \sum_{k=a}^{b-2} w^*(s_k, v) \\ &< w^*(S, v). \end{aligned}$$

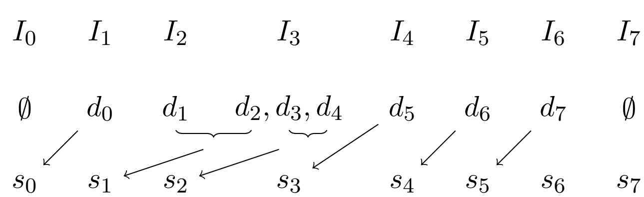

Figure 3.2 Illustration of Case 1 with $a = 3, b = 7$

2. Consider the case when $v \in V_R$ such that $w^*(d_i, v) \leq w^*(d_{i+1}, v)$ for $0 \leq i \leq b-1$. Then $\text{dist}_H(d_{a+1}, s_{a+1}) \geq 2\ell + 1$, which implies that $\sum_{k=a-1}^{a+1} w^*(d_k, v) \leq 3w^*(d_{a+1}, v) < w^*(s_{a+1}, v)$. Also by construction, $w^*(d_i, v) \leq w^*(s_{i+2}, v)$, for $0 \leq i \leq a-2$ and $w^*(d_i, v) \leq w^*(s_i, v)$, for $a+2 \leq i \leq b$. Figure 3.3 visually shows that $w^*(P \cap D, v) < w^*(S, v)$. Then it follows that

$$\begin{aligned} w^*(P \cap D, v) &= \sum_{k=0}^{a-2} w^*(d_k, v) + \sum_{k=a-1}^{a+1} w^*(d_k, v) + \sum_{k=a+2}^{b} w^*(d_k, v) \\ &< \sum_{k=2}^{a} w^*(s_k, v) + w^*(s_{a+1}, v) + \sum_{k=a+2}^{b} w^*(s_k, v) \\ &< w^*(S, v). \end{aligned}$$

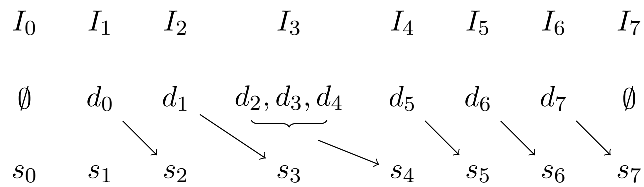

Figure 3.3 Illustration of Case 2 with $a = 3, b = 7$.

3. Consider the case when $v \in V_R$ such that $w^*(d_i, v) \geq w^*(d_{i+1}, v)$ for $0 \leq i < a$ and $w^*(d_i, v) \leq w^*(d_{i+1}, v)$ for $a + 1 \leq i < b - 1$. By construction, $w^*(d_{a-1}, v) + w^*(d_a, v) \leq 2w^*(d_{a-1}, v) \leq w^*(s_{a-1}, v)$. Additionally we have that $w^*(d_i, v) \leq w^*(s_i, v)$, for $0 \leq i \neq a - 1, a \leq b$. Figure 3.4 visually shows that $w^*(P \cap D, v) < w^*(S, v)$. Putting it together gives

$$\begin{aligned} w^*(P \cap D, v) &= \sum_{k=0}^{a-2} w^*(d_k, v) + \sum_{k=a-1}^{a} w^*(d_k, v) + \sum_{k=a+1}^{b} w^*(d_k, v) \\ &< \sum_{k=0}^{a-2} w^*(s_k, v) + w^*(s_{a-1}, v) + \sum_{k=a+1}^{b} w^*(s_k, v) \\ &< w^*(S, v). \end{aligned}$$

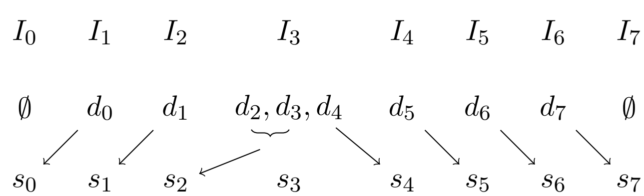

Figure 3.4 Illustration of Case 3 with $a = 3, b = 7$.

4. Consider the case when $v \in V_R$ such that $w^*(d_i, v) \geq w^*(d_{i+1}, v)$ for $0 \leq i < a - 1$ and $w^*(d_i, v) \leq w^*(d_{i+1}, v)$ for $a \leq i < b - 1$. By construction, $w^*(d_a, v) + w^*(d_{a+1}, v) \leq 2w^*(d_{a+1}, v) \leq w^*(s_{a+1}, v)$. Additionally we have that $w^*(d_i, v) \leq w^*(s_i, v)$, for $0 \leq i \neq$

$a, a+1 \leq b$. Figure 3.5 visually shows that $w^*(P \cap D, v) < w^*(S, v)$. Putting it together gives

$$\begin{aligned} &< \textstyle\sum_{k=0}^{a-1} w^*(s_k, v) \quad + \quad w^*(s_{a+1}, v) \quad + \quad \displaystyle\sum_{k=a+2}^{b} w^*(s_k, v) \\ &< w^*(S, v). \end{aligned}$$

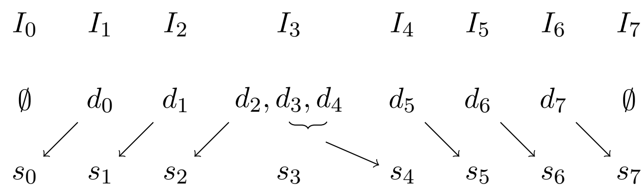

Figure 3.5 Illustration of Case 4 with $a = 3, b = 7$.

In each instance, we have shown that $|D| = |D'|$, $w^*(D', i) \geq 1$ for every $i \in [s_0, s_b]$, and $w^*(D, v) \leq w^*(D', v)$ for every $v \in V'$. Therefore D' is an exponential dominating set. By construction, it follows that I_0 and I_b have that $f_0(D') = f_0(D) + 1$ and $f_b(D') = f_b(D) + 1$. Furthermore, all remaining I_i where $i \neq 0, b$ have that $f_i(D) = f_i(D')$. Therefore $Z(D') = Z(D) - 2$ and $f^*(D) - 2 \leq f^*(D') \leq f^*(D)$. $\square$

The following lemma shows that if $D \subset V(C_{n,[\ell]})$ has the property that one interval contains three members of D, two intervals that contain no members of D, and all remaining intervals have one member of D, then D cannot be an exponential dominating set.

Lemma 3.2.8. *Consider $D \subset V(C_{n,[\ell]})$ and the partition $\mathcal{I}$ such that $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$, where $I_i = [(3\ell+1)i, (3\ell+1)i+3\ell]$ and $f^*(D, \mathcal{I}) = 3$. Assume that $I_i, I_j \subset \mathcal{I}$ are cyclically consecutive intervals for which $f_i(D, \mathcal{I}) = f_j(D, \mathcal{I}) = 0$. Furthermore suppose that there exist $I_k \subset \mathcal{I}$ for which $i < k < j$ and $f_k(D, \ell) = 3$, and all remaining intervals $I_t \subset \mathcal{I}$ have $f_t(D, \mathcal{I}) = 1$. Then D cannot be an exponential dominating set.*

Proof. For the sake of simplicity, let $f^*(D) = f^*(D, \mathcal{I})$ and $Z(D) = Z(D, \mathcal{I})$. Without loss of generality, assume that for $0 < a < b < m$, the intervals I_0 I_a, and I_b have $f_0(D) = f_b(D) = 0$,

$f_a(D) = 3$. Furthermore, assume that the remaining intervals I_t have that $f_t(D) = 1$. Let $I_a \cap D = \{d_1, d_2, d_3\}$ and without loss of generality, consider d_1, d_2. Notice that there is exactly one member of $D \setminus \{d_1, d_2\}$ in every nonempty interval, so $f^*(D \setminus \{d_1, d_2\}) = 1$ and $Z(D \setminus \{d_1, d_2\}) \geq 1$. By (i) of Lemma 3.2.5, it follows that $w^*(D \setminus \{d_1, d_2\}, \ell), w^*(D \setminus \{d_1, d_2\}, (3\ell+1)b+2\ell) < \frac{6}{7}$. This implies that $w^*(\{d_1, d_2\}, \ell)$, $w^*(\{d_1, d_2\}, (3\ell+1)b+2\ell) > \frac{1}{7}$. The only case in which both conditions hold is when $a = 1$ and $b = 2$. Among such D, we choose D' to maximize $w^*(D', \ell) + w^*(D', (3\ell+1)b+2\ell)$. See Figure 3.6 for an illustration of D'. Let $k_0 = \left\lfloor \frac{m}{2} \right\rfloor + 1$ and consider the case when m is odd.

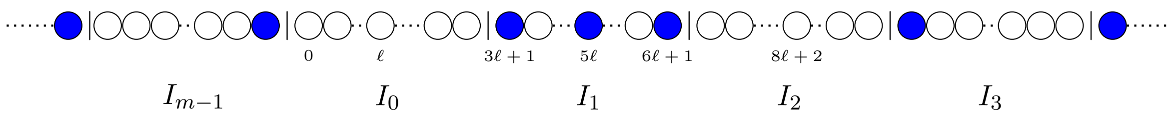

Figure 3.6 Visualization of D', with edges removed and members of D' colored

Then,

$$I_k \cap D' = \begin{cases} \{3\ell+1, 5\ell, 6\ell+1\} & \text{if } k = 1 \\ (3\ell+1)k & \text{if } 2 < k \leq k_0 \\ (3\ell+1)k + 3\ell & \text{if } k_0 < k \leq m. \end{cases}$$

If m is even,

$$I_k \cap D' = \begin{cases} \{3\ell+1, 5\ell, 6\ell+1\} & \text{if } k = 1 \\ (3\ell+1)k & \text{if } 2 < k < k_0 \\ (3\ell+1)k + 2\ell - 1 & \text{if } k = k_0 \\ (3\ell+1)k + 3\ell & \text{if } k_0 < k \leq m. \end{cases}$$

We now compute the length of the shortest path from ℓ to $5\ell, 6\ell+1$ and from $8\ell+2$ to $3\ell+1, 5\ell$. Then notice that with respect to V_H, $\text{dist}_H(5\ell, \ell) = 4\ell$, $\text{dist}_H(6\ell+1, \ell) = 5\ell+1$, $\text{dist}_H(3\ell+1, 8\ell+2) = 5\ell+1$, and $\text{dist}_H(5\ell, 8\ell+2) = 3\ell+2$. Using Remark 3.2.1 and the fact that $1 = \left\lceil \frac{1}{\ell} \right\rceil$ and $1 \leq \left\lceil \frac{2}{\ell} \right\rceil \leq 2$, it follows that

$$\begin{array}{lclcl} w^*(5\ell,\ell) & = & \left(\frac{1}{2}\right)^{\mathrm{dist}(5\ell,\ell)-1} & = & \left(\frac{1}{2}\right)^{3} & = & \frac{1}{8} \\ w^*(6\ell+1,\ell) & = & \left(\frac{1}{2}\right)^{\mathrm{dist}(6\ell+1,\ell)-1} & = & \left(\frac{1}{2}\right)^{4+\lceil\frac{1}{\ell}\rceil} & = & \frac{1}{32} \\ w^*(3\ell+1,8\ell+2) & = & \left(\frac{1}{2}\right)^{\mathrm{dist}(3\ell+1,8\ell+2)-1} & = & \left(\frac{1}{2}\right)^{4+\lceil\frac{1}{\ell}\rceil} & = & \frac{1}{32} \\ w^*(5\ell,8\ell+2) & = & \left(\frac{1}{2}\right)^{\mathrm{dist}(5\ell,8\ell+2)-1} & = & \left(\frac{1}{2}\right)^{2+\lceil\frac{2}{\ell}\rceil} & \leq & \frac{1}{8}. \end{array}$$

Therefore $w^*(\{5\ell,6\ell+1\},\ell), w^*(\{3\ell+1,5\ell\},8\ell+2) \leq \frac{5}{32}$. Regardless of the parity of m, there is exactly one member of $D' \setminus \{3\ell+1,5\ell\}$ and one member of $D' \setminus \{5\ell,6\ell+1\}$ in every nonempty interval. Therefore $f^*(D' \setminus \{3\ell+1,5\ell\}) = f^*(D' \setminus \{5\ell,6\ell+1\}) = 1$ and $Z(D' \setminus \{3\ell+1,5\ell\})$, $Z(D' \setminus \{5\ell,6\ell+1\}) \geq 0$. Then by (iv) and (v) of Lemma 3.2.5, it follows that $w^*(D' \setminus \{5\ell,6\ell+1\},\ell), w^*(D' \setminus \{3\ell+1,5\ell\},8\ell+2) < \frac{377}{448}$. Therefore

$$w^*(D,\ell) \leq w^*(D' \setminus \{5\ell,6\ell+1\},\ell) + w^*(\{5\ell,6\ell+1\},\ell) = \frac{447}{448},$$

$$w^*(D,8\ell+2) \leq w^*(D' \setminus \{3\ell+1,5\ell\},8\ell+2) + w^*(\{3\ell+1,5\ell\},8\ell+2) = \frac{447}{448}.$$

Putting it together gives that $w^*(D,\ell) + w^*(D,8\ell+2) = \frac{447}{224} < 2$. As it is not possible for ℓ and $8\ell+2$ to receive sufficient weight from D', it follows that D cannot be an exponential dominating set. □

Consider the situation when there are at most two members of $D \subset V(C_{n,[\ell]})$ in each interval. The following lemma shows that for every interval that contains no members of D, there must be an adjacent interval that contains two members of D.

Lemma 3.2.9. *Let D be a minimum exponential dominating set for $C_{n,[\ell]}$ and $\mathcal{I}$ be a partition such that $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$, where $I_i = [(3\ell+1)i, (3\ell+1)i+3\ell]$ and $f^*(D,\mathcal{I}) = 2$. Then for every $0 \leq j < m$ for which $f_j(D,\mathcal{I}) = 0$, there exist $k \equiv j \pm 1 \mod m$ such that $f_k(D,\mathcal{I}) = 2$. Moreover, $|\{k : f_k(D,\mathcal{I}) = 2\}| = Z(D,\mathcal{I})$.*

Proof. For the sake of simplicity, let $f(D) = f(D,\mathcal{I})$, $f^*(D) = f^*(D,\mathcal{I})$, $z(D) = z(D,\mathcal{I})$, and $Z(D) = Z(D,\mathcal{I})$. Let $K = \{k : f_k(D) = 2\}$. As Remark 3.2.3 shows that $|D| \leq m$, it follows that for every $k \in K$ there must exist a distinct $z \in z(D)$. Therefore we have that $|K| \leq Z(D)$. Let $I_k \cap D = \{d_k, d'_k\}$ for every $k \in K$ and let $P = P_1 \cup P_2$ such that $P_1 = \{d'_k : k \in K\}$ and $P_2 = \{d_k : k \in K\}$. Without loss of generality suppose that $0 \in z(D)$. Then the interval I_0 has that $f_0(D) = 0$. Let $\hat{D}$ be an exponential dominating set such that $f_0(\hat{D}) = 0$, $f_1(\hat{D}) = f_{m-1}(\hat{D}) = 1$, and $f_i(\hat{D}) = 2$ for $2 \leq i \leq m-2$. Among such $\hat{D}$, choose D' to maximize $w^*(D', 2\ell)$. Let $I_k \cap D' = \{s_k, s'_k\}$ such that $w^*(s_k, 2\ell) \leq w^*(s'_k, 2\ell)$ and without loss of generality, let $P'_1 = \{s'_k : 2 \leq k \leq m-2\}$. By construction, it follows that $w^*(D \setminus P_1, 2\ell) \leq w^*(D' \setminus P'_1, 2\ell)$. Notice that there is exactly one member of $D' \setminus P'_1$ in every nonempty interval, so $f^*(D' \setminus P'_1) = 1$ and $Z(D' \setminus P'_1) \geq 1$. Therefore by Lemma (i) of 3.2.5, $w^*(D' \setminus P'_1, 2\ell) < \frac{6}{7}$. Putting it together gives $w^*(D \setminus P_1, 2\ell) < \frac{6}{7}$. Let $k_0 = \left\lfloor \frac{m}{2} \right\rfloor$, then the choice of D' implies that

$$I_k \cap P'_1 = \begin{cases} (3\ell+1)k & \text{if } 2 \leq k \leq k_0 \\ (3\ell+1)k + 3\ell & \text{if } k_0 < k \leq m-2. \end{cases}$$

Consider $2 \leq k \leq k_0$. Then using (3.2.1) and (3.2.2), it follows that

$$\begin{aligned} \sum_{k=2}^{k_0} w^*(I_k \cap P'_1, 2\ell) &< \sum_{k=1}^{\infty} \left(\frac{1}{2}\right)^{\operatorname{dist}(2\ell,(3\ell+1)k)-1} - \left(\frac{1}{2}\right)^{\operatorname{dist}(2\ell,(3\ell+1))-1} \\ &\leq \frac{4}{7} - \frac{1}{2} \\ &= \frac{1}{14}. \end{aligned} \tag{3.2.5}$$

Now consider $k_0 < k \leq m-2$ and let $k' = m - k$. Notice since k and k' are counters, (3.2.1) and (3.2.4) only differ by ℓ. Furthermore, summing from $k = k_0 + 1$ to $m-2$ is the same as summing k' from 2 to $m - k_0 - 1$, which equals k_0 or $k_0 - 1$. Therefore using (3.2.5), it follows that

$$w^*(P_1, 2\ell) \quad \leq \quad w^*(P'_1, 2\ell) \quad < \quad \frac{3}{2}\sum_{k=2}^{k_0} w^*(I_k \cap P'_1, 2\ell) \quad < \quad \frac{3}{28}. \tag{3.2.6}$$

Thus, $w^*(D, 2\ell) = w^*(D \setminus P_1, 2\ell) + w^*(P_1, 2\ell) < \frac{27}{28}$, which contradicts the assumption that D is an exponential dominating set. Through a symmetric argument, it can be shown that $w^*(D, \ell) < \frac{27}{28}$. Therefore either $1 \in K$, or $m-1 \in K$. In general, this shows that for every $z \in z(D)$, there exist $k \in K$ such that $k \equiv z \pm 1 \mod m$. Without loss of generality, suppose that $k \equiv z+1 \mod n$. Then, $z-1$

mod $n \notin K$, else there will exist $z_0 \in z(D)$ for which $w^*(D,(3\ell+1)z_0+\ell), w^*(D,(3\ell+1)z_0+2\ell) < \frac{27}{28}$. Thus, $|K| = Z(D)$. $\square$

The next lemma extends the result of Lemma 3.2.10 by determining the location of exponential dominating vertices in the intervals to either side of an interval that contains no members of D.

Lemma 3.2.10. *Let D be an exponential dominating set for $C_{n,[\ell]}$ and $\mathcal{I}$ be a partition such that $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$, where $I_i = [(3\ell+1)i, (3\ell+1)i + 3\ell]$ and $f^*(D,\mathcal{I}) = 2$. Consider the interval I_j with $f_j(D,\mathcal{I}) = 0$. Let $k \equiv j-1 \mod m$ and $k' \equiv j+1 \mod m$. Then there are either two members of D contained in $[(3\ell+1)k+2\ell+1, (3\ell+1)k+3\ell]$ and one member of D contained in $[(3\ell+1)k', (3\ell+1)k'+\ell-1]$, or two members of D contained in $[(3\ell+1)k', (3\ell+1)k'+\ell-1]$ and one member of D contained in $[(3\ell+1)k+2\ell+1, (3\ell+1)k+3\ell]$.*

Proof. For the sake of simplicity, let $f(D) = f(D,\mathcal{I})$, $f^*(D) = f^*(D,\mathcal{I})$, $z(D) = z(D,\mathcal{I})$, and $Z(D) = Z(D,\mathcal{I})$. Let $K = \{k : f_k(D) = 2\}$ and let $I_k \cap D = \{d_k, d'_k\}$ for every $k \in K$. Define $P = P_1 \cup P_2$ such that $P_1 = \{d'_k : k \in K\}$ and $P_2 = \{d_k : k \in K\}$. Without loss of generality, consider P_1 and notice that there is exactly one member of $D \setminus P_1$ in every nonempty interval, so $f^*(D \setminus P_1) = 1$ and $Z(D \setminus P_1) \geq 1$. By (i) of Lemma 3.2.5, every $z \in z(D)$ has that $w^*(D \setminus P_1, (3\ell+1)z+\ell), w^*(D \setminus P_1, (3\ell+1)z+2\ell) < \frac{6}{7}$. To maintain that (D, w^*) dominates $C_{n,[\ell]}$, it follows that $w^*(P_1, (3\ell+1)z+\ell), w^*(P_1, (3\ell+1)z+2\ell) > \frac{1}{7}$. Without loss of generality assume $0 \in z(D)$. Then Lemma 3.2.9 shows that either $1 \in K$ or $m-1 \in K$. Suppose $1 \in K$ and consider $d_1 \in P_2$. Then by (ii) of Lemma 3.2.5, $w^*(D \setminus (P_1 \cup d_1), \ell) < \frac{17}{28}$ and $w^*(D \setminus (P_1 \cup d_1), 2\ell) < \frac{5}{14}$. To ensure that ℓ and 2ℓ receive sufficient weight from D, the following conditions must hold

$$w^*(P_1 \cup d_1, \ell) > \frac{11}{28}, \tag{3.2.7}$$

$$w^*(P_1 \cup d_1, 2\ell) > \frac{9}{14}. \tag{3.2.8}$$

Since $w^*(P_1, \ell), w^*(P_1, 2\ell) > \frac{1}{7}$, it follows that $w^*(d_1, \ell) \geq \frac{1}{4}$ and $w^*(d_1, 2\ell) \geq \frac{1}{2}$, satisfy (3.2.7) and (3.2.8). This implies that $d_1 \in [3\ell+1, 4\ell]$. Let $d_{m-1} = I_{m-1} \cap D$ and note that a similar argument gives that $d_{m-1} \in [(3\ell+1)(m-1)+2\ell+1, (3\ell+1)(m-1)+3\ell]$. Additionally, through a similar argument with respect to P_2, it can be shown that $d'_1 \in [3\ell+1, 4\ell]$. Now consider the case when $m-1 \in K$. For $d_{m-1}, d'_{m-1} \in I_{m-1} \cap D$ and $d_1 \in I_1 \cap D$, a symmetric argument gives that $d_{m-1}, d'_{m-1} \in [(3\ell+1)(m-1)+2\ell+1, (3\ell+1)(m-1)+3\ell]$ and $d_1 \in [3\ell+1, 4\ell]$. □

The following lemma shows that if there are two exponential dominating vertices that are within a certain distance of each other, then there exists a shift of these two vertices that creates a new exponential dominating set.

Lemma 3.2.11. *Let D be an exponential dominating set for $C_{n,[\ell]}$. Suppose that there exists $i, j \in D$ such that $i < j$ and $\operatorname{dist}_H(i,j) \leq \ell+1$. Consider $S = (V(C_{n,[\ell]}) \setminus D) \cap [j+\ell, i-\ell]$. Let $a_0, b_0 \in S$ so that $\operatorname{dist}_H(a_0, i-\ell) < \operatorname{dist}_H(a, i-\ell)$ for every $a \in S \setminus a_0$ and $\operatorname{dist}_H(b_0, j+\ell) < \operatorname{dist}_H(b, j+\ell)$ and for every $b \in S \setminus b_0$. Then $D' = (D \setminus \{i,j\}) \cup \{a,b\}$ is an exponential dominating set.*

Proof. Consider $D' = (D \setminus \{i,j\}) \cup \{a_0, b_0\}$. As $\operatorname{dist}_H(i-\ell, j+\ell) \leq 3\ell+1$, Remark 3.2.2 shows that $i-\ell, j+\ell$ exponentially dominates $[i-\ell, j+\ell]$. Then $w^*(D', u) \geq 1$ for all $u \in [a_0, b_0]$. Let $v \in V(C_{n,[\ell]}) \setminus [a_0, b_0]$ and without loss of generality, suppose that $w^*(i,v) \geq w^*(j,v)$. Observe that $w^*(i,v) + w^*(j,v) \leq 2w^*(i,v) \leq w^*(a_0, v)$, which implies that $w^*(D,v) \leq w^*(D',v)$. Thus D' is an exponential dominating set. □

3.2.2 Main Results

The main results of this paper consists of the following two theorems. Theorem 3.2.12 determines the structure of the minimum porous exponential dominating set for $C_{n,[\ell]}$, when $3\ell+1$ divides n. In this proof, all but one case is shown to either have a porous exponential dominating set that is not minimum, or to have a set of vertices that is not a porous exponential dominating set. Theorem 3.1.2 determines the explicit formula for $\gamma_e^*(C_{n,[\ell]})$ and $\gamma_e(C_{n,[\ell]})$. In this proof Theorem 3.2.12 and Remark 3.2.2 to determine a lower bound for $\gamma_e^*(C_{n,[\ell]})$ and upper bound for $\gamma_e(C_{n,[\ell]})$, respectively. Additionally (3.1.1) is used to link $\gamma_e^*(C_{n,[\ell]})$ and $\gamma_e(C_{n,[\ell]})$.

Theorem 3.2.12. *Let $n = (3\ell+1)m \geq 6\ell+2$. Let D be a minimum exponential dominating set for $C_{n,[\ell]}$, and $\mathcal{I}$ be a partition such that $\mathcal{I} = \bigcup_{i=0}^{m-1} I_i$, where $I_i = [(3\ell+1)i, (3\ell+1)i+3\ell]$. Then $f^*(D,\ell) = 1$ and $Z(D,\ell) = 0$ for any partition $\mathcal{I}$. Furthermore, D is unique up to isomorphism.*

Proof. Let D be an exponential dominating set for $C_{n,[\ell]}$. For the sake of simplicity, let $f(D) = f(D,\mathcal{I})$, $f^*(D) = f^*(D,\mathcal{I})$ and $Z(D) = Z(D,\mathcal{I})$. Through induction, we show the contrapositive of the statement: if $2 \leq f^*(D)+Z(D) \leq 3\ell+1$, then D cannot be a minimum exponential dominating set.

BC 1 Suppose that $f^*(D) \geq 2$ and $Z(D) = 0$. Through counting the exponential dominating vertices, it follows that $|D| \geq m+1$. Remark 3.2.3, shows that there exists an exponential dominating set D^* for $C_{n,[\ell]}$ such that $|D^*| = m$. Therefore D cannot be a minimum exponential dominating set.

BC 2 Suppose that $f^*(D) = 1$ and $Z(D) \geq 1$. By (i) of Lemma 3.2.5, there exists $2\ell \in V(C_{n,[\ell]})$ such that $w^*(D,2\ell) < \frac{6}{7}$. Thus it is not possible for 2ℓ to receive sufficient weight from D, which implies that D is not an exponential dominating set.

Assume that if $2 \leq f^*(D)+Z(D) < \alpha$, then D is not a minimum exponential dominating set. Now suppose $f^*(D)+Z(D) = \alpha$. We have the following three cases:

1. Suppose that $4 \leq f^*(D) \leq 3\ell+1$. Then by Remark 3.2.4 we have that $Z(D) \geq 3$. With D and $\mathcal{I}$, we construct $D' \subset V(C_{n,[\ell]})$ using Algorithm 3.2.6. Then Lemma 3.2.7 shows that D' is an exponential dominating set such that $Z(D') = Z(D)-2$, $f^*(D)-2 \leq f^*(D') \leq f^*(D)$, and $|D| = |D'|$. Therefore $Z(D') \geq 1$ and $2 \leq f^*(D)-2 \leq f^*(D') \leq f^*(D)$. This implies that $3 \leq f^*(D')+Z(D') \leq \alpha-2$. By the induction hypothesis, D' is not a minimum exponential dominating set. Thus D cannot be a minimum exponential dominating set.

2. Suppose that $f^*(D) = 3$. Then by Remark 3.2.4, $Z(D) \geq 2$. With D and $\mathcal{I}$, construct $D' \subset V(C_{n,[\ell]})$ using Algorithm 3.2.6. Then Lemma 3.2.7 shows that D' is an exponential dominating set such that $Z(D') = Z(D)-2$, $f^*(D)-2 \leq f^*(D') \leq f^*(D)$, and $|D| = |D'|$. Therefore $Z(D') \geq 0$ and $1 \leq f^*(D') \leq 3$. Consider the following two subcases:

(a) Consider the case when $f^*(D') \geq 1$ and $Z(D') \geq 1$. Then we have that $2 \leq f^*(D') + Z(D') \leq \alpha - 2$. By the induction hypothesis, D' is not a minimum exponential dominating set. Thus D cannot be a minimum exponential dominating set.

(b) Consider the case when $f^*(D') = 1$ and $Z(D') = 0$. This implies that there exists $I_i, I_j, I_k \subset \mathcal{I}$ for which $f_i(D) = f_k(D) = 0$, $f_j(D) = 3$, and $f_t(D) = 1$ for all remaining $I_t \subset \mathcal{I}$. By Lemma 3.2.8, D is not an exponential dominating set.

3. Suppose that $f^*(D) = 2$. Then $Z(D) \geq 1$ by Remark 3.2.4. Without loss of generality we assume that the interval $I_0 = [0, 3\ell]$ has $f_0(D) = 0$. Lemma 3.2.9 show that either the intervals I_1 and I_{m-1} have that $f_1(D) = 2$ or $f_{m-1}(D) = 2$. Without loss of generality, suppose that $f_1(D) = 2$. Let $I_1 \cap D = \{d_0, d_1\}$ and let $d_i = I_i \cap D$ for all $2 \leq i \leq m-1$. Then consider the following two cases:

(a) Suppose that $Z(D) \geq 2$. By Lemma 3.2.10, $d_0, d_1 \in [3\ell + 1, 4\ell]$. Consider $D' = (D \setminus \{d_0, d_1\}) \cup \{d_0 - \ell, d_1 + \ell\}$ and Lemma 3.2.11 shows that D' is an exponential dominating set. By construction, we know that $d_0 - \ell \in I_0$ and $d_1 + \ell \in I_1$. Therefore we have that $|D| = |D'|$, $f_1(D') = 1$, $f_0(D') = 1$, and $f_t(D') = f_t(D)$ for all remaining $I_t \subset \mathcal{I}$. This implies that $Z(D') \geq 1$ and $1 \leq f^*(D') \leq 2$. Then we have that $2 \leq f^*(D') + Z(D') \leq \alpha - 1$. By our induction hypothesis, D' is not a minimum exponential dominating set. Thus, D cannot be a minimum exponential dominating set.

(b) Suppose that $Z(D) = 1$. Then $f_i(D) = 1$ for $2 \leq i \leq m-1$. Observe that by Lemma 3.2.10 the minimum requirement on d_0, d_1, d_{m-1} to ensure that I_0 gets exponentially dominated by D is that $d_0, d_1 \in [4\ell - 1, 4\ell]$ and $d_{m-1} = (3\ell + 1)(m - 1) + 2\ell + 1$. Through symmetry of the above argument, the minimum requirement to ensure that the interval $[4\ell + 1, 7\ell + 1]$ is exponentially dominated is that $d_2 = 8\ell + 1$. Fix j_0 such that $3 \leq j_0 \leq m-1$, and suppose that the interval $[(3\ell+1)(j_0-1)+2\ell, (3\ell+1)j_0+2\ell-1]$ contains no members of D. Let $d \in \{d_0, d_1\}$ and notice that there is one member of $D \setminus d$ in every nonempty interval. Therefore $f^*(D \setminus d) = 1$ and $Z(D \setminus d) \geq 0$. By (i) of Lemma

3.2.5, $w^*(D \setminus d, (3\ell+1)j_0 < \frac{6}{7}$. This condition forces $w^*(d, (3\ell+1)j_0) > \frac{1}{7}$. However $d \in I_1$, so $w^*(d, (3\ell+1)j_0) < \frac{1}{7}$. This gives that $w^*(D, (3\ell+1)j_0) < 1$, a contradiction. Therefore the minimum requirement to ensure $[(3\ell+1)(j_0-1)+2\ell, (3\ell+1)j_0+2\ell-1]$ is exponentially dominated is that $d_{j_0} = (3\ell+1)j_0+2\ell-1$. This implies $(3\ell+1)(m-1)+2\ell-1, (3\ell+1)(m-1)+2\ell \in I_{m-1} \cap D$, which contradicts that $f_{m-1}(D) = 1$. Therefore $(3\ell+1)(m-1)+2\ell-1 \notin D$ and $w^*(D, (3\ell+1)(m-1)) < 1$. Thus it is not possible for $(3\ell+1)(m-2)+3\ell$ to receive sufficient weight from D, which implies that D is not an exponential dominating set.

Through induction, it has been shown that if $f^*(D) + Z(D) \geq 2$, then D is not a minimum exponential dominating set. Therefore if D is an exponential dominating set, then $|D| = m$ such that $f^*(D) = 1$ and $Z(D) = 0$ for all $3\ell+1$ distinct partitions $\mathcal{I}$. What is left to show is that D is unique up to isomorphism. Suppose that $0 \in D$ and fix the remaining members of D. Let $I_0 \in \mathcal{I}$ such that $I_0 = [0, 3\ell]$. Therefore none of the remaining elements of I_0 can be members of D. Shift the partition by one step to construct the interval $I_0' = [1, 3\ell+1]$. Note that $|I_0' \cap D| = 1$ and $2, 3, \ldots, 3\ell \notin D$, so we must have $3\ell+1 \in D$. Continuing this argument gives that $D = \{(3\ell+1)k : 0 \leq k \leq m-1\}$. Thus D is unique up to isomorphism. □

Proof of Theorem 3.1.2: Let $n = (3\ell+1)m + r$ and D be a porous exponential dominating set for $C_{n,[\ell]}$ such that $|D| \leq m$. In the case where $r = 0$, Theorem 3.2.12 shows that D is a minimum porous exponential dominating set such that $|D| = m$. Remark 3.2.2 shows that D forms a non-porous exponential dominating set. Thus using (3.1.1) we have that

$$\frac{n}{3\ell+1} = \gamma_e^*(C_{n,[\ell]}) \leq \gamma_e(C_{n,[\ell]}) \leq \frac{n}{3\ell+1}.$$

Consider the case when $r > 0$. We first partition H into $m+1$ intervals. Then notice that there must be at least one interval that contains no dominating vertices. We choose the partition $\mathcal{I} = \cup_{i=0}^{m} I_i$ around H such that $I_i = [(3\ell+1)i, (3\ell+1)i+3\ell]$ for $0 \leq i \leq m-1$, $I_m = [(3\ell+1)m, (3\ell+1)m+r-1]$, and $f_{m+1}(D) = 0$. Consider the graph $C_{n',[\ell]}$, where $n' = (3\ell+1)m$. We define the vertex map $\varphi : V(C_{n,[\ell]}) \to V(C_{n',[\ell]})$ such that $\varphi(i) = i$ for every $i \in \{0, 1, \ldots, n'-1\}$.

Let $i, j \in V(C_{n,[\ell]})$. As $\text{dist}_H(\varphi(i), \varphi(j)) \leq \text{dist}_H(i, j)$, it follows that D forms an exponential dominating set for $C_{n',[\ell]}$. Theorem 3.2.12 shows that a minimum exponential domination set of for $C_{n',[\ell]}$ must have cardinality m and is unique up to isomorphism. As $|D| \leq m$, D must form a minimum exponential dominating set for $C_{n',[\ell]}$ with $|D| = m$. Without loss of generality, let $D = \{(3\ell + 1)t : 0 \leq t \leq m - 1\}$. See Figure 3.7 for an illustration of D and the mapping φ. With regards to $C_{n,[\ell]}$, D remains fixed since $I_m \cap D = \emptyset$. Consider the intervals $I_0 = [0, 3\ell]$

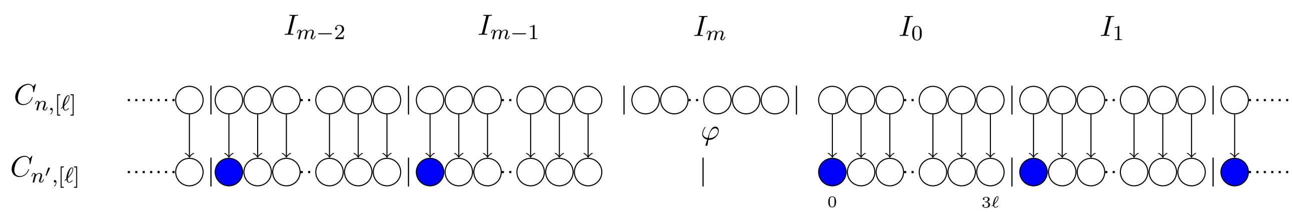

Figure 3.7 Illustration of φ, with edges removed and $D \subset V(C_{n',[\ell]})$ defined

and $I_1 = [3\ell + 1, 6\ell + 1]$. By construction, $0, 3\ell + 1 \in D$. Now shift the partition $\mathcal{I}$ so that $I_k = [(3\ell + 1)k + r + 1, (3\ell + 1)k + 3\ell + r + 1 \mod n]$ for $0 \leq k < m$ and $I_m = [(3\ell + 1)m + r + 1 \mod n, (3\ell + 1)m + 2r \mod n]$. Under φ, notice that $\text{dist}_H(\varphi(0), \varphi(3\ell)) = 3\ell + 1 - r$. This shows that D is not unique up to isomorphism in $C_{n',[\ell]}$, which contradicts Theorem 3.2.12. Therefore D cannot to be an exponential dominating set, see Figure 3.8 for an illustration of this contradiction. Consider $D' = D \cup v$, where $v \in I_m$. Observe that $\text{dist}_H(d_k, d_{k+1}) \leq 3\ell + 1$ for consecutive d_k, d_{k+1}

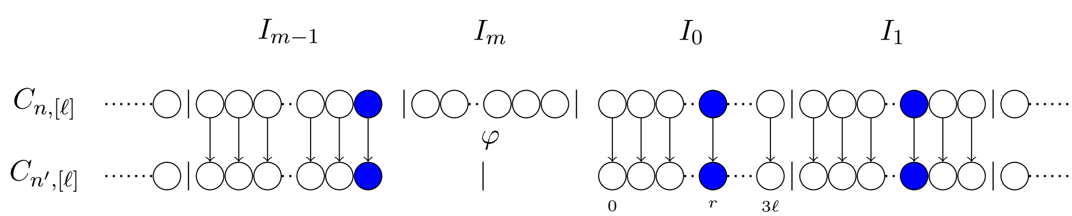

Figure 3.8 Illustration of why D is not an exponential dominating set, with edges removed

mod $n \in D'$. An application of Remark 3.2.2 shows that D' is a porous exponential dominating set for $C_{n,[\ell]}$ where $|D'| = m + 1$. Therefore D' must be minimum. Additionally, Remark 3.2.2 shows that D' forms a non-porous exponential dominating set. Thus using (3.1.1) we have that

$$\left\lceil \frac{n}{3\ell+1} \right\rceil \leq \gamma_e^*(C_{n,[\ell]}) \leq \gamma_e(C_{n,[\ell]}) \leq \left\lceil \frac{n}{3\ell+1} \right\rceil.$$

□

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3.4 Bibliography

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CHAPTER 4. CONCLUDING REMARKS

Overall, exponential domination in graphs is an interesting area within graph theory. The global influence that exponential dominating vertices exert on the other vertices in the graph is the aspect about exponential domination that sets it apart from other variants of domination. Notice that exponential domination is defined with the growth factor of $\frac{1}{2}$. In the literature, this factor has remained constant. One avenue for future work would be to explore exponential domination with a growth factor of p, where $0 < p < 1$.

In Chapter 2, a generalized linear programming method was derived to establish lower bounds for the porous exponential domination number of any graph. In particular for grid graphs, a technique to improve this lower bound using linear programming was discussed. The difference between the upper bound constructions and lower bounds derived from this technique for the porous exponential domination number of the King grid $\mathcal{K}_n$ and Slant grid $\mathcal{S}_n$ were on the order of tenths. We believe that additional constraints added to Mixed Integer Linear Program 2.2.3 will help to further increase the lower bound. In particular see Conjecture 2.5.2, which details the idea of a global distance α needed to ensure a vertex is dominated, and Conjecture 2.5.7, which suggests that there exists a minimum porous exponential dominating set with nonadjacent members, for potential constraints. Notice that the annuli and corresponding partition described in Lemma 2.5.5 and Lemma 2.5.6 did not fully utilize the properties of $\mathcal{K}_\infty$. For future work, other annuli and minimal distance between exponential dominating vertices will be examined.

In Chapter 3, it was shown that $\gamma_e^*(C_{n,[\ell]}) = \gamma_e(C_{n,[\ell]}) = \left\lceil \frac{n}{3\ell+1} \right\rceil$. A natural direction would be to determine bounds or exact values for the porous and non-porous exponential domination numbers of other types of circulant graphs, and see if the equivalence property of $C_{n,[\ell]}$ holds. A *strongly regular* graph G, is defined to be a k-regular graph with $|V(G)| = n$, with the properties that every two adjacent vertices have λ vertices in common, and every two non-adjacent vertices have μ vertices in common. Observe that $C_{n,[\ell]}$ is a strongly regular graph. Another future direction would be to further study exponential domination on strongly regular graphs. One question would be to determine whether there is an explicit formula for the porous or non-porous exponential domination number of a general strongly regular graph. Another interesting question is to find if there exists a family of strongly regular graphs $\mathcal{G}$ for which $\gamma_e(\mathcal{G}) = \gamma_e^*(\mathcal{G})$.