The Inverse Eigenvalue Problem of a Graph and Zero Forcing

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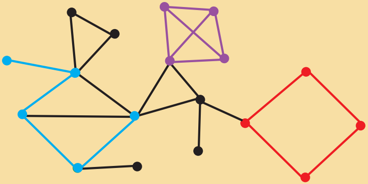
Graphs

- A graph G is a set of V vertices or nodes, together with a set E of edges.
- An edge is a 2 element subset of vertices.
- Edge {v,u} is often denoted vu.
- The order of G, $|G|$, is the number of evertices
- All graphs discussed are simple, undirected and finite.

Example

A graph showing a path P_5, cycle C_4, and complete graph K_4.



Matrices and Graphs

Matrices are real and symmetric ($a_{ji} = a_{ij}$) unless otherwise stated.
$S_n(\mathbb{R})$ is the set of $n \times n$ real symmetric matrices.

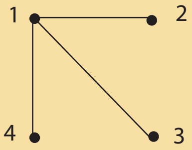
The graph $\mathcal{G}(A) = (V, E)$ of $n \times n$ matrix A is $G = (V, E)$ where

- $V = \{1, ..., n\}$,
- $E = \{ij : a_{ij} \neq 0 \text{ and } i \neq j\}$.
- Diagonal of A is ignored.

Example

$$A = \begin{bmatrix} 2 & -1 & 3 & 5 \\ -1 & 0 & 0 & 0 \\ 3 & 0 & -3 & 0 \\ 5 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{G}(A)$



Inverse Eigenvalue Problem of a Graph (IEP-G)

The family of symmetric matrices described by a graph G is

$$\mathcal{S}(G) = \{A \in S_n(\mathbb{R}) : \mathcal{G}(A) = G\}.$$

The Inverse Eigenvalue Problem of a Graph G is to determine all possible spectra (multisets of eigenvalues) of matrices in $\mathcal{S}(G)$.

Example

A matrix in $\mathcal{S}(P_3)$ is of the form

$$A = \begin{bmatrix} x & a & 0 \\ a & y & b \\ 0 & b & z \end{bmatrix} \text{ where } a, b \neq 0.$$

The possible spectra of matrices in $\mathcal{S}(P_3)$ are all sets of 3 distinct real numbers.

Applications of IEP-G

- IEP-G is motivated by inverse problems arising in the theory of vibrations.
- IEP-G where G is a path corresponds to a discretization of the inverse Sturm-Liouville problem for the string.
- This leads to the classical study of the inverse eigenvalue problem for Jacobi matrices (irreducible tridiagonal matrices).
- IEP-G can be viewed as the inverse problem for a vibrating system with prescribed structure given by G.
- IEP-G (beyond paths) has applications to modeling skyscrapers swaying in the wind.

Graphs for which IEP-G is solved

- Paths P_n.
- Cycles C_n.
- Complete graphs K_n.
- Graphs of order at most 5.

IEP-G for paths, cycles, and complete graphs

Theorem (Hochstadt, 1967)

A multiset Λ of n real numbers is the spectrum of a matrix in $\mathcal{S}(P_n)$ if and only if Λ consists of n distinct numbers.

Theorem (Ferguson 1978; Fernandes, da Fonseca, 2009)

A multiset of real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ is the spectrum of an $n \times n$ matrix $A \in \mathcal{S}(C_n)$ if and only if one of the following two conditions holds:

1) $\lambda_n > \lambda_{n-1} \geq \lambda_{n-2} > \lambda_{n-3} \geq \dots$.
2) $\lambda_n \geq \lambda_{n-1} > \lambda_{n-2} \geq \lambda_{n-3} > \dots$.

Theorem (Barrett, Lazenby, Malloy, Nelson, Sexton, 2013)

A multiset Λ of n real numbers is the spectrum of a matrix in $\mathcal{S}(K_n)$ if and only if Λ contains at least 2 distinct numbers.

Partial answers to IEP-G

- Due to the difficulty of IEP-G, various subquestions have been studied.
- Subquestions of interest include:
  - maximum multiplicity of an eigenvalue
  - minimum number of distinct eigenvalues
  - ordered multiplicity lists
- Answers to such subquestions can provide information that can be used to attack the full IEP-G.

Maximum multiplicity and minimum rank

The maximum multiplicity or maximum nullity of graph G is

$$\begin{aligned} \mathrm{M}(G) &= \max\{\mathrm{mult}_A(\lambda) : A \in \mathcal{S}(G),\ \lambda \in \mathrm{spec}(A)\}. \\ &= \max\{\mathrm{null}\, A : A \in \mathcal{S}(G)\}. \end{aligned}$$

The minimum rank of graph G is

$$\mathrm{mr}(G) = \min_{A \in \mathcal{S}(G)} \mathrm{rank}\, A.$$

By using nullity,

$$\mathrm{M}(G) + \mathrm{mr}(G) = |G|.$$

The Maximum Multiplicity Problem (or Minimum Rank Problem) for a graph G is to determine $\mathrm{M}(G)$ (or $\mathrm{mr}(G)$).

Example

Path (tridiagonal matrix)
$\mathrm{mr}(P_n) = n - 1$

Complete graph
$\mathrm{mr}(K_n) = 1$

$$A = \begin{bmatrix} ? & * & 0 & \dots & 0 & 0 \\ * & ? & * & \dots & 0 & 0 \\ 0 & * & ? & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & ? & * \\ 0 & 0 & 0 & \dots & * & ? \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & 1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & 1 \end{bmatrix}$$

$*$ is nonzero, ? is indefinite

Determination of maximum multiplicity and minimum rank

The Maximum Multiplicity Problem for a graph G is to determine $\mathrm{M}(G)$.

Despite being much simpler than IEP-G, the Maximum Multiplicity Problem has not been solved in general, although the value of $\mathrm{M}(G)$ is known for many families of graphs. For example:

- Trees, cycles, complete graphs, complete bipartite graphs, hypercubes, wheels, necklaces, halfgraphs, etc.
- Cartesian products of paths, cycles, and complete graphs.
- Line graphs of trees and graphs with a Hamiltonian path.
- Complements of trees, cycles, and 2-trees.

http://admin.aimath.org/resources/graph-invariants/minimumrankoffamilies/

Basic properties of minimum rank/maximum multiplicity

- For a graph G of order n with an edge, $1 \le \mathrm{mr}(G) \le n-1$.
- If the connected components of G are $G_1, \ldots, G_c$, then

$$\mathrm{mr}(G) = \sum_{i=1}^{c} \mathrm{mr}(G_i).$$

A graph $H = (V_H, E_H)$ is a subgraph of $G = (V_G, E_G)$, $H \le G$, if $V_H \subseteq V_G$ and $E_H \subseteq E_G$.
A subgraph $H = (V_H, E_H)$ of $G = (V_G, E_G)$ is induced if $u, v \in V_H, uv \in E_G \Rightarrow uv \in E_H$.

- If H is an induced subgraph of G, then $\mathrm{mr}(H) \le \mathrm{mr}(G)$.
- If P_k is an induced subgraph of G, then $k-1 \le \mathrm{mr}(G)$.

Tools for determining maximum nullity/minimum rank

- Edge clique covers.
- Cut vertex reduction.
- Vertex connectivity lower bound.
- Zero forcing upper bound for maximum nullity.
- Colin de Verdière type parameters.
- Software that implements some of these tools:
  https://sage.math.iastate.edu/home/pub/84/

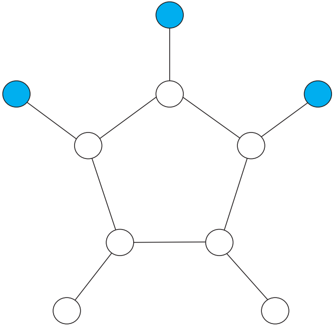
Zero Forcing

Each vertex is colored blue or white.

Color change rule

If G is a graph with each vertex colored either white or blue, u is a blue vertex of G, and exactly one neighbor w of u is white, then change the color of w to blue.

Example



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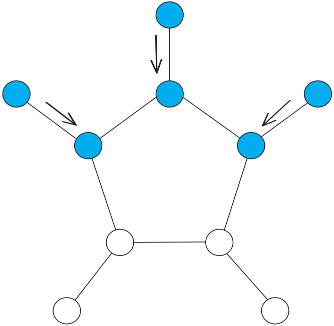

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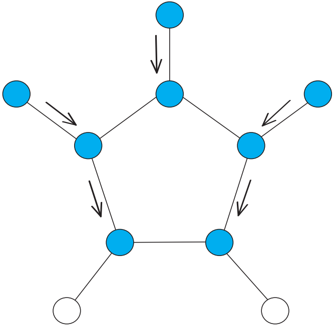

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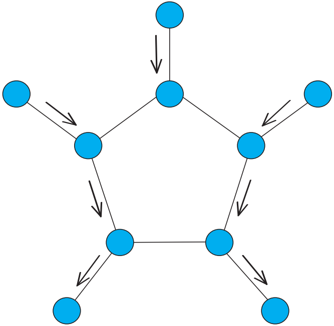


Zero Forcing Number

- Given a coloring of G, the final coloring is the result of applying the color change rule until no more changes result.
- A zero forcing set for a graph G is a subset of vertices B such that if initially the vertices in B are colored blue and the remaining vertices are colored white, the final coloring of G is all blue.
- The zero forcing number $\mathrm{Z}(G)$ is the minimum of $|B|$ over all zero forcing sets $B \subseteq V(G)$.
- 'Zero forcing' refers to forcing zeros in a null vector of a matrix described by the graph.

Theorem (AIM Special Graphs Workgroup, 2008)

For every graph G, $\mathrm{M}(G) \leq \mathrm{Z}(G)$.

Strong Arnold Property

Matrix A has the Strong Arnold Property (SAP) if the zero matrix is the only real symmetric matrix X such that

- $A \circ X = O$, $I_n \circ X = O$, and
- $AX = O$.

A Colin de Verdière type parameter of graph G is the maximum nullity of (real symmetric) matrices A having $\mathcal{G}(A) = G$, satisfying SAP, and possibly other properties.

$\xi(G) = \max\{\text{null}\, A : \mathcal{G}(A) = G \text{ and } A \text{ has SAP}\}$.

$$\xi(G) \leq \text{M}(G).$$

Subgraphs and minors

- A contraction of edge uv identifies vertices u and v; any loops or duplicate edges that arise in the process are deleted.
- H is a minor of G ($H \preceq G$) if H can be obtained from G by performing a sequence of deletions of edges, deletions of isolated vertices, and/or contractions of edges.
- A subgraph is a minor.
- A graph parameter ζ is minor monotone if $H \preceq G \Rightarrow \zeta(H) \leq \zeta(G)$.
- [Barioli, Fallat, H, 2005] ξ is minor monotone.

If we can find a subgraph (or other minor) H of G for which we know $\xi(H)$, then

$$\xi(H) \leq \xi(G) \leq \mathrm{M}(G).$$

Illustration using ξ

Example

$\mathrm{M}(K_3) = 2$.

$K_3 = C_3$ is a minor of C_n for $n \geq 3$, so $2 \leq \mathrm{M}(C_n)$.

$C_n - v = P_{n-1}$, so $n - 2 \leq \mathrm{mr}(C_n)$ and $2 \geq \mathrm{M}(C_n)$.

SAP, manifolds, and generalizations

- Matrix A has SAP if and only if the constant rank manifold and the constant pattern manifold $\mathcal{S}(\mathcal{G}(A))$ intersect transversally at A.
- SAP is used to guarantee minor monotonicity.
- SAP and the Colin de Verdière type parameters are easier to compute than other minor monotone parameters related to maximum multiplicity, such as minor monotone floor of maximum multiplicity.
- We consider the transverse intersection of other relevant manifolds to obtain 'subgraph monotonicity' and a form of 'minor monotonicity' for additional spectral properties.
- The properties can be applied to the inverse eigenvalue problem.

Strong Spectral Property (SSP)

- Λ is a multiset of real numbers of cardinality n.
- The set of all $n \times n$ real symmetric matrices with spectrum Λ is denoted by $\mathcal{E}_\Lambda$.
- $\mathcal{E}_{\text{spec}(A)}$ is the set of all symmetric matrices cospectral with A.
- It is well known that $\mathcal{E}_\Lambda$ is a manifold.
- The commutator $AB - BA$ of two matrices is denoted by $[A, B]$.

Matrix A has the Strong Spectral Property (SSP) if the zero matrix is the only symmetric matrix X satisfying

- $A \circ X = O$, and $I \circ X = O$
- $[A, X] = O$.

Since $AX = O$ implies $[A, X] = O$, if A has the SSP, then A has the SAP.

Ordered multiplicity lists

- Suppose the distinct eigenvalues of A are $\mu_1 < \mu_2 < \cdots < \mu_q$ and the multiplicity of these eigenvalues are $m_1, m_2, \ldots, m_q$.
- The ordered multiplicity list of A is $\mathbf{m}(A) = (m_1, m_2, \ldots, m_q)$.

Ordered Multiplicity List Problem: Given a graph G, determine which ordered multiplicity lists arise among the matrices in $\mathcal{S}(G)$.

- The Ordered Multiplicity List Problem lies in between the Inverse Eigenvalue Problem and the Maximum Multiplicity Problem.
- The Ordered Multiplicity List Problem has been solved for paths, cycles, and complete graphs (because the IEP-G is solved for these graphs).
- The Ordered Multiplicity List Problem has been solved for all graphs of order 6.

Strong Multiplicity Property (SMP)

- $\mathbf{m} = (m_1, \ldots, m_q)$ is an ordered list of positive integers with $m_1 + m_2 + \cdots + m_q = n$.
- $\mathcal{U}_{\mathbf{m}}$ is the set of all symmetric matrices whose ordered multiplicity list is $\mathbf{m}$.
- $\mathcal{U}_{\mathbf{m}(A)}$ is the set of symmetric matrices has the same ordered multiplicity list as A.
- It follows from results of Arnold that $\mathcal{U}_{\mathbf{m}}$ is a manifold.
- The $n \times n$ symmetric matrix A satisfies the Strong Multiplicity Property (SMP) provided no nonzero symmetric matrix X satisfies
  - $A \circ X = O$, and $I \circ X = O$,
  - $[A, X] = O$, and
  - $\text{tr}(A^i X) = 0$ for $i = 0, \ldots, n-1$.
- SSP implies SMP.

Subgraph monotonicity

Theorem (Barrett, Fallat, Hall, H, Lin, Shader, 2017)

G is a graph of order n, $\widehat{G}$ is of order $\hat{n}$, $G \leq \widehat{G}$.

- *If $A \in \mathcal{S}(G)$ has SSP, then there exists $\widehat{A} \in \mathcal{S}(\widehat{G})$ with SSP such that*
$$\operatorname{spec}(\widehat{A}) = \operatorname{spec}(A) \cup \Lambda$$
where Λ is any set of $\hat{n} - n$ distinct real numbers such that $\operatorname{spec}(A) \cap \Lambda = \emptyset$.
- *If $A \in \mathcal{S}(G)$ has SMP, then there exists $\widehat{A} \in \mathcal{S}(\widehat{G})$ with SMP such that $\mathbf{m}(\widehat{A})$ is obtained from $\mathbf{m}(A)$ by extending with 1s in any positions.*

Application of the SSP to distinct eigenvalues

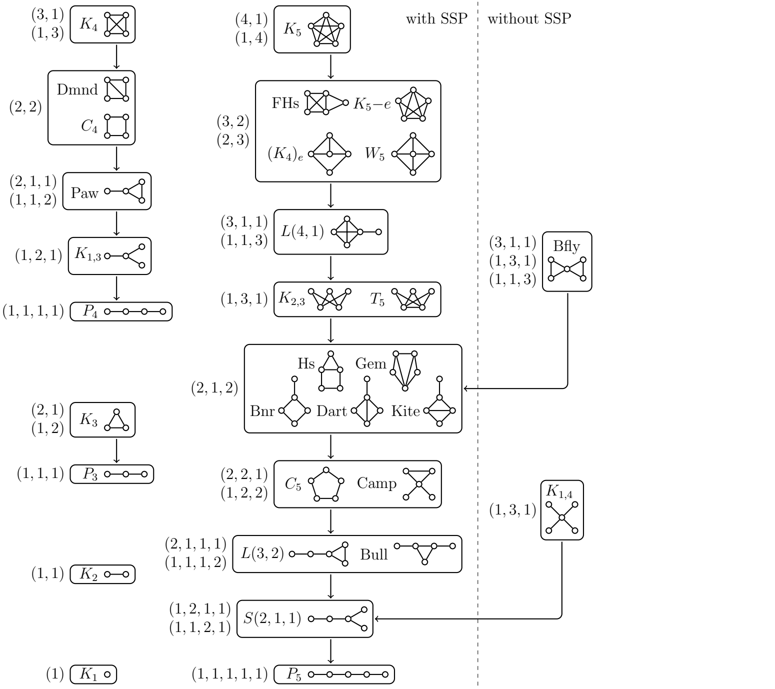
- A diagonal matrix D with distinct eigenvalues has SSP: $DX = XD$ implies all off-diagonal entries of X are zero.
- For any graph on n vertices and any set Λ of n distinct real numbers, there is a realization A that has SSP and $\operatorname{spec}(A) = \Lambda$.
- The existence of such a matrix was proved in [Monfarad, Shader, 2015] via a different method.
- $\operatorname{mr}(P_n) = n - 1$ and $\operatorname{M}(P_n) = 1$.
- The set of possible spectra of P_n is any set of n distinct real numbers (originally shown in [Hochstadt, 1967])

Theorem (Barrett, Butler, Fallat, Hall, H, Lin, Shader, Young)

Suppose H is a minor of G obtained by contraction of r edges, deletion of s vertices, and deletion of any number of edges, and $A \in \mathcal{S}(H)$.

SMP *If A has SMP and $\mathbf{m}(A) = (m_1, \ldots, m_t)$, then there is a matrix $A' \in \mathcal{S}(G)$ having SMP with $\mathbf{m}(A')$ obtained from $\mathbf{m}(A)$ by adding $r + s$ ones, with at most s of these between m_1 and m_t.*

SSP *If in addition A has SSP, then A' can be chosen to have SSP, $\operatorname{spec}(A) \subseteq \operatorname{spec}(A')$, the remaining eigenvalues are simple, and s of the additional simple eigenvalues can be chosen to have any values (including between $\lambda_{\min(A)}$ and $\lambda_{\max(A)}$).*



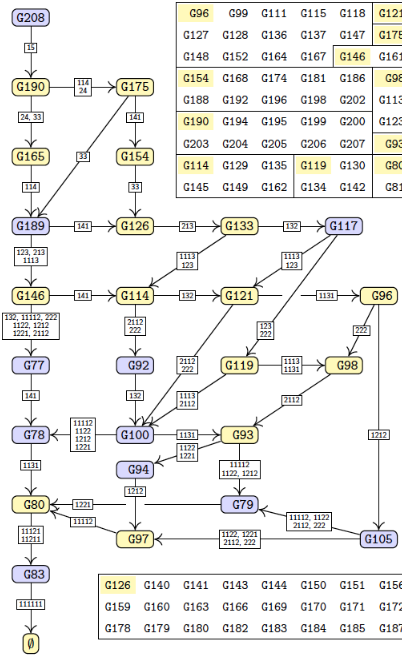
with SSP
without SSP
(3, 1) (1, 3) K_4
(2, 2) Dmnd C_4
(2, 1, 1) (1, 1, 2) Paw
(1, 2, 1) $K_{1,3}$
(1, 1, 1, 1) P_4
(2, 1) (1, 2) K_3
(1, 1, 1) P_3
(1, 1) K_2
(1) K_1
(4, 1) (1, 4) K_5
(3, 2) (2, 3) FHs K_5-e $(K_4)_e$ W_5
(3, 1, 1) (1, 1, 3) $L(4, 1)$
(1, 3, 1) $K_{2,3}$ T_5
(2, 1, 2) Hs Gem Bnr Dart Kite
(2, 2, 1) (1, 2, 2) C_5 Camp
(2, 1, 1, 1) (1, 1, 1, 2) $L(3, 2)$ Bull
(1, 2, 1, 1) (1, 1, 2, 1) $S(2, 1, 1)$
(1, 1, 1, 1, 1) P_5
(3, 1, 1) (1, 3, 1) (1, 1, 3) Bfly
(1, 3, 1) $K_{1,4}$

IEP-G solution for order ≤ 5

- The diagram shows the connected graphs of order at most 5 with their ordered multiplicity lists.
- If a box is joined to another box by a line then the graphs in the upper box can realize every ordered multiplicity list of the graphs in the lower box (including other boxes below connected with lines to lower boxes).
- Every ordered multiplicity list is spectrally arbitrary for the graphs that attain it.
- The proof uses the subgraph monotonicity of SSP and minimal subgraphs for each ordered multiplicity list, together with spectrally arbitrary matrices for each ordered multiplicity list/graph.

Theorem (Barrett, Butler, Fallat, Hall, H, Lin, Shader, Young)

The diagram is correct (it lists all ordered multiplicity lists for each connected graph of order $n \leq 5$).



G96	G99	G111	G115	G118	G121	G125	G138
G127	G128	G136	G137	G147	G175	G197	G201
G148	G152	G164	G167	G146	G161	G165	G191
G154	G168	G174	G181	G186	G98	G103	G112
G188	G192	G196	G198	G202	G113	G120	G122
G190	G194	G195	G199	G200	G123	G124	G139
G203	G204	G205	G206	G207	G93	G95	G104
G114	G129	G135	G119	G130	G80	G97	G133
G145	G149	G162	G134	G142	G81	G102	G153

G126	G140	G141	G143	G144	G150	G151	G156	G157	G158
G159	G160	G163	G166	G169	G170	G171	G172	G173	G177
G178	G179	G180	G182	G183	G184	G185	G187	G193	

OMLP solution for order ≤ 6

- 26 equivalence classes:
  - Those in blue have one graph.
  - Those in yellow have their full membership given in the boxes.
- To determine attainable ordered multiplicity lists of a graph G:
  - Find the number of G in *Atlas of Graphs*.
  - Find equivalence class of G in the diagram.
  - Take any (directed) path to $\emptyset$.
  - The multiplicity lists (and their reversals) that occur on the edges of the path are the only ones attainable.

Theorem (Ahn, Alar, Bjorkman, Butler, Carlson, Goodnight, Knox, Monroe, Wigal)

The diagram is correct (it lists all ordered multiplicity lists for each connected graph of order $n \leq 6$).

Minimum number of distinct eigenvalues

- For a matrix A, $q(A)$ is the number of distinct eigenvalues of A.
- For a graph G, the minimum number of distinct eigenvalues of G is
$$q(G) := \min\{q(A) : A \in \mathcal{S}(G)\}.$$
- Minimum Number of Distinct Eigenvalues Problem:
$$\text{Determine } q(G).$$
- Determining $q(G)$ is a subproblem if IEP-G.

Minimum number of distinct eigenvalues

$$q_{SMP}(G) := \min\{q(A) : A \in \mathcal{S}(G) \text{ and } A \text{ has SMP}\}$$

$$q_{SSP}(G) := \min\{q(A) : A \in \mathcal{S}(G) \text{ and } A \text{ has SSP}\}.$$

$$q(G) \le q_{SMP}(G) \le q_{SSP}(G).$$

Theorem (Barrett, Fallat, Hall, H, Lin, Shader, 2017)

If G is a subgraph of $\widehat{G}$, $|G| = n$, and $|\widehat{G}| = \hat{n}$, then:

$$q(\widehat{G}) \le q_{SSP}(\widehat{G}) \le \hat{n} - (n - q_{SSP}(G)).$$

$$q(\widehat{G}) \le q_{SMP}(\widehat{G}) \le \hat{n} - (n - q_{SMP}(G)).$$

If $\hat{n} = n$,

$$q(\widehat{G}) \le q_{SMP}(G).$$

High minimum number of distinct eigenvalues

Proposition

Let G be a graph. Then the following are equivalent:

(a) $q(G) = |G|$,

(b) $\mathrm{M}(G) = 1$,

(c) *G is a path.*

Theorem (Barrett, Fallat, Hall, H, Lin, Shader, 2017)

A graph G has $q(G) \geq |G| - 1$ if and only if G is one of the following:

(a) *a path,*

(b) *the disjoint union of a path and an isolated vertex,*

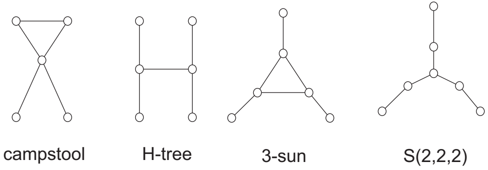
(c) *a path with one leaf attached to an interior vertex,*

(d) *a path with an extra edge joining two vertices at distance 2.*

Key ideas for the proof of $q(G) \geq |G| - 1$ theorem

Proposition (Barrett, Fallat, Hall, H, Lin, Shader, 2017)

Let G be one of the graphs H-tree, campstool, $S(2,2,2)$-tree, or 3-sun shown below. Then $q_{SSP}(G) \leq |G| - 2$.



Theorem (Barrett, Fallat, Hall, H, Lin, Shader, 2017)

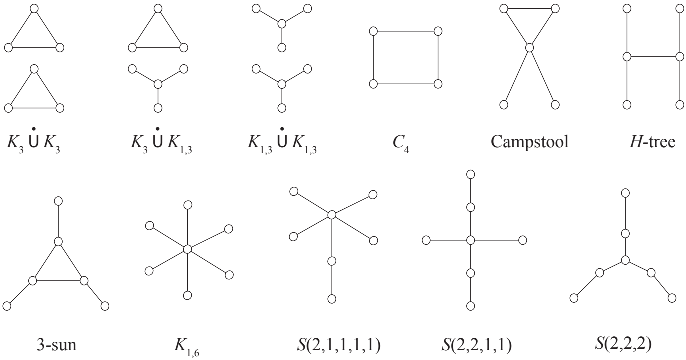
Let C_n be the cycle on $n \geq 3$ vertices. Then

$$q_{SMP}(C_n) = \left\lceil \frac{n}{2} \right\rceil.$$

Forbidden minors for at most one multiple eigenvalue

Theorem (Barrett, Butler, Fallat, Hall, H, Lin, Shader, Young)

If G is a connected graph and none of the eleven graphs shown below is a minor of G, then any matrix $A \in \mathcal{S}(G)$ has at most one multiple eigenvalue.



Verification: Show matrix A has SSP (naively)

Example

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$X^T = X$, $A \circ X = 0$, $I \circ X = O$ imply

$$X = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & u & v \\ 0 & u & 0 & w \\ 0 & v & w & 0 \end{pmatrix}.$$

$[A, X] = 0$ implies X has all row sums and column sums equal to zero, which in turn implies $X = O$. Thus, A has SSP.

In general, one can start with a matrix X of variables that satisfies $A \circ X = O = I \circ X$ and show $[A, X] = O$ implies $X = O$.

Verification matrix for SSP

- E_{ij} is the $n \times n$ matrix with a 1 in position (i, j) and 0 elsewhere.
- K_{ij} is the $n \times n$ skew-symmetric matrix $E_{ij} - E_{ji}$.
- Let H be a graph of order n with edge set $\{e_1, \dots, e_p\}$ where $e_k = i_k j_k$. For $B = (b_{ij}) \in S_n(\mathbb{R})$, $\text{vec}_H(A)$ is the p-vector whose k-th coordinate is b_{i_k, j_k}.
- Let $A \in S_n(\mathbb{R})$, let p be the number of off-diagonal zero pairs in A (so $p = \#$ edges in $\overline{\mathcal{G}(A)}$). The SSP verification matrix of A, $\Psi_S(A)$, is the $p \times \binom{n}{2}$ submatrix whose columns are $\text{vec}_{\overline{\mathcal{G}(A)}}(AK_{ij} - K_{ij}A)$ for $1 \le i < j \le n$.

Theorem (Barrett, Fallat, Hall, H, Lin, Shader, 2017)

Let $A \in \mathcal{S}(G)$ and let p be the number of edges in $\overline{G}$. Then A has SSP if and only if $\Psi_S(A)$ has rank p.

There is an analogous verification matrix Ψ_M for SMP.

Verification of SSP for matrix A

Example

$$A = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \qquad [A, K_{1,2}] = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 2 & \mathbf{1} & \mathbf{1} \\ 0 & 1 & 0 & \mathbf{0} \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

$$[A, K_{1,3}] = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1} & \mathbf{0} \\ 0 & 1 & 2 & \mathbf{1} \\ 0 & 0 & 1 & 0 \end{pmatrix}, [A, K_{1,4}] = \begin{pmatrix} -2 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} & \mathbf{1} \\ 0 & 0 & 0 & \mathbf{1} \\ 0 & 1 & 1 & 2 \end{pmatrix}$$

$$\text{rank} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} = 3 \text{ so } A \text{ has SSP.}$$

SSP, SMP, and manifolds

Theorem (Barrett, Fallat, Hall, H, Lin, Shader, 2017)

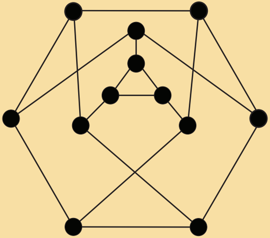
- *Matrix A has the SSP if and only if the manifolds $\mathcal{S}(\mathcal{G}(A))$ and $\mathcal{E}_{\mathrm{spec}(A)}$ intersect transversally at A.*
- *Matrix A has the SMP if and only if the manifolds $\mathcal{S}(\mathcal{G}(A))$ and $\mathcal{U}_{\mathbf{m}(A)}$ intersect transversally at A.*

Differences between SSP and SMP

It is easy to find a graph where one matrix has the SMP but not the SSP, but the next example is more interesting.

Example

Here is a graph and ordered multiplicity list $\mathbf{m}$ such that there exists $B \in \mathcal{S}(G)$ with $\mathbf{m}(B) = \mathbf{m}$ and B has the SMP, but no matrix $A \in \mathcal{S}(G)$ with $\mathbf{m}(A) = \mathbf{m}$ has the SSP:



$$\mathbf{m} = (3, 5, 4)$$

Rigid linkages

G is a graph and $\alpha, \beta \subseteq V(G)$.

- A linkage in G is a subgraph whose connected components are paths.
- The order of a linkage is its number of components.
- A linkage $\mathcal{P}$ is an (α, β)-linkage if α consists of one endpoint of each path in $\mathcal{P}$ and β consists of the other endpoints of the paths. (If the path is a single vertex v, then $v \in \alpha \cap \beta$.
- A linkage $\mathcal{P}$ is (α, β)-rigid if $\mathcal{P}$ is the unique (α, β)-linkage in G. A linkage $\mathcal{P}$ is rigid if $\mathcal{P}$ is (α, β)-rigid for some α and β such that $\mathcal{P}$ is an (α, β)-linkage.

Rigid linkages and other graph parameters

- Rigid linkages have connections to the unique linkages of Robertson and Seymour in the classic work on graph minors.
- Rigid linkages can be thought of as forcing chains produced by partial zero forcing.
- Rigid linkages have applications to the IEP-G.

Theorem (Ferrero, Flagg, Hall, H, Lin, Meyer, Nasserasr, Shader, 2019)

If G is a graph, $\mathcal{P}$ is an order t spanning rigid linkage in G, then $\text{tw}(G) \leq t$.

Theorem (Ferrero, Flagg, Hall, H, Lin, Meyer, Nasserasr, Shader, 2019)

A set of forcing chains is a rigid linkage in G.

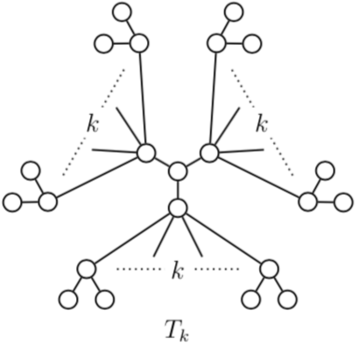
Application of rigid linkages to the IEP-G

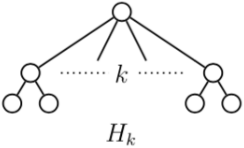
Theorem (Ferrero, Flagg, Hall, H, Lin, Meyer, Nasserasr, Shader, 2019)

Let $\mathcal{P}$ be an (α, β)-rigid linkage of order t in a graph G. Then, for any $A \in \mathcal{S}(G)$ and eigenvalue λ of A

$$\text{mult}_{A(V(\mathcal{P}))}(\lambda) \geq \text{mult}_A(\lambda) - t.$$

This result can be viewed as a generalization of eigenvalue interlacing.





Proposition (Ferrero, Flagg, Hall, H, Lin, Meyer, Nasserasr, Shader, 2019)

Let $k \geq 2$. If B is the matrix obtained from the adjacency matrix of T_k by replacing its $(1,1)$-entry by $\sqrt{2}$, then

$$\operatorname{spec}(B) = \{0^{(3k+2)}, (\sqrt{2})^{(3k-2)}, (-\sqrt{2})^{(3k-3)}, (\sqrt{k+2})^{(2)}, (-\sqrt{k+2})^{(2)}, \lambda_6^{(1)}, \lambda_7^{(1)}, \lambda_8^{(1)}\}$$

where $\lambda_6 + \lambda_7 + \lambda_8 = 0$.

Application of rigid linkages to the IEP-G

Theorem (Ferrero, Flagg, Hall, H, Lin, Meyer, Nasserasr, Shader, 2019)

Let $k \geq 3$, and suppose that $B \in \mathcal{S}(T_k)$ has spectrum

$$\{\lambda_1^{(3k+2)}, \lambda_2^{(3k-2)}, \lambda_3^{(3k-3)}, \lambda_4^{(2)}, \lambda_5^{(2)}, \lambda_6^{(1)}, \lambda_7^{(1)}, \lambda_9^{(1)}\}.$$

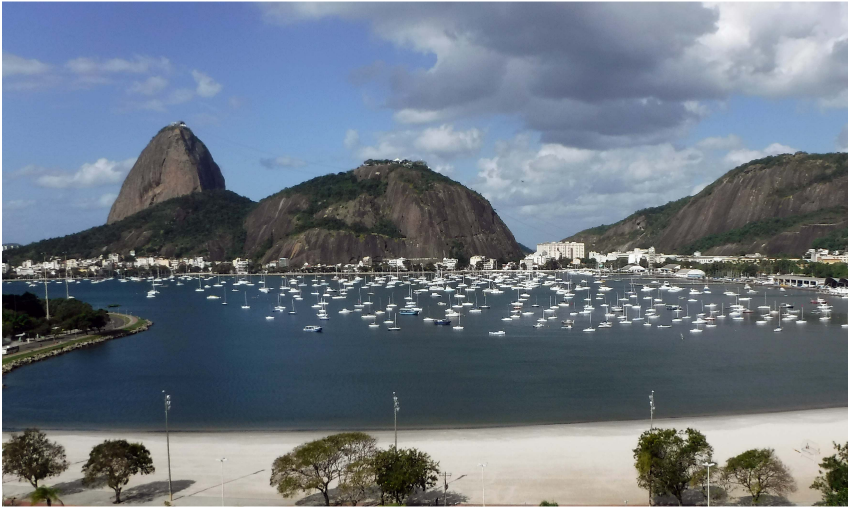
Then

$$\lambda_1 + 3\lambda_2 + 3\lambda_3 = 2\lambda_4 + 2\lambda_5 + \lambda_6 + \lambda_7 + \lambda_8.$$

This result generalizes results of Barioli and Fallat, 2003, that shows that the solution of the OML Problem is not equivalent to the IEP-G for trees.

Parameters related to zero forcing

- Zero forcing has applications to mathematical physics, monitoring electric power networks, graph searching etc.
- There are many types of zero forcing, including PSD forcing, an upper bound to the PSD IEP-G.
- Related parameters:
- Propagation time (the time needed to color all vertices blue performing independent forces simultaneously).
- Throttling (minimizing the sum of the number of blue vertices and the time propagation time of the set of blue vertices).
- MS-21 Zero Forcing, Propagation, Throttling:
  II This afternoon, III Tomorrow afternoon.



Thank you!

MATHEMATICS
RESEARCH COMMUNITIES

- AMS Mathematics Research Community 2020
  Finding Needles in Haystacks: IEP-G and Zero Forcing etc.
- June 14-21, 2020, near Providence, RI, USA
- Support limited to US-based participants not more than 5 years post-PHD or 2 years pre-PhD.

`www.ams.org/programs/research-communities/mrc-20`