The distance matrix and its variants for graphs and digraphs

by

Carolyn Reinhart

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Leslie Hogben, Major Professor
Steve Butler
Jonas Hartwig
Sung-Yell Song
Michael Young

The student author, whose presentation of the scholarship herein was approved by the program of study committee, is solely responsible for the content of this dissertation. The Graduate College will ensure this dissertation is globally accessible and will not permit alterations after a degree is conferred.

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ABSTRACT

The distance matrix $\mathcal{D}(G)$ of a connected graph G is the matrix whose entries are the pairwise distances between vertices. The distance matrix was defined by Graham and Pollak in 1971 in order to study the problem of loop switching in routing messages through a network. Since then, variants such as the distance Laplacian and distance signless Laplacian have been introduced and studied. This dissertation will study various properties of the distance matrix and its Laplacians.

First, a new distance matrix variant, the normalized distance Laplacian, denoted $\mathcal{D}^{\mathcal{L}}(G)$, is introduced and is defined analogously to the normalized Laplacian matrix, $\mathcal{L}(G)$. Bounds on the $\mathcal{D}^{\mathcal{L}}(G)$ spectral radius and connections with the normalized Laplacian matrix are presented. The number of graphs with $\mathcal{D}^{\mathcal{L}}$-cospectral mates is determined for all graphs on 10 and fewer vertices, providing evidence that the normalized distance Laplacian has fewer cospectral pairs than other matrices.

Various graph parameters have been shown to be preserved or not preserved by cospectrality for the distance matrix and its variants. We summarize known results and show several parameters are not preserved by cospectrality for the distance matrix, the signless distance Laplacian, the distance Laplacian, and the normalized distance Laplacian. Furthermore, we prove that two transmission regular graphs which are distance cospectral must have the same transmission and thus the same Wiener index.

The distance matrix of a digraph is the matrix whose ijth entry is the distance from vertex v_i to vertex v_j. In order for this matrix to be defined, we consider only strongly connected digraphs, i.e., digraphs for which there is a dipath from v_i to v_j for every pair of vertices. The number of digraphs with a distance cospectral mate is found for 6 and fewer vertices. A cospectral construction is described that produces pairs of distance cospectral digraphs from a digraph with certain structural properties.

CHAPTER 1. GENERAL INTRODUCTION

Spectral graph theory is the study of matrices defined in terms of a graph and how the eigenvalues of the matrices relate to various properties of the graphs. Originally, the adjacency matrix was studied, along with its variants, including the combinatorial Laplacian, the signless Laplacian, and the normalized Laplacian. In 1971, Graham and Pollak introduced the distance matrix in order to study the problem of loop switching in routing messages through a network [8]. Much work has been done to study the spectra of distance matrices of graphs; for a survey see [3]. Recently, several variants of the distance matrix have been introduced. In [2], Aouchiche and Hansen defined the distance Laplacian and the signless distance Laplacian. Another variant, the normalized distance Laplacian, was introduced in [13].

A topic that has received much attention in the study of spectral graph theory is bounding the spectral radius of each matrix and characterizing which graphs achieve the minimum and maximum values. It has been shown that the complete graph K_n uniquely achieves the minimum value of the spectral radius for the distance matrix ([12]), the distance Laplacian ([2]), and the signless distance Laplacian ([4]). It is also known that the path graph P_n uniquely achieves the maximum value of the spectral radius for the distance matrix ([12]), and for the distance Laplacian and the signless distance Laplacian ([7]). In Chapter 2, the spectral radius of the normalized distance Laplacian is bounded and the complete graph K_n is shown to achieve the minimum value. Furthermore, it shown that any other graph achieving this minimum spectral radius value would share a spectrum with K_n. Unlike the other distance matrices, the normalized distance Laplacian spectral radius is shown not to be maximized by the path graph. Instead, a barbell type graph is conjectured to achieve the maximum.

Another one of the biggest questions in spectral graph theory is: When can the spectrum of a matrix be used to determine if two graphs are isomorphic? No matrix has yet been found whose spectrum uniquely determines all graphs. Thus, it is natural to examine the instances where the spectra of a given matrix fails to differentiate. Two non-isomorphic graphs G and H are *M-cospectral* if $\mathrm{spec}(M(G)) = \mathrm{spec}(M(H))$; if G and H are M-cospectral we call them *M-cospectral mates*. A *M-cospectral construction* is a process by which M-cospectral graphs can be produced. A graph parameter is said to be *preserved by M-cospectrality* if two graphs that are M-cospectral must share the same value for that parameter.

Cospectrality and the preservation of parameters have been studied to varying degrees for the distance matrices and its Laplacians. The number of connected graphs with a cospectral mate on ten or fewer was computed for the distance, distance Laplacian, and the signless distance Laplacian matrices in [2]. Cospectral constructions have been found for the distance matrix such as those in [11], [1], [9], and [10] and for the distance Laplacian in [6]. Preservation of graph parameters by cospectrality has been studied for the distance matrix ([9], [1]), the distance Laplacian ([5], [6]), and the signless distance Laplacian ([5]).

In Chapter 2, we examine cospectrality for the normalized distance Laplacian matrix. We compute the number of cospectral mates on 10 or fewer vertices, providing evidence that it has fewer instances of cospectrality than other well studied matrices. We also show several parameters are not preserved by cospectrality for the normalized distance Laplacian.

In Chapter 3, we examine parameters that are preserved and not-preserved by cospectrality for the distance and its Laplacians. Specifically, we show that girth is not preserved by distance cospectrality and that planarity, degree sequence, and transmission sequence are not preserved by distance or signless distance Laplacian cospectrality. We also show that the number of connected components of the graph complement is not preserved by cospectrality for the distance, signless distance Laplacian, or normalized distance Laplacian matrix. Finally, we prove that transmission regular graphs that are distance cospectral must have the same transmission and Wiener index.

In Chapter 4, we study distance cospectrality for digraphs. We compute the number of digraphs with cospectral mates on six or fewer vertices. We observe cospectral pairs are easily produced by reversing all the arcs of the digraphs. Furthermore, a cospectral construction for the distance matrix is described that produces cospectral digraphs that differ by more than arc reversal.

1.1 Definitions for graphs

A *graph* is a pair $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ is the set of *vertices* and E is the set of *edges*. Each edge is an unordered set of two distinct vertices $\{v_i, v_j\}$, usually denoted as just v_iv_j, for $1 \leq i \neq j \leq n$. A graph G is *connected* if for all $u, v \in V(G)$, there exists a path from u to v. Since the study of distance matrices requires it, all graphs in this dissertation are assumed to be connected unless otherwise stated.

The *adjacency matrix* of a graph G, denoted $A(G)$ is the real symmetric matrix whose ijth entry is 1 if and only if ij is an edge, and 0 otherwise. The *degree matrix* is the diagonal matrix $D(G) = \operatorname{diag}(\deg(v_1), \ldots, \deg(v_n))$. The *combinatorial Laplacian matrix* of a graph G is defined such that $L(G) = D(G) - A(G)$ and the *signless Laplacian* is defined such that $Q(G) = D(G) + A(G)$. The *normalized Laplacian matrix* of a graph G without isolated vertices, has entries

$$(\mathcal{L}(G))_{ij} = \begin{cases} -\frac{1}{\sqrt{\deg(v_i)\deg(v_j)}} & ij \in E(G) \\ 1 & i = j \\ 0 & \text{otherwise} \end{cases}.$$

Observe that $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2} = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$.

The eigenvalues of A are called the *adjacency eigenvalues* and are denoted $\lambda_1 \leq \cdots \leq \lambda_n$, the eigenvalues of L are called the *combinatorial Laplacian eigenvalues* and are denoted $\phi_1 \leq \cdots \leq \phi_n$, the eigenvalues of Q are called the *signless Laplacian eigenvalues* and are denoted $q_1 \leq \cdots \leq q_n$, and the eigenvalues of $\mathcal{L}$ are called the *normalized Laplacian eigenvalues* and are denoted $\mu_1 \leq \cdots \leq \mu_n$. The matrices $A(G)$ and $Q(G)$ are non-negative so $\rho_{A(G)} = \lambda_n$ and $\rho_{Q(G)} = q_n$. The matrices $L(G)$ and $\mathcal{L}(G)$ are positive semidefinite, so $\rho_{L(G)} = \phi_n$, and $\rho_{\mathcal{L}(G)} = \mu_n$.

In a graph G, the *distance* between vertices v_i and v_j, denoted, $d(v_i, v_j)$, is the number of edges in a shortest path between v_i and v_j. The *transmission* of a vertex $v \in V(G)$, denoted $\mathrm{t}_G(v)$, is the sum of the distances from v to all other vertices, i.e. $\mathrm{t}_G(v) = \sum_{u_i \in V(G)} d(v, u_i)$. A graph is *$k$-transmission regular* if $\mathrm{t}(v) = k$ for all $v \in V$.

In [8], the *distance matrix*, denoted $\mathcal{D}(G)$, was defined and has entries $(\mathcal{D}(G))_{ij} = d(v_i, v_j)$. In order to ensure $d(v_i, v_j)$ is finite for every pair of vertices $v_i, v_j \in V(G)$, we require the graph G be connected. The *transmission matrix* is the diagonal matrix $T(G) = \mathrm{diag}(\mathrm{t}(v_1), \dots, \mathrm{t}(v_n))$. For a connected graph G, Aouchiche and Hansen ([2]) defined the *distance Laplacian matrix*, denoted $\mathcal{D}^L(G)$, such that $\mathcal{D}^L(G) = T(G) - \mathcal{D}(G)$ and the *signless distance Laplacian*, denoted $\mathcal{D}^Q(G)$, such that $\mathcal{D}^Q(G) = T(G) + \mathcal{D}(G)$. The *normalized distance Laplacian matrix* of a connected graph G, was defined in [13] and has entries

$$(\mathcal{D}^{\mathcal{L}}(G))_{ij} = \begin{cases} -\frac{1}{\sqrt{\mathrm{t}(v_i)\,\mathrm{t}(v_j)}} & i \neq j \\ 1 & i = j \end{cases}.$$

Observe that $\mathcal{D}^{\mathcal{L}}(G) = T(G)^{-1/2}\mathcal{D}^L(G)T(G)^{-1/2} = I - T(G)^{-1/2}\mathcal{D}(G)T(G)^{-1/2}$.

The eigenvalues of $\mathcal{D}(G)$ are called *distance eigenvalues* and are denoted $\partial_1 \leq \cdots \leq \partial_n$, the eigenvalues of $\mathcal{D}^L(G)$ are called *distance Laplacian eigenvalues* and are denoted $\partial_1^L \leq \cdots \leq \partial_n^L$, the eigenvalues of $\mathcal{D}^Q(G)$ are called the *signless distance Laplacian eigenvalues* and are denoted $\partial_1^Q \leq \cdots \leq \partial_n^Q$, and the eigenvalues of $\mathcal{D}^{\mathcal{L}}(G)$ are called the *normalized distance Laplacian eigenvalues* and are denoted $\partial_1^{\mathcal{L}} \leq \cdots \leq \partial_n^{\mathcal{L}}$. The matrices $\mathcal{D}(G)$ and $\mathcal{D}^Q(G)$ are non-negative and irreducible so $\rho_{\mathcal{D}(G)} = \partial_n$ and $\rho_{\mathcal{D}^Q(G)} = \partial_n^Q$. The matrices $\mathcal{D}^L(G)$ and $\mathcal{D}^{\mathcal{L}}(G)$ are positive semidefinite, so $\rho_{\mathcal{D}^L(G)} = \partial_n^L$, and $\rho_{\mathcal{D}^{\mathcal{L}}(G)} = \partial_n^{\mathcal{L}}$.

1.2 Digraphs

A *digraph* is a pair $\Gamma = (V, E)$ where $V = \{v_1, \dots, v_n\}$ is the set of *vertices* and E is the set of arcs. Each arc is an ordered pair of two distinct vertices (v_i, v_j), usually denoted as just $v_i v_j$, for $1 \leq i \neq j \leq n$. A digraph Γ is *strongly connected* if for all $u, v \in V(G)$, there exists a path from u to v. Since the study of distance matrices requires it, all digraphs in this dissertation are assumed to be strongly connected unless otherwise stated.

In a digraph Γ, the *distance* between vertices v_i and v_j, denoted, $d(v_i, v_j)$, is the number of arcs in a shortest path from v_i to v_j. The *transmission* of a vertex $v \in V(\Gamma)$, denoted $\mathrm{t}_\Gamma(v)$, is the sum of the distances from v to all other vertices, i.e. $\mathrm{t}_\Gamma(v) = \sum_{u_i \in V(\Gamma)} d(v, u_i)$. This value is also sometimes called the *out-transmission*; the *in-transmission* of a vertex v is $\sum_{u_i \in V(\Gamma)} d(u_i, v)$. A digraph is *k-transmission regular* or *k-out-transmission-regular* if $\mathrm{t}(v) = k$ for all $v \in V$. It is not necessary for a digraph Γ to be *k-in-transmission-regular* to be considered k-transmission regular.

The *adjacency matrix of a digraph* Γ, denoted $A(\Gamma)$ is the real matrix whose ijth entry is 1 if and only if ij is an arc, and 0 otherwise. The *distance matrix of a strongly connected digraph* Γ, denoted $D(\Gamma)$, is the real matrix whose ijth entry is $d(v_i, v_j)$. The adjacency and distance Laplacians are all defined analogously as they are for graphs. However, we note that unlike for graphs, these matrices need not be symmetric. This presents a particular challenge, as many techniques for graphs rely heavily on the basis of eigenvectors that is guaranteed by symmetry. In Chapter 4, we study distance matrices for digraphs and use the distance characteristic polynomial to show two digraphs are cospectral.

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CHAPTER 2. THE NORMALIZED DISTANCE LAPLACIAN

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Carolyn Reinhart

Department of Mathematics, Iowa State University

Abstract

The distance matrix $\mathcal{D}(G)$ of a connected graph G is the matrix containing the pairwise distances between vertices. The transmission of a vertex v_i in G is the sum of the distances from v_i to all other vertices and $T(G)$ is the diagonal matrix of transmissions of the vertices of the graph. The normalized distance Laplacian, $\mathcal{D}^{\mathcal{L}}(G) = I - T(G)^{-1/2}\mathcal{D}(G)T(G)^{-1/2}$, is introduced. This is analogous to the normalized Laplacian matrix, $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$, where $D(G)$ is the diagonal matrix of degrees of the vertices of the graph and $A(G)$ is the adjacency matrix. Bounds on the spectral radius of $\mathcal{D}^{\mathcal{L}}$ and connections with the normalized Laplacian matrix are presented. Twin vertices are used to determine eigenvalues of the normalized distance Laplacian. The distance generalized characteristic polynomial is defined and its properties established. Finally, $\mathcal{D}^{\mathcal{L}}$-cospectrality and lack thereof are determined for all graphs on 10 and fewer vertices, providing evidence that the normalized distance Laplacian has fewer cospectral pairs than other matrices.

Keywords: Normalized Laplacian, distance matrices, cospectrality, generalized characteristic polynomial

2.1 Introduction

Spectral graph theory is the study of matrices defined in terms of a graph, specifically relating the eigenvalues of the matrix to properties of the graph. Many such matrices are studied and they can often be used in applications. The normalized Laplacian, a matrix popularized by Fan Chung in her book Spectral Graph Theory, has applications in random walks [10]. The distance matrix was defined by Graham and Pollak in [14] in order to study the problem of loop switching in routing telephone calls through a network. In this paper, we introduce the normalized distance Laplacian, which is defined analogously to the normalized Laplacian but incorporates distances between each pair of vertices in the graph.

A *weighted graph* is a graph with vertices V and a weight function w that assigns a nonnegative real number to each pair of vertices in the graph. If v_iv_j is an edge in the graph, $w(v_i, v_j)$ is positive and if it is not an edge, $w(v_i, v_j) = 0$. The *degree* of a vertex in a weighted graph is the sum of the weights of the edges incident to it. Any unweighted graph G may be seen as a weighted graph with edge weights equal to 1.

The *adjacency matrix* of a weighted graph G is the real symmetric matrix defined by $(A(G))_{ij} = w(v_i, v_j)$. The eigenvalues of $A(G)$ are called the *adjacency eigenvalues* and are denoted $\lambda_1 \leq \cdots \leq \lambda_n$. The *degree matrix* is the diagonal matrix $D(G) = \operatorname{diag}(\deg(v_1), \ldots, \deg(v_n))$. The *combinatorial Laplacian matrix* of a weighted graph G, denoted $L(G)$, has entries

$$(L(G))_{ij} = \begin{cases} -w(v_i, v_j) & i \neq j \\ \deg(v_i) & i = j \end{cases},$$

and it is easy to observe that $L(G) = D(G) - A(G)$. The eigenvalues of $L(G)$ are called the *combinatorial Laplacian eigenvalues* and are denoted $\phi_1 \leq \cdots \leq \phi_n$. Since $L(G)$ is a positive semidefinite matrix, $\rho_{L(G)} = \phi_n$. The matrix $Q(G) = D(G) + A(G)$ is called the *signless Laplacian.* The eigenvalues of $Q(G)$ are called the *signless Laplacian eigenvalues* and are denoted $q_1 \leq \cdots \leq q_n$. The *normalized Laplacian matrix* of a weighted graph G without isolated vertices, denoted $\mathcal{L}(G)$, has entries

$$(\mathcal{L}(G))_{ij} = \begin{cases} -\frac{w(v_i,v_j)}{\sqrt{\deg(v_i)\deg(v_j)}} & i \neq j \\ 1 & i = j \end{cases}.$$

Observe that $\mathcal{L}(G) = D(G)^{-1/2}L(G)D(G)^{-1/2} = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$. The eigenvalues of $\mathcal{L}(G)$ are called the *normalized Laplacian eigenvalues* and are denoted $\mu_1 \leq \cdots \leq \mu_n$. Since $\mathcal{L}(G)$ is a positive semidefinite matrix, $\rho_{\mathcal{L}(G)} = \mu_n$.

The four matrices are also denoted A, L, Q, and $\mathcal{L}$ when the intended graph is clear. Note that while all these matrices are defined for graphs in general, in this paper we consider them for connected graphs only, unless otherwise stated.

The *distance matrix*, denoted $\mathcal{D}(G)$, has entries $(\mathcal{D}(G))_{ij} = d(v_i, v_j)$ where $d(v_i, v_j)$ is the *distance* (number of edges in a shortest path) between v_i and v_j. Much work has been done to study the spectra of distance matrices; for a survey see [2]. Requiring that every graph G be connected ensures that $d(v_i, v_j)$ is finite for every pair of vertices $v_i, v_j \in V(G)$. The eigenvalues of $\mathcal{D}(G)$ are called *distance eigenvalues* and are denoted $\partial_1 \leq \cdots \leq \partial_n$. In a graph G, the *transmission* of a vertex $v \in V(G)$, denoted $\mathrm{t}_G(v)$ or $\mathrm{t}(v)$ when the intended graph is clear, is defined as $\mathrm{t}_G(v) = \sum_{u_i \in V(G)} d(v, u_i)$. A graph is *k-transmission regular* if $\mathrm{t}(v) = k$ for all $v \in V$. The *transmission matrix* is the diagonal matrix $T(G) = \mathrm{diag}(\mathrm{t}(v_1), \ldots, \mathrm{t}(v_n))$.

In [1], Aouchiche and Hansen defined the distance Laplacian and the signless distance Laplacian. The *distance Laplacian matrix*, denoted $\mathcal{D}^L(G)$, has entries

$$(\mathcal{D}^L(G))_{ij} = \begin{cases} -d(v_i, v_j) & i \neq j \\ \mathrm{t}(v_i) & i = j \end{cases},$$

and $\mathcal{D}^L(G) = T(G) - \mathcal{D}(G)$. The eigenvalues of $\mathcal{D}^L(G)$ are called *distance Laplacian eigenvalues* and are denoted $\partial_1^L \leq \cdots \leq \partial_n^L$. Since $\mathcal{D}^L(G)$ is a positive semidefinite matrix, $\rho_{\mathcal{D}^L(G)} = \partial_n^L$. The matrix $\mathcal{D}^Q(G) = T(G) + \mathcal{D}(G)$ is called the *signless distance Laplacian*. The eigenvalues of $\mathcal{D}^Q(G)$ are called the *signless distance Laplacian eigenvalues* and are denoted $\partial_1^Q \leq \cdots \leq \partial_n^Q$. The matrices are also denoted as just $\mathcal{D}, \mathcal{D}^L$, and $\mathcal{D}^Q$ when the intended graph is clear.

In Section 2.2, we define the normalized distance Laplacian and show that its spectral radius is strictly less than 2, in contrast with the normalized Laplacian whose spectral radius is equal to 2 when the graph is bipartite. We also find bounds on the normalized distance Laplacian eigenvalues and provide data that leads to conjectures about the graphs achieving the maximum and minimum spectral radius. Methods using twin vertices to determine eigenvalues for the normalized distance Laplacian are described in Section 2.3 and applied to determine the spectrum of several families of graphs. In Section 2.4, we define the distance generalized characteristic polynomial. We show that if the polynomial is equal for two non-isomorphic graphs, they have the same $\mathcal{D}$, $\mathcal{D}^L$, $\mathcal{D}^Q$, and $\mathcal{D}^{\mathcal{L}}$ spectra and the same multiset of transmissions, extending concepts from the generalized characteristic polynomial.

Two non-isomorphic graphs G and H are *M-cospectral* if $\operatorname{spec}(M(G)) = \operatorname{spec}(M(H))$; if G and H are M-cospectral we call them *M-cospectral mates* (or just cospectral mates if the choice of M is clear). A graph parameter is said to be *preserved by M-cospectrality* if two graphs that are M-cospectral must share the same value for that parameter (can be numeric or true/false). Cospectral graphs and the preservation of parameters has been studied for many different matrices. Godsil and McKay were the first to produce an adjacency cospectrality construction [12] but many other papers study cospectrality of the normalized Laplacian (see, for example, [6],[8],[9],[18]). Several of these papers also discuss preservation by M-cospectrality; for a table summarizing preservation by A, L, Q and $\mathcal{L}$ cospectrality of some well known graph parameters, see [7].

Cospectrality of $\mathcal{D}, \mathcal{D}^L$, and $\mathcal{D}^Q$ was studied by Aouchiche and Hansen in [3] and cospectral constructions have been found for the distance matrix in [16]. Cospectral constructions for the distance Laplacian matrix were exhibited in [4] and several graph parameters were shown to not be preserved by $\mathcal{D}^L$-cospectrality. In Section 2.5, we find all cospectral graphs on 10 or fewer vertices for the normalized distance Laplacian and show how some of the graph pairs could be constructed using $\mathcal{D}^L$-cospectrality constructions. We also use examples of graphs on 9 and 10 vertices to show several parameters are not preserved by normalized distance Laplacian cospectrality and provide evidence that cospectral mates are rare for this matrix. Since graphs with different spectra cannot possibly be isomorphic, the spectrum of graphs with respect to matrices can be thought of as a tool to differentiate between graphs. Because of this, the rarity of cospectrality is beneficial.

Throughout the paper, we use the following standard definitions and notation. A *graph* G is a pair $G = (V, E)$, where $V = \{v_1, \dots, v_n\}$ is the set of *vertices* and E is the set of edges. An *edge* is a two element subset of vertices $\{v_i, v_j\}$, also denoted as just v_iv_j. We use $n = |V|$ to denote the *order* of G and assume all graphs G are connected and simple (i.e. no loops or multiedges). Two vertices v_i and v_j are *neighbors* if $v_iv_j \in E(G)$ and the *neighborhood* $N(v)$ of a vertex v is the set of its neighbors. The *degree* of a vertex v is $\deg(v) = |N(v)|$. A graph is k-*regular* if $\deg(v_i) = k$ for all $1 \leq i \leq n$. Let $\operatorname{spec}(M)$ denote the *spectrum* of a matrix M and let $p_M(x)$ denote the *characteristic polynomial* of matrix M. The *spectral radius* of a matrix M with eigenvalues $\nu_1 \leq \cdots \leq \nu_n$ is $\rho_M = \max_{1 \leq i \leq n} |\nu_i|$. An $n \times n$ real symmetric matrix M is *positive semidefinite* if $x^T M x \geq 0$ for all $x \in \mathbb{R}^n$. Equivalently, a real symmetric matrix M is positive semidefinite if and only if all its eigenvalues are non-negative. If all eigenvalues are non-negative, observe $\rho_M = \nu_n$. Note all matrices we will consider are real and symmetric.

2.2 The normalized distance Laplacian

As with the combinatorial Laplacian matrix, it is natural to define a normalized version of the distance Laplacian matrix. In this section, we introduce this new matrix and derive many proprieties of its eigenvalues.

Definition 2.2.1. The *normalized distance Laplacian matrix*, denoted $\mathcal{D}^{\mathcal{L}}(G)$, or just $\mathcal{D}^{\mathcal{L}}$, is the matrix with entries

$$(\mathcal{D}^{\mathcal{L}}(G))_{ij} = \begin{cases} \frac{-d(v_i,v_j)}{\sqrt{\mathrm{t}(v_i)\,\mathrm{t}(v_j)}} & i \neq j \\ 1 & i = j \end{cases}.$$

Observe that $\mathcal{D}^{\mathcal{L}}(G) = T(G)^{-1/2}\mathcal{D}^L(G)T(G)^{-1/2} = I - T(G)^{-1/2}\mathcal{D}(G)T(G)^{-1/2}$. We call the eigenvalues of $\mathcal{D}^{\mathcal{L}}(G)$ the *normalized distance Laplacian eigenvalues* and denote them $\partial_1^{\mathcal{L}} \leq \cdots \leq \partial_n^{\mathcal{L}}$.

It is easy to draw parallels between the properties of A, L, Q, and $\mathcal{L}$ and the properties of $\mathcal{D}$, $\mathcal{D}^L, \mathcal{D}^Q$, and $\mathcal{D}^{\mathcal{L}}$. In the remainder of this section, we present results that are known to hold for the adjacency matrix and its Laplacians, followed by their generalizations to the distance matrices.

Both the normalized Laplacian and the normalized distance Laplacian include square roots (unless the graph is regular or transmission regular, respectively). This can make computation of eigenvalues difficult with these matrices. Because of this, we can turn to similar matrices that make computation slightly easier. In [10], Chung introduces the matrix $D^{-1}L$, which one can easily see is similar to $\mathcal{L}$ by the similarity matrix $D^{-1/2}$. The eigenvectors $\mathbf{v_i}$ of $D^{-1}L(G)$ are called the *harmonic eigenvectors of* $\mathcal{L}(G)$ and $\mathbf{v_i} = D^{-1/2}\mathbf{u_i}$ where $\mathbf{u_i}$ is an eigenvector of $\mathcal{L}$. We now show an analogous similar matrix for $\mathcal{D}^{\mathcal{L}}$.

Proposition 2.2.2. *For all eigenvalues* $\partial_i^{\mathcal{L}}$ *of* $\mathcal{D}^{\mathcal{L}}$ *and associated eigenvectors* $\mathbf{x_i}$*,* $\partial_i^{\mathcal{L}}$ *is also an eigenvalue of* $T^{-1}\mathcal{D}^{L}$ *with associated eigenvector* $\mathbf{y_i} = T^{-1/2}\mathbf{x_i}$*.*

Proof. Note

$$\begin{aligned} T^{-1}\mathcal{D}^{L}\mathbf{y}_i &= T^{-1}\mathcal{D}^{L}T^{-1/2}\mathbf{x_i} \\ &= T^{-1/2}\mathcal{D}^{\mathcal{L}}\mathbf{x_i} \\ &= T^{-1/2}\partial_i^{\mathcal{L}}\mathbf{x_i} \\ &= \partial_i^{\mathcal{L}}\mathbf{y_i}. \end{aligned}$$

So $\partial_i^{\mathcal{L}}$ is an eigenvalue of $T^{-1}\mathcal{D}^{L}$ with associated eigenvector $\mathbf{y_i}$, as desired. □

Call the eigenvectors $\mathbf{y_i}$ of $T^{-1}\mathcal{D}^{L}(G)$ the *harmonic eigenvectors of* $\mathcal{D}^{\mathcal{L}}(G)$.

The following relationship between the eigenvalues of $A(G)$ and $\mathcal{L}(G)$ can be observed using Sylvester's law of inertia.

Proposition 2.2.3. [5, p. 14] *The multiplicity of* 0 *as an eigenvalue of* $A(G)$ *is the multiplicity of* 1 *as an eigenvalue of* $\mathcal{L}(G)$*, the number of negative eigenvalues for* $A(G)$ *is the number of eigenvalues greater than* 1 *for* $\mathcal{L}(G)$*, and the number of positive eigenvalues for* $A(G)$ *is the number of eigenvalues less than* 1 *for* $\mathcal{L}(G)$*.*

The analogous result for $\mathcal{D}$ and $\mathcal{D}^{\mathcal{L}}$ can be shown using the proof technique suggested by Butler in [5]. Two matrices A and B are *congruent* if there exists an invertible matrix P such that $P^TAP = B$. Sylvester's law of inertia states that any two real symmetric matrices that are congruent have the same number of positive, negative, and zero eigenvalues.

Proposition 2.2.4. *The multiplicity of* 0 *as an eigenvalue of* $\mathcal{D}(G)$ *is the multiplicity of* 1 *as an eigenvalue of* $\mathcal{D}^{\mathcal{L}}(G)$*, the number of negative eigenvalues for* $\mathcal{D}(G)$ *is the number of eigenvalues greater than* 1 *for* $\mathcal{D}^{\mathcal{L}}(G)$*, the number of positive eigenvalues for* $\mathcal{D}(G)$ *is the number of eigenvalues greater than* 1 *for* $\mathcal{D}^{\mathcal{L}}(G)$*.*

Proof. Since $(T(G)^{-1/2})^T = T(G)^{-1/2}$, $\mathcal{D}(G)$ is congruent to $T(G)^{-1/2}\mathcal{D}(G)T(G)^{-1/2}$, and therefore they have the same number of positive, negative, and zero eigenvalues. It is easy to see 0 is an eigenvalue of $T(G)^{-1/2}\mathcal{D}(G)T(G)^{-1/2}$ if and only if 1 is an eigenvalue of $\mathcal{D}^{\mathcal{L}}(G)$. If $\nu < 0$ is an eigenvalue of $T(G)^{-1/2}\mathcal{D}(G)T(G)^{-1/2}$, then $1-\nu > 1$ is an eigenvalue of $\mathcal{D}^{\mathcal{L}}(G)$. Similarly, if $\nu > 0$ is an eigenvalue of $T(G)^{-1/2}\mathcal{D}(G)T(G)^{-1/2}$, then $1-\nu < 1$ is an eigenvalue of $\mathcal{D}^{\mathcal{L}}(G)$. □

In special cases, we may deduce an exact relationship between the eigenvalues of various matrices. The following facts are easy to observe and well-known in the literature.

Observation 2.2.5. *For a r-regular weighted graph, $D(G) = rI$ so for every adjacency eigenvalue λ_i, $\phi_{n-i+1} = r - \lambda_i$, $q_i = r + \lambda_i$, and $\mu_{n-i+1} = 1 - \frac{1}{r}\lambda_i$. Similarly, for a k-transmission regular graph, $T(G) = kI$ so for every distance eigenvalue ∂_i, $\partial^L_{n-i+1} = k - \partial_i$ and $\partial^Q_i = k + \partial_i$.*

For k-transmission regular graphs, the relationships between the eigenvalues of $\mathcal{D}^{\mathcal{L}}$ and $\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q$ are also easily observed.

Observation 2.2.6. *For a k-transmission regular graph G, the normalized distance Laplacian eigenvalues are $\partial^{\mathcal{L}}_i = \frac{1}{k}\partial^L_i = 1 - \frac{1}{k}\partial_{n-i+1} = 2 - \frac{1}{k}\partial^Q_{n-i+1}$.*

This observation can be applied to compute the $\mathcal{D}^{\mathcal{L}}$-spectrum for some transmission regular graph families. The spectrum of $\mathcal{D}^L(K_n)$ is $\left\{0, n^{(n-1)}\right\}$ [1] and the complete graph is $n-1$-transmission regular, so it is easy to observe $\operatorname{spec}(\mathcal{D}^{\mathcal{L}}(K_n)) = \left\{0, \frac{n}{n-1}^{(n-1)}\right\}$.

In [1], the distance Laplacian eigenvalues are given for a cycle. For even length cycles where $n = 2p$,

$$\operatorname{spec}(\mathcal{D}^L(C_n)) = \left\{0, \left(\frac{n^2}{4}\right)^{(p-1)}, \frac{n^2}{4} + \csc^2\left(\frac{\pi(2j-1)}{n}\right)\right\} \text{ for } j = 1, \ldots, p,$$

and for odd length cycles where $n = 2p+1$,

$$\operatorname{spec}(\mathcal{D}^L(C_n)) = \left\{0, \frac{n^2-1}{4} + \frac{1}{4}\sec^2\left(\frac{\pi j}{n}\right), \frac{n^2-1}{4} - \frac{1}{4}\sec^2\left(\frac{\pi(2j-1)}{2n}\right)\right\} \text{ for } j = 1, \ldots, p.$$

The cycle is a transmission regular graph with transmission $\frac{n^2}{4}$ when n is even and transmission $\frac{n^2-1}{4}$ when n is odd. So we can apply Observation 2.2.6 to these known spectra to obtain the eigenvalues of $\mathcal{D}^{\mathcal{L}}(C_n)$.

Proposition 2.2.7. *Let C_n be the cycle on n vertices. Then if $n = 2p$ is even,*

$$\operatorname{spec}(\mathcal{D}^{\mathcal{L}}(C_n)) = \left\{0, 1^{(p-1)}, 1 + \frac{4}{n^2}\csc^2\left(\frac{\pi(2j-1)}{n}\right)\right\} \text{ for } j = 1, \ldots, p,$$

and if $n = 2p+1$ is odd,

$$\operatorname{spec}(\mathcal{D}^{\mathcal{L}}(C_n)) = \left\{0, 1 + \frac{1}{n^2-1}\sec^2\left(\frac{\pi j}{n}\right), 1 - \frac{1}{n^2-1}\sec^2\left(\frac{\pi(2j-1)}{2n}\right)\right\} \text{ for } j = 1, \ldots, p.$$

In her book that describes the normalized Laplacian matrix [10], Chung finds many bounds on the eigenvalues of $\mathcal{L}$. We now show similar results hold for the eigenvalues of the normalized distance Laplacian. The first result provides a range in which all eigenvalues of $\mathcal{L}$ lie and notes that both bounds are achieved. The result appears with proof in [10] and is stated without proof for weighted graphs in [7]; one can verify the proof from [10] remains valid for weighted graphs.

Theorem 2.2.8. [10] *For all weighted connected graphs G,*

$$0 = \mu_1 < \mu_2 \leq \cdots \leq \mu_n \leq 2,$$

with $\mu_n = 2$ if and only if G is non-trivial and bipartite.

This result generalizes with one notable difference: The normalized distance Laplacian never achieves 2 as an eigenvalue for $n \geq 3$. Observe the normalized distance Laplacian of a graph G is the normalized Laplacian of the weighted complete graph $W(G)$ with edges weights $w_{W(G)}(v_i, v_j) = d_G(v_i, v_j)$. Then the degree of a vertex v_i in $W(G)$ is the transmission of v_i in G. The next result is an application of Theorem 2.2.8 to $W(G)$ along with the observation that complete graphs on $n \geq 3$ vertices are not bipartite.

Corollary 2.2.9. *For all graphs G,*

$$0 = \partial_1^{\mathcal{L}} < \partial_2^{\mathcal{L}} \leq \cdots \leq \partial_n^{\mathcal{L}} = \rho_{\mathcal{D}^{\mathcal{L}}} \leq 2,$$

and for $n \geq 3$, $\partial_n^{\mathcal{L}} < 2$.

Since this bound is not tight, a natural next question is: Which graphs have the largest spectral radius? Using a *Sage* search [19], the maximum spectral radius of any graph on a given numbers of vertices was determined for $n \leq 10$. The results are listed in Table 2.1 below. Define KPK_{n_1,n_2,n_3} for $n_1, n_3 \geq 1$, $n_2 \geq 2$ to be the graph formed by taking the vertex sum of a vertex in K_{n_1} with one end of the path P_{n_2} and the vertex sum of a vertex in K_{n_3} with the other end of P_{n_2}. Note the number of vertices is $n = n_1 + n_2 + n_3 - 2$ and $KPK_{1,n,1} = KPK_{2,n-1,1} = KPK_{2,n-2,2} = P_n$.

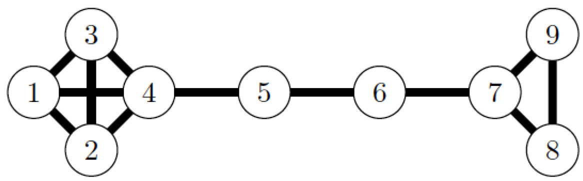

Figure 2.1 $KPK_{4,4,3}$

Table 2.1 The maximum $\rho_{\mathcal{DL}}$ and graph that achieves it for all graphs on 10 or fewer vertices

n	$\rho_{\mathcal{DL}}$	Graph
2	2	$KPK_{1,2,1}$
3	1.666	$KPK_{2,2,1}$
4	1.614	$KPK_{2,2,2}$
5	1.589	$KPK_{2,3,2}$
6	1.578	$KPK_{3,3,2}$
7	1.586	$KPK_{3,3,3}$
8	1.590	$KPK_{3,4,3}$
9	1.594	$KPK_{4,4,3}$
10	1.603	$KPK_{4,4,4}$

It is natural to conjecture a pattern from the graphs given in Table 2.1. For example, for $n = 3\ell$, one might conjecture that the graph achieving the maximum value of $\rho_{\mathcal{DL}}$ is $KPK_{\ell+1,\ell+1,\ell}$. However, as n grows larger, this pattern does not hold. For example, when $n = 15$, $\rho_{\mathcal{DL}}(KPK_{6,5,6}) > \rho_{\mathcal{DL}}(KPK_{6,6,5})$. In Table 2.2 we provide evidence that $\rho_{\mathcal{DL}}(KPK_{n_1,n_2,n_3})$ tends towards 2 as n becomes large for some n_1, n_2, n_3. Note that these graphs were the graphs with largest $\rho_{\mathcal{DL}}$ found by checking several graphs in the KPK_{n_1,n_2,n_3} family on *Sage*, and are not guaranteed to have the largest $\rho_{\mathcal{D}\mathcal{L}}$ of all graphs on n vertices or even within the family KPK_{n_1,n_2,n_3}. This data leads to the next conjecture.

Table 2.2 Evidence that maximum $\rho_{\mathcal{D}\mathcal{L}}$ approaches 2 as n becomes large, data from *Sage* [19]

n	$\rho_{\mathcal{D}\mathcal{L}}$	Graph
15	1.634	$KPK_{6,5,6}$
20	1.661	$KPK_{8,6,8}$
25	1.682	$KPK_{10,7,10}$
50	1.748	$KPK_{21,10,21}$
100	1.808	$KPK_{43,16,43}$
200	1.857	$KPK_{90,22,90}$
400	1.895	$KPK_{184,34,184}$
600	1.913	$KPK_{280,42,280}$
800	1.924	$KPK_{377,48,377}$

Conjecture 2.2.10. *The maximum $\mathcal{D}^{\mathcal{L}}$ spectral radius achieved by a graph on n vertices tends to 2 as $n \to \infty$ and is achieved by KPK_{n_1,n_2,n_3} for some $n_1 + n_2 + n_3 = n + 2$.*

This family also shows that while $\rho_{\mathcal{D}L}$ is subgraph monotonically increasing (see [1, Theorem 3.5]), $\rho_{\mathcal{D}\mathcal{L}}$ is not. Specifically, we can see that P_n is a subgraph of KPK_{n_1,n_2,n_3} for all $n_1+n_2+n_3 = n+2$. However, it has been verified using *Sage* [19] for $n \leq 20$ that $\rho_{\mathcal{D}\mathcal{L}}(P_n) < \rho_{\mathcal{D}\mathcal{L}}(KPK_{n_1,n_2,n_3})$ for some $n_1 + n_2 + n_3 = n + 2$.

The following result provides a bound for the smallest non-zero eigenvalue and the largest eigenvalue with respect to $\mathcal{L}$.

Theorem 2.2.11. [10, Lemma 1.7(ii)] *For all graphs G on $n \geq 2$ vertices,*

$$\mu_2 \leq \frac{n}{n-1},$$

with equality holding if and only if G is K_n. Also,

$$\rho = \mu_n \geq \frac{n}{n-1}.$$

The proof of this result can be used to prove nearly the same result for $\mathcal{D}^{\mathcal{L}}$. Note the proof of equality of the first inequality if and only if G is K_n could not be generalized, since the proof relies on $\mathcal{L}$ having 0 entries corresponding to non-adjacencies.

Theorem 2.2.12. *For a graph G on $n \geq 2$ vertices,*

$$\partial_2^{\mathcal{L}} \leq \frac{n}{n-1} \text{ and } \rho_{\mathcal{D}^{\mathcal{L}}} = \partial_n^{\mathcal{L}} \geq \frac{n}{n-1}.$$

Proof. Observe $\sum\limits_{i=1}^{n} \partial_i^{\mathcal{L}} = \sum\limits_{i=2}^{n} \partial_i^{\mathcal{L}} = \operatorname{trace}(\mathcal{D}^{\mathcal{L}}) = n$. Then since $\partial_2^{\mathcal{L}}$ is the smallest non-zero eigenvalue, $\partial_2^{\mathcal{L}}(n-1) \leq \sum\limits_{i=2}^{n} \partial_i^{\mathcal{L}} = n$ so $\partial_2^{\mathcal{L}} \leq \frac{n}{n-1}$. Similarly, since $\partial_n^{\mathcal{L}}$ is the largest eigenvalue $\partial_n^{\mathcal{L}}(n-1) \geq \sum\limits_{i=2}^{n} \partial_i^{\mathcal{L}} = n$ so $\partial_n^{\mathcal{L}} \geq \frac{n}{n-1}$. □

We can see that Theorem 2.2.12 provides a lower bound on the spectral radius of $\mathcal{D}^{\mathcal{L}}$. As previously computed, this is the spectral radius of the complete graph, so this minimum is achieved by K_n. In fact, we can prove the following stronger statement.

Theorem 2.2.13. *If any graph G has $\mathcal{D}^{\mathcal{L}}$ spectral radius $\frac{n}{n-1}$,* $\operatorname{spec}_{\mathcal{D}^{\mathcal{L}}}(G) = \left\{0, \frac{n}{n-1}^{(n-1)}\right\}$.

Proof. For a graph G, let $\rho_{\mathcal{D}^{\mathcal{L}}} = \partial_n^{\mathcal{L}} = \frac{n}{n-1}$. Recall $\partial_1^{\mathcal{L}} = 0$ for all graphs and obviously $\partial_2^{\mathcal{L}} \leq \cdots \leq \partial_{n-1}^{\mathcal{L}} \leq \frac{n}{n-1}$ by definition. As in the proof of the previous theorem, we have $\sum\limits_{i=2}^{n} \partial_i^{\mathcal{L}} = \operatorname{trace}(\mathcal{D}^{\mathcal{L}}) = n$ so $\sum\limits_{i=2}^{n-1} \partial_i^{\mathcal{L}} = n - \frac{n}{n-1} = \frac{n(n-2)}{n-1}$. If $\partial_i^{\mathcal{L}} < \frac{n}{n-1}$ for some $2 \leq i \leq n-1$, $\sum\limits_{i=2}^{n-1} \partial_i^{\mathcal{L}} < \frac{n(n-2)}{n-1}$. So $\partial_i^{\mathcal{L}} = \frac{n}{n-1}$ for all $2 \leq i \leq n$. □

Theorem 2.2.13 shows that any other graph achieving minimal spectral radius would be $\mathcal{D}^{\mathcal{L}}$-cospectral to the complete graph K_n. The next conjecture would follow if it was shown that K_n has no $\mathcal{D}^{\mathcal{L}}$-cospectral mates. Using *Sage* [19], we can verify that K_n is the only graph achieving minimum $\rho_{\mathcal{D}^{\mathcal{L}}}$ for $n \leq 10$.

Conjecture 2.2.14. *For a graph on n vertices,*

$$\rho_{\mathcal{D}^{\mathcal{L}}} = \partial_n^{\mathcal{L}} = \frac{n}{n-1},$$

if and only if G is the complete graph K_n, and so K_n is the only graph achieving minimum spectral radius with respect to $\mathcal{D}^{\mathcal{L}}$.

We may also bound the eigenvalues of one matrix in terms of the other. Butler described a relationship between the eigenvalues of L and $\mathcal{L}$. The next result appears with proof in [5] and is stated without proof for weighted graphs in [7]; one can verify the proof from [5] remains valid for weighted graphs.

Theorem 2.2.15. [5, Theorem 4] *Let G be a weighted graph with Δ the maximum degree of a vertex in G and δ the minimum degree of a vertex in G. Then for $1 \le i \le n$,*

$$\frac{1}{\Delta}\phi_i \le \mu_i \le \frac{1}{\delta}\phi_i.$$

Since the above result holds for weighted graphs, we can again apply the result to the weighted complete graph $W(G)$ to obtain a similar result for $\mathcal{D}^{\mathcal{L}}$. Note the distance Laplacian of G is the combinatorial Laplacian of the weighted complete graph $W(G)$, so $\phi_i(W(G)) = \partial_i^L(G)$.

Corollary 2.2.16. *Let G be a graph with t_{max} the maximum transmission of a vertex in G and t_{min} the minimum transmission of a vertex in G. Then for $1 \le i \le n$,*

$$\frac{1}{t_{max}}\partial_i^L \le \partial_i^{\mathcal{L}} \le \frac{1}{t_{min}}\partial_i^L.$$

2.3 Using twin vertices to determine eigenvalues

A pair of vertices u and v in G are called *twins* if they have the same neighborhood, and the same edge weights in the case of a weighted graph. If uv is an edge in G, they are called *adjacent twins* and if uv is not an edge in G, they are called *non-adjacent twins.* Twins have proved very useful in the study of spectra. In this section, we show how twin vertices can be used to compute eigenvalues of $\mathcal{D}^{\mathcal{L}}$ and apply these results to compute the spectra for several families of matrices.

Theorem 2.3.1. [7] *If a weighted graph G has a set of two or more nonadjacent twins, then 1 is an eigenvalue of $\mathcal{L}(G)$ and 0 is an eigenvalue of $A(G)$. If a weighted graph G has a set of two* *or more adjacent twins of degree d, then $\frac{d+1}{d}$ is an eigenvalue of $\mathcal{L}(G)$ and -1 is an eigenvalue of $A(G)$.*

Applying part of this result to $W(G)$ gives the analogous result for adjacent twins. In the weighted complete graph $W(G)$, v_1 and v_2 are adjacent twins with degree k. Then by Theorem 2.3.1, $\frac{k+1}{k}$ is an eigenvalue of $\mathcal{L}(W(G))$ and thus of $\mathcal{D}^{\mathcal{L}}(G)$.

Corollary 2.3.2. *Let G be a graph with $v_1, v_2 \in V(G)$ such that v_1 and v_2 are adjacent twins and $\mathrm{t}(v_1) = \mathrm{t}(v_2) = k$. Then $\frac{k+1}{k}$ is an eigenvalue of $\mathcal{D}^{\mathcal{L}}(G)$.*

Theorem 2.3.1 cannot be used to prove anything for non-adjacent twins, since all vertices are adjacent in the weighted complete graph. However, the proof of the following result adapts the method used to prove Theorem 2.3.1 to the normalized distance Laplacian.

Theorem 2.3.3. *Let G be a graph with $v_1, v_2 \in V(G)$ such that v_1 and v_2 are non-adjacent twins and $\mathrm{t}(v_1) = \mathrm{t}(v_2) = k$. Then $\frac{k+2}{k}$ is an eigenvalue of $\mathcal{D}^{\mathcal{L}}(G)$ with eigenvector $\mathbf{x} = [1, -1, 0, \ldots, 0]^T$.*

Proof. Observe for $i = 3, \ldots, n$, $\mathcal{D}^{\mathcal{L}}_{1,i} = \mathcal{D}^{\mathcal{L}}_{i,1} = \mathcal{D}^{\mathcal{L}}_{2,i} = \mathcal{D}^{\mathcal{L}}_{i,2} = -\frac{d(v_1,v_i)}{\sqrt{kt(v_i)}} = -\frac{d(v_2,v_i)}{\sqrt{kt(v_i)}}$ so the first and second rows and the first and second columns are the same except for in the 2×2 submatrix indexed by v_1, v_2. This submatrix is $\begin{bmatrix} 1 & -\frac{2}{k} \\ -\frac{2}{k} & 1 \end{bmatrix}$. Multiplying $\mathcal{D}^{\mathcal{L}}$ by $\mathbf{x}$ gives

$$\mathcal{D}^{\mathcal{L}}(G)\mathbf{x} = \begin{bmatrix} 1 + \frac{2}{k} \\ -\frac{2}{k} - 1 \\ \mathcal{D}^{\mathcal{L}}_{3,1} - \mathcal{D}^{\mathcal{L}}_{3,2} \\ \vdots \\ \mathcal{D}^{\mathcal{L}}_{n,1} - \mathcal{D}^{\mathcal{L}}_{n,2} \end{bmatrix} = \begin{bmatrix} \frac{k+2}{k} \\ -\frac{k+2}{k} \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \frac{k+2}{k}\mathbf{x}.$$

So we see $\frac{k+2}{k}$ is an eigenvalue of $\mathcal{D}^{\mathcal{L}}(G)$ with eigenvector $\mathbf{x}$, as desired. □

We can now apply Corollary 2.3.2 and Theorem 2.3.3 to compute the $\mathcal{D}^{\mathcal{L}}$-spectrum of some well known families.

Theorem 2.3.4. *The complete bipartite graph on $n+m$ vertices $K_{n,m}$ has*

$$\operatorname{spec}\left(\mathcal{D}^{\mathcal{L}}(K_{n,m})\right) = \left\{0, \left(\frac{2n+m}{2n+m-2}\right)^{(n-1)}, \left(\frac{n+2m}{n+2m-2}\right)^{(m-1)}, \frac{2(n^2+m^2+mn-n-m)}{(2n+m-2)(n+2m-2)}\right\}.$$

Proof. Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_m\}$ be the partite sets of $K_{n,m}$. Observe every pair of vertices in A are non-adjacent twins with transmission $2(n-1)+m = 2n+m-2$. By Theorem 2.3.3, every pair a_1, a_j yields the eigenvalue $\frac{2n+m}{2n+m-2}$ and there are $n-1$ such pairs. Similarly, every pair of vertices in B are non-adjacent twins with transmission $n+2(m-1) = n+2m-2$. By Theorem 2.3.3, every pair b_1, b_j yields the eigenvalue $\frac{n+2m}{n+2m-2}$ and there are $m-1$ such pairs. We also have that 0 is an eigenvalue. We have accounted for $n+m-1$ eigenvalues so only one eigenvalue remains, denote this eigenvalue ν. Observe $\sum_{i=1}^{n+m} \partial_i^{\mathcal{L}} = \operatorname{trace}(\mathcal{D}^{\mathcal{L}}) = n+m$ so

$$n+m = 0 + \frac{2n+m}{2n+m-2}(n-1) + \frac{n+2m}{n+2m-2}(m-1) + \nu.$$

By computation,

$$\nu = \frac{2(n^2+m^2+mn-n-m)}{(2n+m-2)(n+2m-2)}.$$

□

Corollary 2.3.5. *The star graph on n vertices S_n has* $\operatorname{spec}\left(\mathcal{D}^{\mathcal{L}}(S_n)\right) = \left\{0, \frac{2n-2}{2n-3}, \left(\frac{2n-1}{2n-3}\right)^{(n-2)}\right\}$.

Theorem 2.3.6. *The complete graph on n vertices with one edge removed, $K_n - e$, has*

$$\operatorname{spec}\left(\mathcal{D}^{\mathcal{L}}(K_n - e)\right) = \left\{0, \frac{n^2-n+2}{n(n-1)}, \left(\frac{n}{n-1}\right)^{(n-3)}, \frac{n+2}{n}\right\}.$$

Proof. Let $V(K_n) = V(K_n - e) = \{v_1, \dots, v_n\}$. By vertex transitivity of the complete graph, let $e = v_1v_2$. Then it is easy to observe v_1 and v_2 are non-adjacent twins while $v_3, \dots, v_n$ are adjacent twins. Since $\mathrm{t}(v_1) = \mathrm{t}(v_2) = n$ and $\mathrm{t}(v_i) = n-1$ for $i = 3, \dots, n$, Theorem 2.3.3 shows $\frac{n+2}{n}$ is an eigenvalue with multiplicity 1 while Corollary 2.3.2 shows $\frac{n}{n-1}$ is an eigenvalue with multiplicity $n-3$. Since 0 is always an eigenvalue, we have accounted for $n-3+1+1 = n-1$ eigenvalues. Calculation shows the remaining eigenvalue is

$$\nu = \frac{n^2-n+2}{n(n-1)}.$$

□

2.4 Characteristic polynomials

An alternative to direct computation for determining eigenvalues of a matrix is to compute the characteristic polynomial of the matrix. In this section we define the distance generalized characteristic polynomial and show if the polynomial is equal for two non-isomorphic graphs then the graphs must have the same transmission sequence. Then we generalize a method of computing the $\mathcal{L}$ characteristic polynomial to the $\mathcal{D}^{\mathcal{L}}$ characteristic polynomial.

Let $N(\lambda, r, G) = \lambda I_n - A(G) + rD(G)$. The *generalized characteristic polynomial* is $\phi(\lambda, r, G) = \det(N(\lambda, r, G))$. When the parameters are clear, we also use the notation $N(G)$ and $\phi(G)$ or N and ϕ if the graph is also clear. It is known that if $\phi(G) = \phi(H)$ then G and H are A, L, Q, and $\mathcal{L}$ cospectral. We now define an analogous polynomial for the distance matrices $\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q$, and $\mathcal{D}^{\mathcal{L}}$.

Definition 2.4.1. Let $N^{\mathcal{D}}(\lambda, r, G) = \lambda I_n - \mathcal{D}(G) + rT(G)$. The *distance generalized characteristic polynomial* is $\phi^{\mathcal{D}}(\lambda, r, G) = \det(N^{\mathcal{D}}(\lambda, r, G))$.

When the parameters intended are clear, we also use the notation $N^{\mathcal{D}}(G)$ and $\phi^{\mathcal{D}}(G)$ or $N^{\mathcal{D}}$ and $\phi^{\mathcal{D}}$ if graph is also clear.

Theorem 2.4.2. *From $\phi^{\mathcal{D}}(\lambda, r, G)$ we can recover the characteristic polynomials for $\mathcal{D}$ and $\mathcal{D}^Q$, characteristic polynomial of $\mathcal{D}^L$ up to sign and the characteristic polynomial of $\mathcal{D}^{\mathcal{L}}$ up to a constant.*

Proof. We show that through proper choices of λ and r, we can obtain the desired polynomials. First, for $\mathcal{D}$, choose $\lambda = x$ and $r = 0$ and observe

$$\phi^{\mathcal{D}}(x, 0, G) = \det(xI_n - \mathcal{D}(G)) = p_{\mathcal{D}(G)}(x).$$

For $\mathcal{D}^L$, choose $\lambda = -x$ and $r = 1$, which gives

$$\phi^{\mathcal{D}}(-x, 1, G) = \det(-xI_n - \mathcal{D}(G) + T(G)) = (-1)^n p_{\mathcal{D}^L(G)}(x).$$

For $\mathcal{D}^Q$, choose $\lambda = x$ and $r = -1$, resulting in

$$\phi^{\mathcal{D}}(x, -1, G) = \det(xI_n - \mathcal{D}(G) - T(G)) = p_{\mathcal{D}^Q(G)}(x).$$

Finally, for $\mathcal{D}^{\mathcal{L}}$ we choose $\lambda = 0$ and $r = -x + 1$, so

$$\begin{aligned}\phi^{\mathcal{D}}(0, -x+1, G) &= \det(-\mathcal{D}(G) + (-x+1)T(G)) \\ &= \det(T(G)) \det(T(G)^{-1/2}(\mathcal{D}^L(G) - xT(G))T(G)^{-1/2}) \\ &= \det(T(G)) \det(\mathcal{D}^{\mathcal{L}}(G) - xI_n)) \\ &= (-1)^n \det(T(G)) p_{\mathcal{D}^{\mathcal{L}}(G)}(x).\end{aligned}$$

□

Corollary 2.4.3. *If $\phi^{\mathcal{D}}(G) = \phi^{\mathcal{D}}(H)$ for two graphs G and H, then G and H are $\mathcal{D}$, $\mathcal{D}^L$, $\mathcal{D}^Q$, and $\mathcal{D}^{\mathcal{L}}$ cospectral.*

Proof. In Theorem 2.4.2, we showed the characteristic polynomials of $\mathcal{D}$ and $\mathcal{D}^Q$ can be recovered exactly, so it is clear that G and H are $\mathcal{D}$ and $\mathcal{D}^Q$ cospectral. Let G have order n and H have order n', then if $\phi^{\mathcal{D}}(G) = \phi^{\mathcal{D}}(H)$, necessarily $n = n'$ since otherwise the polynomials would not have the same degree. Therefore $(-1)^n p_{\mathcal{D}^L(G)}(x) = (-1)^{n'} p_{\mathcal{D}^L(H)}(x)$ implies $p_{\mathcal{D}^L(G)}(x) = p_{\mathcal{D}^L(H)}(x)$ so G and H are $\mathcal{D}^L$ cospectral. The leading term for all graphs G of $p_{\mathcal{D}^{\mathcal{L}}(G)}(x)$ is x^n, so for some constants c_1, c_2, $c_1 p_{\mathcal{D}^{\mathcal{L}}(G)}(x) = c_2 p_{\mathcal{D}^{\mathcal{L}}(H)}(x)$ implies $c_1 = c_2$ and $p_{\mathcal{D}^{\mathcal{L}}(G)}(x) = p_{\mathcal{D}^{\mathcal{L}}(H)}(x)$. Therefore G and H are $\mathcal{D}^{\mathcal{L}}$ cospectral. □

In [17], the authors explore properties of non-isomorphic graphs G and H for which $\phi(G) = \phi(H)$. The next theorem is one of their main results.

Theorem 2.4.4. [17, Theorem 2.1] *If $\phi(G) = \phi(H)$, then graphs G and H have the same degree sequence.*

The proof of the above theorem uses [17, Lemma 2.3], which holds for any diagonal matrix but is applied to the degree matrix. It also uses [17, Lemma 2.4], which is stated specifically for the adjacency matrix. However, this lemma holds for all real symmetric matrices; we state the lemma in its full generality next from its original source.

Lemma 2.4.5. [13, p. 186] *Let $\{\mathbf{y_i}\}$ be a set of orthonormal eigenvectors of the real symmetric matrix M with associated eigenvalues λ_i $(i = 1, 2, ..., n)$. Then $(\lambda I_n - M)^{-1} = \sum_{i=1}^{n} \frac{\mathbf{y_i}\mathbf{y_i}^T}{\lambda - \lambda_i}$.*

We now observe the proof given for [17, Theorem 2.1] can be used to show the following more general result.

Theorem 2.4.6. *Let M_1 and M_2 be $n \times n$ real symmetric matrices and let D_1 and D_2 be $n \times n$ diagonal matrices. If* $\det(\lambda I_n - M_1 + rD_1) = \det(\lambda I_n - M_2 + rD_2)$*, then* $\operatorname{spec}(M_1) = \operatorname{spec}(M_2)$ *and* $\operatorname{spec}(D_1) = \operatorname{spec}(D_2)$*.*

Proof. That $\operatorname{spec}(M_1) = \operatorname{spec}(M_2)$ is immediate by letting $r = 0$. The proof that the degree sequences are the same (Theorem 2.4.4) is by showing $D(G)$ and $D(H)$ are similar matrices, which here shows $\operatorname{spec}(D_1) = \operatorname{spec}(D_2)$. $\square$

Applying Theorem 2.4.6 to the real symmetric matrices $\mathcal{D}(G)$ and $\mathcal{D}(H)$ and the diagonal matrices $T(G)$ and $T(H)$, we obtain a result for $\phi^{\mathcal{D}}$ as a corollary.

Corollary 2.4.7. *If $\phi^{\mathcal{D}}(G) = \phi^{\mathcal{D}}(H)$, then graphs G and H have the same transmission sequence.*

Characteristic polynomials are often difficult to calculate. Because of this, many reduction formulas exist. In [18], Osborne provides one such reduction algorithm for the generalized characteristic polynomial $\phi(G)$. For a matrix $M(G)$ and a subset of vertices $\alpha \subset V$, let $M_\alpha(G)$ be the submatrix obtained by deleting the rows and columns corresponding to the vertices in α from M.

Theorem 2.4.8. [18] *Let u be a vertex in G, let $\mathcal{C}(u)$ be the collection of cycles in G containing u. Then*

$$\phi(\lambda, r, G) = (\lambda + \deg(u)r)\det(N_u(G)) - \sum_{w \sim u} \det(N_{\{u,w\}}(G)) - 2\sum_{Z \in \mathcal{C}(u)} \det(N_Z(G)).$$

We now prove a reduction result for $\phi^{\mathcal{D}}(G)$ using similar proof techniques.

Theorem 2.4.9. *Let u be a vertex in G, let $\mathcal{CP}(u)$ denote the cyclic permutations of S_n that do not fix u, and let $V(\sigma)$ denote vertices not fixed by a permutation σ. Then,*

$$\phi^{\mathcal{D}}(\lambda, r, G) = (\lambda + t(u)r)\det(N_u^{\mathcal{D}}(G)) - \sum_{\substack{\sigma \in \mathcal{CP}(u) \\ |\sigma| = k}} d(u, \sigma(u))d(\sigma(u), \sigma^2(u)) \dots d(\sigma^{k-1}(u), u)\det(N_{V(\sigma)}(G)).$$

Proof. Let the vertices of G be $1 = u, 2, \ldots, n$ and let $(N(G))_{ij} = n_{ij}$. It is clear

$$n_{ij} = \begin{cases} \lambda + t(i)r & i = j \\ -d(i,j) & \text{else} \end{cases},$$

and $\phi^{\mathcal{D}}(G) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} n_{i\sigma(i)}$. Partition S_n into P_1 and P_2 such that $\sigma \in P_1$ if $\sigma(1) = 1$ and otherwise $\sigma \in P_2$. Write σ as a product of cycles $\sigma = \sigma_1\sigma_2 \ldots \sigma_\ell$ such that $1 \in \sigma_1$. Clearly,

$$\phi^{\mathcal{D}}(G) = \sum_{\sigma \in P_1} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} n_{i\sigma(i)} + \sum_{\sigma \in P_2} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} n_{i\sigma(i)}.$$

If $\sigma \in P_1$, then $\operatorname{sgn}(\sigma) = \operatorname{sgn}(\sigma_2 \ldots \sigma_\ell)$ and

$$\begin{aligned} \sum_{\sigma \in P_1} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} n_{i\sigma(i)} &= \sum_{\sigma \in P_1} (\lambda + t(1)r) \operatorname{sgn}(\sigma_2 \ldots \sigma_\ell) \prod_{i=2}^{n} n_{i\sigma(i)} \\ &= (\lambda + t(1)r) \sum_{\sigma \in S_{n-1}} \operatorname{sgn}(\sigma) \prod_{i=2}^{n} n_{i\sigma(i)} \\ &= (\lambda + t(1)r) \det(N_1^{\mathcal{D}}(G)). \end{aligned}$$

If $\sigma \in P_2$, let $\sigma_1 = (1, \sigma(1), \ldots, \sigma^{k-1}(1))$. Then

$$\begin{aligned} &\sum_{\sigma \in P_2} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} n_{i\sigma(i)} \\ = &\sum_{\sigma \in P_2} \operatorname{sgn}(\sigma_1)(-1)^k d(1,\sigma(1)) d(\sigma(1),\sigma^2(1)) \ldots d(\sigma^{k-1}(1),1) \operatorname{sgn}(\sigma_2 \ldots \sigma_k) \prod_{i \in V(\sigma_2 \ldots \sigma_k)} n_{i\sigma(i)} \\ = &\sum_{\sigma \in P_2} (-1)^{k-1}(-1)^k d(1,\sigma(1)) d(\sigma(1),\sigma^2(1)) \ldots d(\sigma^{k-1}(1),1) \operatorname{sgn}(\sigma_2 \ldots \sigma_k) \prod_{i \in V(\sigma_2 \ldots \sigma_k)} n_{i\sigma(i)}. \end{aligned}$$

Fixing σ_1, consider all other permutations of the remaining vertices. This gives

$$\begin{aligned} &-\sum_{\substack{\sigma \in \mathcal{CP}(1) \\ |\sigma| = k}} d(1,\sigma(1)) d(\sigma(1),\sigma^2(1)) \ldots d(\sigma^{k-1}(1),1) \sum_{\tau \in S_{n-|V(\sigma)|}} \operatorname{sgn}(\tau) \prod_{i \in V(\tau)} n_{i\tau(i)} \\ = &-\sum_{\substack{\sigma \in \mathcal{CP}(1) \\ |\sigma| = k}} d(1,\sigma(1)) d(\sigma(1),\sigma^2(1)) \ldots d(\sigma^{k-1}(1),1) \det(N_{V(\sigma)}(G)). \end{aligned}$$

□

We now shift our focus back to the standard characteristic polynomial. Methods of computing characteristic polynomials have been found for various matrices associated with graphs. Such a method was found for the weighted normalized Laplacian in [9] and is given below. A decomposition D of an undirected weighted graph G is a subgraph consisting of disjoint edges and cycles. Let $s(D)$ denote the number of cycles of length at least three in D, let $e(D)$ denote the number of cycles in D that have an even number of vertices (here, consider an edge to be a cycle of length two), and let $F(D)$ be the set of isolated edges in the decomposition. Note a decomposition need not be spanning, and in fact the empty decomposition is included.

Theorem 2.4.10. [9] *Let G be a weighted graph on n vertices. Then the characteristic polynomial of the normalized Laplacian matrix is*

$$p(x) = \sum_{D} (-1)^{e(D)} 2^{s(D)} \frac{\prod_{v_iv_j \in E(D)} w(v_i, v_j) \prod_{v_iv_j \in F(D)} w(v_i, v_j)}{\prod_{v_i \in V(D)} \deg(v_i)} (x-1)^{n-|V(D)|},$$

where the sum runs over all decompositions D of the graph G.

Applying this result to the normalized distance Laplacian viewed as a weighted normalized Laplacian, we obtain the following formula for computing its characteristic polynomial. Note $w_{W(G)}(v_i, v_j) = d_G(v_i, v_j)$ and $\deg_{W(G)}(v_i) = \mathrm{t}_G(v_i)$.

Corollary 2.4.11. *Let G be a graph on n vertices. Then the characteristic polynomial of the normalized distance Laplacian matrix is*

$$p(x) = \sum_{D} (-1)^{e(D)} 2^{s(D)} \frac{\prod_{v_iv_j \in E(D)} d_G(v_i, v_j) \prod_{v_iv_j \in F(D)} d_G(v_i, v_j)}{\prod_{v_i \in V(D)} \mathrm{t}_G(v_i)} (x-1)^{n-|V(D)|},$$

where the sum runs over all decompositions D of the complete graph K_n.

2.5 Cospectral graphs

In this section, we show cospectral graphs with respect to the normalized distance Laplacian are rare. In Section 2.5.1 we exhibit and discuss the 5 cospectral pairs on 8 and 9 vertices as well as interesting examples of cospectral pairs on 10 vertices. We show the number of edges in a graph, degree sequence, transmission sequence, girth, Weiner index, planarity, k-regularity, and k-transmission regularity are not preserved by $\mathcal{D}^{\mathcal{L}}$-cospectrality. We compare $\mathcal{D}^{\mathcal{L}}$-cospectralitity with $\mathcal{D}^L$-cospectrality and provide examples of pairs of graphs that are both $\mathcal{D}^{\mathcal{L}}$-cospectral and $\mathcal{D}^L$-cospectral, as well as pairs only cospectral with respect to one of the matrices. We also exhibit graphs that are cospectral only for $\mathcal{D}^{\mathcal{L}}$ and graphs that are cospectral for all matrices $A, L, Q, \mathcal{L}, \mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q, \mathcal{D}^{\mathcal{L}}$. In Section 2.5.2, the number of graphs on 10 or fewer vertices with a $\mathcal{D}^{\mathcal{L}}$-cospectral pair is computed and compared with the number of graphs with a M-cospectral pair, where $M = A, L, Q, \mathcal{L}, \mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q$. That section also includes a discussion of computational methods.

2.5.1 Cospectral pairs on 10 or fewer vertices

The first instance of cospectral graphs with respect to the normalized distance Laplacian occurs on 8 vertices and there is only one such pair, shown in Figure 2.2. Using *Sage* [20] we can compute their $\mathcal{D}^{\mathcal{L}}$ characteristic polynomial: $p_{\mathcal{D}^{\mathcal{L}}}(x) = x^8 - 8x^7 + \frac{317947}{11616}x^6 - \frac{5428399}{104544}x^5 + \frac{24668087}{418176}x^4 - \frac{4196075}{104544}x^3 + \frac{575771}{38016}x^2 - \frac{85211}{34848}x$. The only difference between the graphs is the light colored edge, and we refer to the maximal shared subgraph (i.e. the graph that results in removing the light colored edge from either graph) as the *base graph*.

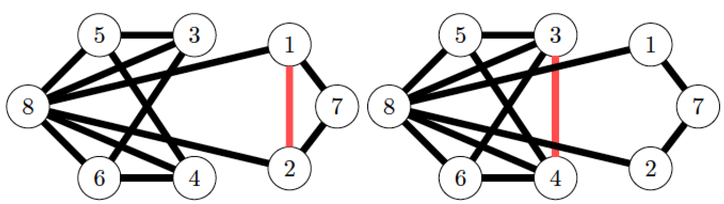

Figure 2.2 The only $\mathcal{D}^{\mathcal{L}}$-cospectral pair on 8 vertices

If a graph G has vertices v_1, v_2, v_3, and v_4 such that v_1 and v_2 are non-adjacent twins, v_3 and v_4 are non-adjacent twins, and $\mathrm{t}(v_1) = \mathrm{t}(v_3)$ then we say $\{\{v_1, v_2\}, \{v_3, v_4\}\}$ is a set of *co-transmission twins*. In [4], a cospectral construction is described for the distance Laplacian using co-transmission twins. If a graph G has co-transmission twins $\{\{v_1, v_2\}, \{v_3, v_4\}\}$, then $G + v_1v_2$ and $G + v_3v_4$ are $\mathcal{D}^L$-cospectral. Note that the graphs in Figure 2.2 can be constructed this way from their base graph with co-transmission twins $\{\{1, 2\}, \{3, 4\}\}$, so they are $\mathcal{D}^L$-cospectral as well. Their $\mathcal{D}^L$

characteristic polynomial is $p_{\mathcal{D}^L}(x) = x^8 - 94x^7 + 3756x^6 - 82728x^5 + 1084992x^4 - 8473984x^3 + 36492288x^2 - 66834432x$. However, this construction does not always find $\mathcal{D}^{\mathcal{L}}$-cospectral graphs. The graphs in Figure 2.3 can be constructed from their base graph using co-transmission twins $\{\{1,2\},\{3,4\}\}$, so they are cospectral with respect to $\mathcal{D}^L$, but they are not cospectral with respect to $\mathcal{D}^{\mathcal{L}}$.

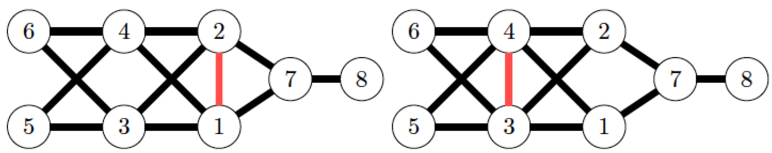

Figure 2.3 Graphs that are $\mathcal{D}^L$ but not $\mathcal{D}^{\mathcal{L}}$-cospectral using the co-transmission twins construction

There are only four pairs of $\mathcal{D}^{\mathcal{L}}$-cospectral graphs on 9 vertices. Three of the pairs that are $\mathcal{D}^{\mathcal{L}}$-cospectral differ by only one edge and have related base graphs. In Figure 2.4, the three pairs can be seen by including 0, 1, or 2 of the dashed edges $\{1,4\}$ and $\{2,3\}$ (note including just the edge $\{1,4\}$ or just the edge $\{2,3\}$ creates isomorphic graphs, so we only need consider one of these cases). When 0 dashed edges are included, their $\mathcal{D}^{\mathcal{L}}$ characteristic polynomial is $p_{\mathcal{D}^{\mathcal{L}}}(x) = x^9 - 9x^8 + \frac{1926013}{54450}x^7 - \frac{259072321}{3267000}x^6 + \frac{2717888893}{24502500}x^5 - \frac{233194363}{2352240}x^4 + \frac{243851297233}{4410450000}x^3 - \frac{587831111}{33412500}x^2 + \frac{674126228}{275653125}x$. When 1 dashed edge is included, their $\mathcal{D}^{\mathcal{L}}$ characteristic polynomial is $p_{\mathcal{D}^{\mathcal{L}}}(x) = x^9 - 9x^8 + \frac{1926211}{54450}x^7 - \frac{9598153}{121000}x^6 + \frac{1812984073}{16335000}x^5 - \frac{24314025553}{245025000}x^4 + \frac{67829453381}{1225125000}x^3 - \frac{10796929657}{612562500}x^2 + \frac{758404}{309375}x$. When both dashed edges are included, their $\mathcal{D}^{\mathcal{L}}$ characteristic polynomial is $p_{\mathcal{D}^{\mathcal{L}}}(x) = x^9 - 9x^8 + \frac{3852769}{108900}x^7 - \frac{21601501}{272250}x^6 + \frac{27208546}{245025}x^5 - \frac{6083465273}{61256250}x^4 + \frac{11318237801}{204187500}x^3 - \frac{1228870232}{69609375}x^2 + \frac{15544256}{6328125}x$.

Again, we see these cospectral pairs may be constructed using a $\mathcal{D}^L$-cospectrality construction from [4]. Let G be a graph of order at least five with $v_1, v_2, v_3, v_4 \in V(G)$. Let $C = \{\{v_1, v_2\}, \{v_3, v_4\}\}$ and $U(C) = V(G) \backslash \{v_1, v_2, v_3, v_4\}$. Then C is a set of *cousins* in G if for all $u \in U(G)$, $d_G(u, v_1) = d_G(u, v_2)$, $d_G(u, v_3) = d_G(u, v_4)$, and $\sum_{u \in U(C)} d_G(u, v_1) = \sum_{u \in U(C)} d_G(u, v_3)$.

Theorem 2.5.1. [4, Theorem 3.10] *Let G be a graph with a set of cousins $C = \{\{v_1, v_2\}, \{v_3, v_4\}\}$ satisfying the following conditions:*

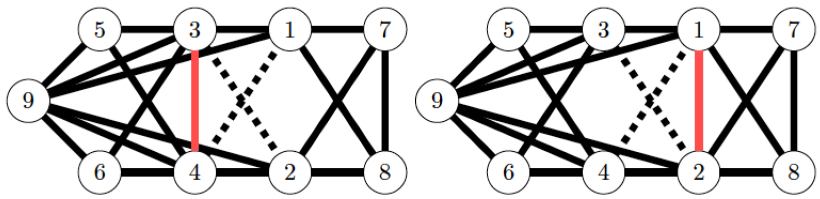

Figure 2.4 Three $\mathcal{D}^{\mathcal{L}}$-cospectral pairs on 9 vertices

- *$v_1v_2, v_3v_4 \notin E(G)$,*

- *and the subgraph of $G+v_1v_2$ induced by $\{v_1, v_2, v_3, v_4\}$ is isomorphic to the subgraph of $G+v_3v_4$ induced by $\{v_1, v_2, v_3, v_4\}$.*

If $G + v_1v_2$ and $G + v_3v_4$ are not isomorphic, then they are $\mathcal{D}^L$-cospectral.

One can easily verify that in each of the three base graphs in Figure 2.4, $C = \{\{1, 2\}, \{3, 4\}\}$ is a set of cousins and the subgraph of $G + \{1, 2\}$ induced by $\{1, 2, 3, 4\}$ is isomorphic to the subgraph of $G + \{3, 4\}$ induced by $\{1, 2, 3, 4\}$. Therefore all three pairs of cospectral graphs can be constructed by Theorem 2.5.1 and so are $\mathcal{D}^L$-cospectral. When 0 dashed edges are included, their $\mathcal{D}^L$ characteristic polynomial is $p_{\mathcal{D}^L}(x) = x^9 - 116x^8 + 5853x^7 - 167806x^6 + 2990335x^5 - 33920980x^4 + 239222875x^3 - 959072786x^2 + 1673692704x$. When 1 dashed edge is included, their $\mathcal{D}^L$ characteristic polynomial is $p_{\mathcal{D}^L}(x) = x^9 - 114x^8 + 5648x^7 - 158862x^6 + 2774997x^5 - 30830726x^4 + 212786586x^3 - 834230170x^2 + 1422606240x$. When both dashed edges are included, their $\mathcal{D}^L$ characteristic polynomial is $p_{\mathcal{D}^L}(x) = x^9 - 112x^8 + 5447x^7 - 150274x^6 + 2572751x^5 - 27995116x^4 + 189113161x^3 - 725242914x^2 + 1209121056x$.

Again, we can see this does not always work. The pair of graphs in Figure 2.5 can be constructed in the way described in Theorem 2.5.1 using their base graph and the set of cousins $\{\{1, 2\}, \{3, 4\}\}$, so they are $\mathcal{D}^L$-cospectral. However, they are not $\mathcal{D}^{\mathcal{L}}$-cospectral.

The last $\mathcal{D}^{\mathcal{L}}$-cospectral pair on 9 vertices is shown in Figure 2.6. While the other three $\mathcal{D}^{\mathcal{L}}$-cospectral pairs involve edge switching, in this pair two additional edges (light colored) are added to the first graph to obtain the second. The $\mathcal{D}^{\mathcal{L}}$ characteristic polynomial of the graphs is $p_{\mathcal{D}^{\mathcal{L}}}(x) =$

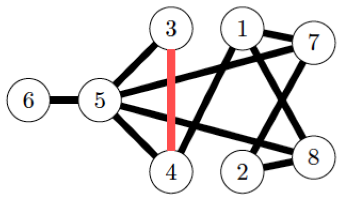

Figure 2.5 Graphs that are $\mathcal{D}^L$ but not $\mathcal{D}^{\mathcal{L}}$-cospectral using the cousins construction

$x^9 - 9x^8 + \frac{23884}{675}x^7 - \frac{7232482}{91125}x^6 + \frac{30369859}{273375}x^5 - \frac{27161183}{273375}x^4 + \frac{15156922}{273375}x^3 - \frac{1608332}{91125}x^2 + \frac{74536}{30375}x$. The graphs in Figure 2.6 show that the number of edges, the degree sequence, and the transmission sequence are not preserved by $\mathcal{D}^{\mathcal{L}}$-cospectrality. The *degree sequence* of a graph G is the list of degrees of the vertices in G and the *transmission sequence* of a graph G is the list of transmissions of the vertices in G. The transmission sequence of G_1 is $[9, 9, 9, 9, 10, 10, 10, 10, 12]$ and the transmission sequence of G_2 is $[9, 9, 9, 9, 9, 9, 10, 10, 10]$. The degree sequence of G_1 is $[4, 6, 6, 6, 6, 7, 7, 7, 7]$ and the degree sequence of G_2 is $[6, 6, 6, 7, 7, 7, 7, 7, 7]$. This pair also provides an example of a cospectral pair that is $\mathcal{D}^{\mathcal{L}}$-cospectral but not $\mathcal{D}^L$-cospectral.

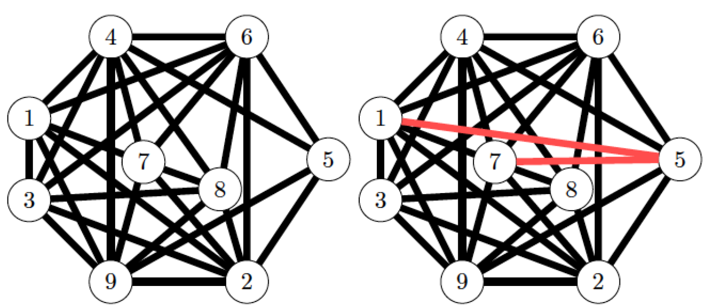

Figure 2.6 G_1 and G_2, $\mathcal{D}^{\mathcal{L}}$-cospectral pair on 9 vertices with a different number of edges, different degree sequences, and different transmission sequences

On 10 vertices, there are 3763 pairs of $\mathcal{D}^{\mathcal{L}}$-cospectral graphs and 4 triples. A graph is *planar* if it can be drawn in the plane without any edges crossing. In Figure 2.7, H_1 is planar and H_2 is not, and the graphs share the $\mathcal{D}^{\mathcal{L}}$ characteristic polynomial $p_{\mathcal{D}^{\mathcal{L}}}(x) = x^{10} - 10x^9 + \frac{11760575}{264992}x^8 - \frac{14698252437}{128107070}x^7 +$

$\frac{1561159495967}{8198852480}x^6 - \frac{56057989 73451}{26646270560}x^5 + \frac{7068694654043}{45679320960}x^4 - \frac{11682868723247}{159877623360}x^3 + \frac{535639931153}{26646270560}x^2 - \frac{65401424433}{26646270560}x$.
Therefore planarity is not preserved by $\mathcal{D}^{\mathcal{L}}$-cospectrality.

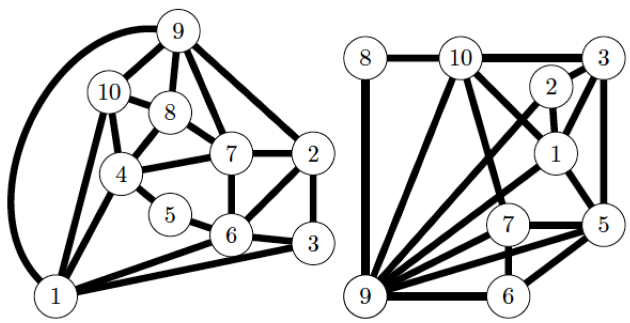

Figure 2.7 H_1 and H_2, $\mathcal{D}^{\mathcal{L}}$ cospectral graphs where one is planar and one is not

The *Weiner index* of a graph G with vertices $V(G) = \{v_1, \ldots, v_n\}$ is $W(G) = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} d(v_i, v_j)$ and the *girth* of a graph is the length of the shortest cycle in the graph. The pair of graphs in Figure 2.8 show girth, Wiener index, k-regularity, and k-transmission regularity are not preserved by $\mathcal{D}^{\mathcal{L}}$-cospectrality. This is in contrast to $\mathcal{D}^{L}$-cospectrality, which preserves the Weiner index (observe $W(G) = \frac{1}{2}\operatorname{trace}\mathcal{D}^{L}(G)$ [3]). While F_1 has girth 5, Weiner index 75, is 3-regular and is 10-transmission regular, F_2 has girth 3, Weiner index 50, is 8-regular, and is 15-transmission regular. Note that while this shows k-regularity and k-transmission regularity are not preserved by $\mathcal{D}^{\mathcal{L}}$-cospectrality, both graphs are still regular and transmission regular so it does not show regularity or transmission regularity are not preserved. These graphs share the $\mathcal{D}^{\mathcal{L}}$ characteristic polynomial $p_{\mathcal{D}^{\mathcal{L}}}(x) = x^{10} - 10x^9 + \frac{222}{5}x^8 - \frac{2872}{25}x^7 + \frac{23861}{125}x^6 - \frac{660126}{3125}x^5 + \frac{486504}{3125}x^4 - \frac{230256}{3125}x^3 + \frac{63504}{3125}x^2 - \frac{7776}{3125}x$.

The *diameter* of a graph G is the maximum distance between any pair of vertices in the graph. In [11], the authors show that for r-regular graphs with diameter at most 2, if $\lambda_1 \leq \cdots \leq \lambda_{n-1} \leq \lambda_n = r$, then $\partial_i = -\lambda_i - 2$ for $1 \leq i \leq n-1$ and $\partial_n = 2n - 2 - r$. So in the case of graphs with diameter at most two that are regular and transmission regular, the eigenvalues of $A, L, Q, \mathcal{L}, \mathcal{D}, \mathcal{D}^{L}$, and $\mathcal{D}^{Q}$ can all be obtained from each other using the above result and Observations 2.2.5 and 2.2.6. Since both of the graphs in Figure 2.8 are regular, transmission regular, and have diameter 2, we

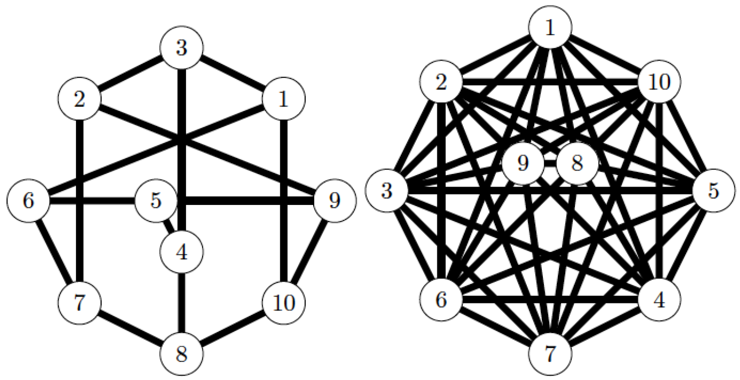

Figure 2.8 F_1 and F_2 (the Petersen graph and cocktail party graph, respectively), $\mathcal{D}^{\mathcal{L}}$ cospectral graphs with different girth, Weiner indexes, k-regularity, and k-transmission regularity and are only M-cospectral for $M = \mathcal{D}^{\mathcal{L}}$

can easily calculate their spectra for $A, L, Q, \mathcal{L}, \mathcal{D}, \mathcal{D}^L$, and $\mathcal{D}^Q$ from their $\mathcal{D}^{\mathcal{L}}$-spectrum. However, since the two graphs have different regularity and different transmission regularity, it is clear that their spectra will be different for every other matrix. Therefore F_1 and F_2 serve as an example of graphs that are only cospectral with respect to $\mathcal{D}^{\mathcal{L}}$.

We can also find a pair of graphs that are M-cospectral for all $M = A, L, Q, \mathcal{L}, \mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q$ and $\mathcal{D}^{\mathcal{L}}$. In Figure 2.9, the two graphs are both diameter 2, 5-regular, and 13-transmission regular and they are $\mathcal{D}^{\mathcal{L}}$-cospectral, so they will be M-cospectral for all $M = A, L, Q, \mathcal{L}, \mathcal{D}, \mathcal{D}^L$, and $\mathcal{D}^Q$ as well. We can also see this by noting that $\phi(\lambda, r, L_1) = \phi(\lambda, r, L_2)$ and $\phi^{\mathcal{D}}(\lambda, r, L_1) = \phi^{\mathcal{D}}(\lambda, r, L_2)$.

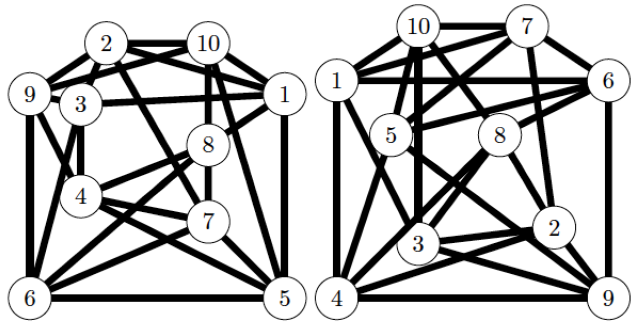

Figure 2.9 L_1 and L_2, Graphs that are M cospectral for $M = A, L, Q, \mathcal{L}, \mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q$ and $\mathcal{D}^{\mathcal{L}}$

2.5.2 The number of graphs with a cospectral mate

The number of graphs that have cospectral mates has been computed for all graphs on 10 and fewer vertices for all matrices discussed in this paper (except for $\mathcal{L}$, for which the number of graphs with a $\mathcal{L}$-cospectral mate has only been computed for 9 and fewer vertices). These values are given in Tables 2.3 and 2.4. In Table 2.5, the percentage of graphs that have a cospectral mate is given for each matrix. It is obvious that $\mathcal{D}^{\mathcal{L}}$ has significantly fewer graphs with a cospectral mate than any previously studied matrix on 10 or fewer vertices and we conjecture this pattern continues into larger number of vertices. This makes the normalized distance Laplacian a useful tool for determining if two connected graphs are isomorphic.

Table 2.3 Number of graphs with a cospectral mate with respect to each matrix. Counts for A, L, Q from [15], counts for $\mathcal{L}$ from [8].

n	# graphs	A	L	Q	$\mathcal{L}$
3	4	0	0	0	0
4	11	0	0	2	2
5	34	2	0	4	4
6	156	10	4	16	14
7	1,044	110	130	102	52
8	12,346	1,722	1,767	1,201	201
9	274,668	51,038	42,595	19,001	1,092
10	12,005,168	2,560,516	1,412,438	636,607	

Table 2.4 Number of connected graphs with a cospectral pair with respect to each matrix. Counts for $\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q$ from [3].

n	# connected graphs	$\mathcal{D}$	$\mathcal{D}^L$	$\mathcal{D}^Q$	$\mathcal{D}^{\mathcal{L}}$
3	2	0	0	0	0
4	6	0	0	0	0
5	21	0	0	2	0
6	112	0	0	6	0
7	853	22	43	38	0
8	11,117	658	745	453	2
9	261,080	25,058	19,778	8,168	8
10	11,716,571	1,389,984	787,851	319,324	7538

Table 2.5 Percent of total graphs (for $A, L, Q, \mathcal{L}$) / connected graphs (for $\mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q, \mathcal{D}^{\mathcal{L}}$) that have a cospectral mate with respect to each matrix.

n	A	L	Q	$\mathcal{L}$	$\mathcal{D}$	$\mathcal{D}^L$	$\mathcal{D}^Q$	$\mathcal{D}^{\mathcal{L}}$
3	0	0	0	0	0	0	0	0
4	0	0	18.1818%	18.1818%	0	0	0	0
5	5.8824%	0	11.7647%	11.7647%	0	0	9.5238%	0
6	6.4103%	2.5641%	10.2564%	8.9744%	0	0	5.3571%	0
7	10.5354%	12.4521%	9.7701%	4.9808%	2.5791%	5.0410%	4.4549%	0
8	13.9478%	14.3123%	9.7278%	1.6281%	5.91886%	6.7014%	4.0748%	0.0180%
9	18.5817%	15.5078%	6.9178%	0.3976%	9.5978%	7.5755%	3.1285%	0.0031%
10	21.3284%	11.7653%	5.3028%		11.8634%	6.7242%	2.7254%	0.0643%

The similar matrix $T^{-1}\mathcal{D}^L$ was used for all computations rather than $\mathcal{D}^{\mathcal{L}}$ since it does not include square roots and runs more quickly on *Sage*. To find $\mathcal{D}^{\mathcal{L}}$-cospectral graphs graphs on 8 and fewer vertices, it was sufficient to use the *Sage* command cospectral.graphs(), which takes as its input any matrix defined with respect to a graph. However, this method was too computationally slow for 9 and 10 vertices.

The method used for 9 and 10 vertices is a multi-step process that sorts the graphs into groups of potentially cospectral graphs with an approximation of their characteristic polynomials using double precision decimal arithmetic. Each characteristic polynomial is evaluated at a large number, and then the floor and ceiling of the result is taken modulo a large prime number. The graphs are then sorted into groups by this value. Using both the floor and ceiling is to ensure cospectral graphs end up in the same group at least once, despite any numerical approximation error. Within these groups, potential cospectral graphs are found by evaluating each approximated characteristic polynomial at a prime number, and searching for pairs of graphs for which this value is within an $\epsilon = 0.00005$ tolerance. Then each of these pairs is checked for cospectrality using their exact characteristic polynomial.

2.6 Concluding remarks

In this paper we introduced the normalized distance Laplacian. In Section 2.2, we derive bounds on its eigenvalues. Most notably, we show $\partial^{\mathcal{L}} < 2$ for all graphs G on $n \geq 3$ vertices, in contrast to the normalized Laplacian, which has the property $\mu = 2$ if and only if the graph is bipartite. It is natural to ask the following further questions: What is the maximum $\mathcal{D}^{\mathcal{L}}$ spectral radius achieved by a graph on n vertices and which graphs achieve it? Based on the behavior of the family KPK_{n_1,n_2,n_3} for large n_1, n_2, n_3, we conjecture that the maximum $\mathcal{D}^{\mathcal{L}}$ spectral radius tends to 2 as n becomes large and that this value is achieved by some graph in the family KPK_{n_1,n_2,n_3}. We also found that the complete graph K_n achieves the minimal spectral radius and conjecture that is the only such graph.

It would be interesting to find methods for constructing $\mathcal{D}^{\mathcal{L}}$-cospectral graphs. Since in Section 2.5 we show examples of $\mathcal{D}^{L}$-cospectral constructions producing $\mathcal{D}^{\mathcal{L}}$-cospectral graphs, it seems likely that a suitable additional restriction placed on a $\mathcal{D}^{L}$-cospectral construction may provide a $\mathcal{D}^{\mathcal{L}}$-cospectral construction method.

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CHAPTER 3. A NOTE ON THE PRESERVATION OF GRAPH PARAMETERS BY COSPECTRALITY FOR THE DISTANCE MATRIX AND ITS VARIANTS

Modified from a manuscript which has been submitted for publication

Carolyn Reinhart

Department of Mathematics, Iowa State University

Abstract

Cospectrality for the distance matrix and its Laplacians has been shown to preserve or not preserve various graph parameters. We summarize known results and show several parameters are not preserved by cospectrality for the distance matrix, the signless distance Laplacian, the distance Laplacian, and the normalized distance Laplacian. Furthermore, we prove that two transmission regular graphs which are distance cospectral must have the same transmission and thus the same Wiener index.

Keywords: Distance matrices, cospectrality, parameter preservation, graph complement

3.1 Introduction

Graph cospectrality and the preservation of parameters by cospectrality has been the focus of much study in recent years. For a matrix M, two graphs G and H are *M-cospectral* if they have the same M spectrum. A graph parameter is *preserved by M-cospectrality* if two graphs that are M-cospectral must share the same value for that parameter (can be numeric or true/false). In this chapter, we summarize some known results about parameter preservation for the distance matrix and its variants. Furthermore, we provide additional examples that show parameters are not preserved for various matrices.

In a graph G, the *distance* between vertices v_i and v_j, denoted, $d(v_i, v_j)$, is the number of edges in a shortest path between v_i and v_j. The *diameter* of a graph is the maximum distance between two vertices in the graph. The *transmission* of a vertex $v \in V(G)$, denoted $\mathrm{t}_G(v)$, is the sum of the distances from v to all other vertices, i.e. $\mathrm{t}_G(v) = \sum_{u_i \in V(G)} d(v, u_i)$. A graph is *$k$-transmission regular* if $\mathrm{t}(v) = k$ for all $v \in V$.

In [5], the *distance matrix*, denoted $\mathcal{D}(G)$, was defined and has entries $(\mathcal{D}(G))_{ij} = d(v_i, v_j)$. In order to ensure $d(v_i, v_j)$ is finite for every pair of vertices $v_i, v_j \in V(G)$, we require the graph G be connected. The *transmission matrix* is the diagonal matrix $T(G) = \operatorname{diag}(\mathrm{t}(v_1), \ldots, \mathrm{t}(v_n))$. For a connected graph G, Aouchiche and Hansen [2] defined the *distance Laplacian matrix*, denoted $\mathcal{D}^L(G)$, such that $\mathcal{D}^L(G) = T(G) - \mathcal{D}(G)$ and the *signless distance Laplacian*, denoted $\mathcal{D}^Q(G)$, such that $\mathcal{D}^Q(G) = T(G) + \mathcal{D}(G)$. The *normalized distance Laplacian matrix* of a connected graph G, is defined in [13] and has entries

$$(\mathcal{D}^{\mathcal{L}}(G))_{ij} = \begin{cases} -\frac{1}{\sqrt{\mathrm{t}(v_i)\,\mathrm{t}(v_j)}} & i \neq j \\ 1 & i = j \end{cases}.$$

Observe that $\mathcal{D}^{\mathcal{L}}(G) = T(G)^{-1/2}\mathcal{D}^L(G)T(G)^{-1/2} = I - T(G)^{-1/2}\mathcal{D}(G)T(G)^{-1/2}$.

The following definitions are standard in graph theory and will be used throughout. A *graph* is a pair $G = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ is the set of *vertices* and E is the set of *edges*. Each edge is an unordered set of two distinct vertices $\{v_i, v_j\}$, usually denoted as just $v_i v_j$, for $1 \leq i \neq j \leq n$. The *complement* of a graph G, denoted $\overline{G}$, is the graph with $v_i v_j \in E(\overline{G})$ if and only if $v_i v_j \notin E(G)$. A graph G is *connected* if for all $u, v \in V(G)$, there exists a path from u to v. A connected component of a graph G is a maximal subgraph that is connected; a connected graph is considered to have one connected component. Since the study of distance matrices requires it, all graphs G considered for cospectrality will be connected. However, we note that the graph complement $\overline{G}$ is frequently disconnected.

For an $n \times n$ real symmetric matrix M and a n-vector $\mathbf{x}$, the quotient $\frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$ is a *Rayleigh quotient* of M. The Rayleigh quotients of a matrix can be used to determine the determine eigenvalues of the matrix. It particular, if λ_n is the largest eigenvalue of M, $\lambda_n = \max_{\mathbf{x} \neq 0} \frac{\mathbf{x}^T M \mathbf{x}}{\mathbf{x}^T \mathbf{x}}$.

3.2 Preservation of parameters

A great many parameters have been shown to be preserved or not preserved by M-cospectrality for $M = \mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q, \mathcal{D}^{\mathcal{L}}$. It is obvious that two graphs G and H must have the same order to be M cospectral for any matrix M, therefore graph order is preserved by M-cospectrality for $M = \mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q, \mathcal{D}^{\mathcal{L}}$. Similarly, the trace of a matrix M, $\text{tr}(M)$, must be preserved by M-cospectrality since it is equal to the sum of the eigenvalues. Some known results are summarized in Table 3.1; in this table, a question mark indicates that it has been verified that no example of non-preservation exists on ten or fewer vertices. This verification was performed using *Sage* [8]. Next, we will list sources or examples for each non-trivial answer.

Table 3.1 Some parameters that are known to preserved or not preserved by M-cospectrality for $M = \mathcal{D}, \mathcal{D}^L, \mathcal{D}^Q, \mathcal{D}^{\mathcal{L}}$.

	$\mathcal{D}$	$\mathcal{D}^L$	$\mathcal{D}^Q$	$\mathcal{D}^{\mathcal{L}}$
# Edges	No	No	?	No
Diameter	No	No	?	?
Girth	No	No	?	No
Planarity	No	No	No	No
Wiener index	No	Yes	Yes	No
Degree sequence	No	No	No	No
Transmission sequence	No	No	No	No
Transmission regularity	?	?	Yes	?
# connected components in $\overline{G}$	No	Yes	No	No

The number of edges in a graph is not preserved by cospectrality for $\mathcal{D}$ ([6]), $\mathcal{D}^L$ ([4]), or $\mathcal{D}^{\mathcal{L}}$ ([7]) and the diameter of a graph has been shown not to be preserved by cospectrality for $\mathcal{D}$ ([1]) and $\mathcal{D}^L$ ([4]). The girth of a graph is the length of the shortest cycle in the graph. Girth was shown not to be preserved by cospectrality for $\mathcal{D}^L$ ([4]) and $\mathcal{D}^{\mathcal{L}}$ ([7]); we show now in Example 3.2.1 that girth is not preserved by $\mathcal{D}$-cospectrality.

Example 3.2.1. The graphs G_1 and G_2 in Figure 3.1 are $\mathcal{D}$-cospectral with distance characteristic polynomial $p_{\mathcal{D}}(x) = x^9 - 112x^7 - 758x^6 - 1994x^5 - 2010x^4 + 184x^3 + 1262x^2 + 193x - 222$. The girth of G_1 is 4 and the girth of G_2 is 3.

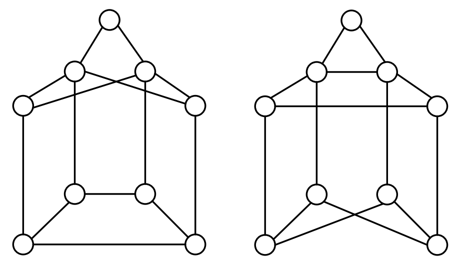

Figure 3.1 Pair of $\mathcal{D}$-cospectral graphs that show the girth of G is not preserved by $\mathcal{D}$-cospectrality.

A graph is planar if it can be drawn in a way such that no edges intersect each other. Planarity was shown not to be preserved by $\mathcal{D}^L$-cospectrality in [4] and $\mathcal{D}^{\mathcal{L}}$-cospectrality in [7]. We show it is not preserved by $\mathcal{D}$-cospectrality in Example 3.2.3 and by $\mathcal{D}^Q$-cospectrality in Example 3.2.2.

Example 3.2.2. The graphs G_1 and G_2 in Figure 3.2 are $\mathcal{D}^Q$-cospectral with signless distance Laplacian characteristic polynomial $p_{\mathcal{D}^Q}(x) = x^8 - 88x^7 + 3296x^6 - 69002x^5 + 886299x^4 - 7169822x^3 + 35735188x^2 - 100453184x + 122045040$. G_1 is planar and G_2 is not planar.

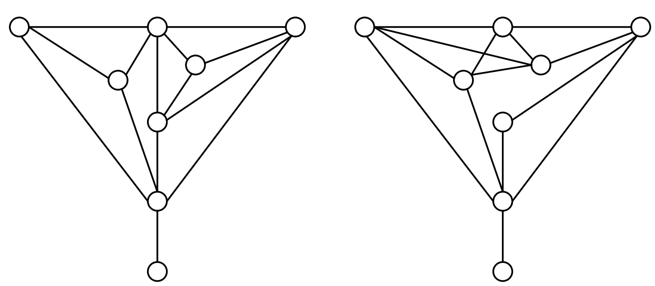

Figure 3.2 Pair of $\mathcal{D}^Q$-cospectral graphs that show planarity is not preserved by $\mathcal{D}^Q$-cospectrality.

The Wiener index of a graph is the sum of all pairs of distances in G, i.e., $\mathrm{W}(G) = \frac{1}{2}\sum_{u,v\in V(G)} d(u,v)$. One can observe that $\mathrm{W}(G) = \frac{1}{2}\sum_{v\in V(G)} \mathrm{t}(v) = \frac{1}{2}\operatorname{tr}(\mathcal{D}^L(G)) = \frac{1}{2}\operatorname{tr}(\mathcal{D}^Q(G))$, and so Wiener index is preserved by $\mathcal{D}^L$ and $\mathcal{D}^Q$ cospectrality. However, it was shown in [1] that the Wiener index is not preserved by $\mathcal{D}$-cospectrality and it was shown in [7] that it is not preserved by $\mathcal{D}^{\mathcal{L}}$-cospectrality.

The degree sequence of a graph is the list of degrees of vertices in the graph in increasing order and the transmission sequence of a graph is the list of transmissions of vertices in the graph in increasing order. The degree sequence and transmission sequence were shown not to be preserved by M-cospectrality for $\mathcal{D}^L$ in [4] and $\mathcal{D}^{\mathcal{L}}$ in [7]. In Examples 3.2.3 and 3.2.4, respectively, we show the degree sequence and transmission sequence of a graph are not preserved by $\mathcal{D}$ and $\mathcal{D}^Q$ cospectrality.

In [3], it is shown that the property of being transmission regular is preserved by $\mathcal{D}^Q$-cospectrality and the number of connected components of the graph complement $\overline{G}$ is shown to be preserved by $\mathcal{D}^L$-cospectrality in [3]. In Examples 3.2.3, 3.2.4, and 3.2.5, we show the number of connected components of $\overline{G}$ is not preserved by cospectrality for $\mathcal{D}$, $\mathcal{D}^Q$, or $\mathcal{D}^{\mathcal{L}}$.

Example 3.2.3. The graphs G_1 and G_2 in Figure 3.3 are $\mathcal{D}$-cospectral with distance characteristic polynomial $p_{\mathcal{D}}(x) = x^7 - 39x^5 - 142x^4 - 180x^3 - 72x^2$. The complement of G_1 has one connected component and the complement of G_2 has two connected components. G_1 is planar and G_2 is not planar. Finally, the degree sequences of G_1 and G_2, respectively, are $[3, 4, 4, 4, 5, 5, 5]$ and $[4, 4, 4, 4, 4, 4, 6]$ and the transmission sequences are $[7, 7, 7, 8, 8, 8, 9]$ and $[6, 8, 8, 8, 8, 8, 8]$.

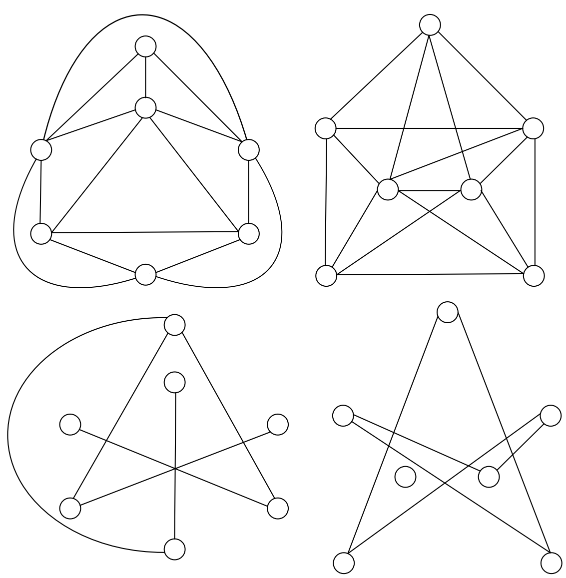

Figure 3.3 Pair of $\mathcal{D}$-cospectral graphs (above) and their complements (below) that show the number of connected components of $\overline{G}$, planarity, the degree sequence, and the transmission sequence are not preserved by $\mathcal{D}$-cospectrality.

Example 3.2.4. The graphs G_1 and G_2 in Figure 3.4 are $\mathcal{D}^Q$-cospectral with signless distance Laplacian characteristic polynomial $p_{\mathcal{D}^Q}(x) = x^5 - 26x^4 + 249x^3 - 1132x^2 + 2480x - 2112$. The complement of G_1 has two connected components and the complement of G_2 has three connected components. The degree sequences of G_1 and G_2, respectively, are $[1, 3, 3, 3, 4]$ and $[2, 2, 2, 4, 4]$ and the transmission sequences are $[4, 5, 5, 5, 7]$ and $[4, 4, 6, 6, 6]$.

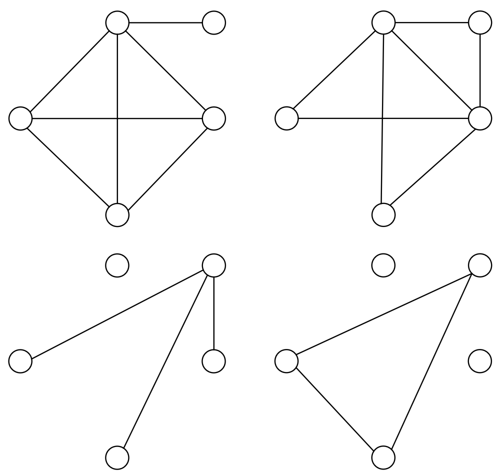

Figure 3.4 Pair of $\mathcal{D}^Q$-cospectral graphs (above) and their complements (below) that show the number of connected components of $\overline{G}$, the degree sequence, and the transmission sequence are not preserved by $\mathcal{D}^Q$-cospectrality.

Example 3.2.5. The graphs G_1 and G_2 in Figure 3.5 are $\mathcal{D}^{\mathcal{L}}$-cospectral with normalized distance Laplacian characteristic polynomial $p_{\mathcal{D}^{\mathcal{L}}}(x) = x^{10} - 10x^9 + 222/5x^8 - 2872/25x^7 + 23861/125x^6 - 660126/3125x^5 + 486504/3125x^4 - 230256/3125x^3 + 63504/3125x^2 - 7776/3125x$. The complement of G_1 has one connected components and the complement of G_2 has five connected components.

In [3], it is shown that the property of being transmission regular is preserved by $\mathcal{D}^Q$-cospectrality. This same proof can not be applied to the distance matrix; however, a weaker result holds. In [1], it was shown that if two k-transmission regular graphs are $\mathcal{D}$-cospectral, then they have the same Wiener index. We improve this result in Proposition 3.2.7. But first, we prove a corollary.

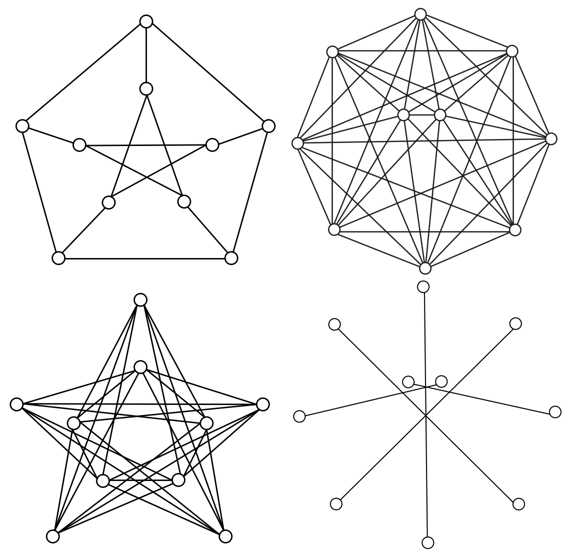

Figure 3.5 A pair of $\mathcal{D}^{\mathcal{L}}$-cospectral graphs G_1 and G_2 (above), the Petersen graph and cocktail party graph, respectively, and their complements $\overline{G_1}$ and $\overline{G_2}$ (below), which show that the number of connected components of $\overline{G}$ is not preserved by $\mathcal{D}^{\mathcal{L}}$-cospectrality.

Corollary 3.2.6. *For a connected graph G, let $t_{\min}$ be the minimum transmission, $t_{\max}$ be the maximum transmission, and $\bar{t}$ be the average transmission. Then*

$$t_{min} \leq \bar{t} \leq \rho\left(\mathcal{D}(G)\right) \leq t_{\max}$$

and equality holds if and only if G is transmission regular.

Proof. Applying the Rayleigh quotient and considering the vector $\mathbb{1}$ of all 1s, we have

$$\rho\left(\mathcal{D}(G)\right) = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T \mathcal{D}(G)\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geq \frac{\mathbb{1}^T \mathcal{D}(G)\mathbb{1}}{\mathbb{1}^T\mathbb{1}} = \frac{1}{n}\sum_{i=1}^{n} t(v_i).$$

Observe $\frac{1}{n}\sum_{i=1}^{n} t(v_i) = \bar{t} \geq t_{\min}$. To obtain the upper bound, observe $t_{\max}$ is the maximum row sum of $\mathcal{D}(G)$. It is clear $\bar{t} = t_{\max}$ only if G is transmission regular and if G is transmission regular, $\mathbb{1}$ is an eigenvector for the eigenvalue $t(v) = \rho\left(\mathcal{D}(G)\right)$. □

Now we apply Corollary 3.2.6 to prove the result.

Proposition 3.2.7. *Let G_1 and G_2 be transmission regular with transmissions t_1 and t_2 respectively. If G_1 and G_2 are $\mathcal{D}$-cospectral, then $t_1 = t_2 = \rho\left(\mathcal{D}(G_1)\right) = \rho\left(\mathcal{D}(G_2)\right)$ and thus $\mathrm{W}(G_1) = \mathrm{W}(G_2)$.*

Proof. Applying Corollary 3.2.6, we see $\rho(\mathcal{D}(G_1)) = t_1$ and $\rho(\mathcal{D}(G_2)) = t_2$. By $\mathcal{D}$-cospectrality, $\rho(\mathcal{D}(G_1)) = \rho(\mathcal{D}(G_2))$. Finally, $\mathrm{W}(G_1) = \frac{n}{2}t_1 = \frac{n}{2}t_2 = \mathrm{W}(G_2)$. □

3.3 Concluding remarks

This chapter showed that girth is not preserved by distance cospectrality and that planarity, degree sequence, and transmission sequence are not preserved by distance or signless distance Laplacian cospectrality. We also showed that the number of connected components of the graph complement is not preserved by cospectrality for the distance, signless distance Laplacian, or normalized distance Laplacian matrix. Thus, the distance Laplacian is unique in preserving this parameter among the distance matrix and its Laplacians. Finally, we proved that transmission regular graphs that are distance cospectral must have the same transmission and Wiener index.

As denoted by question marks in Table 3.1, many open questions remain. In particular, it would be interesting to determine if transmission regularity is preserved for $\mathcal{D}$, $\mathcal{D}^Q$, and $\mathcal{D}^{\mathcal{L}}$. Since no example exhibiting non-preservation exists on 10 or fewer vertices, it would be useful to find a cospectral construction which produces a transmission regular and non-transmission regular graph.

3.4 References

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CHAPTER 4. DISTANCE COSPECTRALITY IN DIGRAPHS

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Carolyn Reinhart

Department of Mathematics, Iowa State University

Abstract

The distance matrix of a digraph Γ, $\mathcal{D}(\Gamma)$, is the matrix whose i, jth entry is the distance from vertex v_i to vertex v_j. In order for this matrix to be defined, we consider only strongly connected digraphs, i.e., digraphs for which there is a dipath from v_i to v_j for every pair of vertices. In this paper, the number of digraphs with a distance cospectral mate is found for 6 and fewer vertices. It is observed that many instances of cospectrality can be accounted for by arc reversal. Using generalized cycle decompositions, cospectral constructions are found that do not rely on arc reversal and produce pairs of distance cospectral digraphs from a digraph containing twin vertices with certain structural properties.

Keywords: Distance matrix, digraphs, cospectrality, twins

4.1 Introduction

The distance matrix of a connected graph was defined in 1971 by Graham and Pollak [5] in order to study the problem of loop switching in routing messages through a network. Since then, the distance matrix has been studied extensively for graphs. A 2014 survey by Aouchiche and Hansen cites over 150 papers and includes results on the distance characteristic polynomial, spectral radius, and general eigenvalue bounds [2]. Cospectral constructions have been found for the distance matrix such as those in [12], [1], [6], and [11]. Less work has been done thus far regarding the distance matrix of digraphs. Bounds on the the distance spectral radius of digraphs and extremal digraphs achieving these bounds have been found for various classes of graphs, such as in [9], [8], and [10]. The distance spectra of digraph products was studied in [4].

In a digraph Γ, the *distance* between vertices v_i and v_j, denoted, $d_\Gamma(v_i, v_j)$, is the number of arcs in a shortest path from v_i to v_j. The *distance matrix of a strongly connected digraph* Γ, denoted $D(\Gamma)$, is the real matrix whose ijth entry is $d(v_i, v_j)$. Note that unlike for graphs, the distance matrix of a digraph need not be symmetric. This presents a particular challenge, as many techniques for graphs rely heavily on the basis of eigenvectors that is guaranteed by symmetry.

Two non-isomorphic digraphs Γ_1 and Γ_2 are $\mathcal{D}$*-cospectral* if they have the same $\mathcal{D}$-spectrum. In this case, Γ_1 and Γ_2 are called $\mathcal{D}$*-cospectral mates*. In Section 4.2, the number of digraphs with a cospectral mate is determined for digraphs on 6 and fewer vertices. It is observed that most digraphs have a $\mathcal{D}$-cospectral mate through arc reversal and the number of digraphs that have a $\mathcal{D}$-cospectral mate not accounted for by arc reversal is also determined for 6 and fewer vertices.

Twin vertices are a useful tool in the study of spectra of graphs. They can be used to determine the eigenvalues of the graph, for more information see [7]. Cospectral constructions have also been found using twin vertices, such as in [4], which describes a distance Laplacian cospectral construction using twins. However, all of these proofs utilize symmetry and the basis of eigenvectors it guarantees and can not be applied directly to digraphs. In a digraph Γ, vertices u and w are called *out-twins* if $uv \in E(\Gamma)$ if and only if $wv \in E(\Gamma)$ for all $v \in V(\Gamma)$ such that $v \neq u, w$. If $vu \in E(\Gamma)$ if and only if $vw \in E(\Gamma)$ for all $v \in V(\Gamma)$ such that $v \neq u, w$, then u and w are called *in-twins*. Vertices u and w are called *twins* if they are out-twins and in-twins. If $uw, wu \in E(\Gamma)$, u and w are *doubly adjacent* (in/out)-twins and $uw, wu \notin E(\Gamma)$, u and w are *non-adjacent* (in/out)-twins. In Section 4.3, (in/out)-twins are used to produce $\mathcal{D}$-cospectral digraphs using generalized cycle decompositions to compute the distance characteristic polynomial.

The following definitions and notations are standard and will be used throughout. A *digraph* is a pair $\Gamma = (V, E)$ where $V = \{v_1, \ldots, v_n\}$ is the set of *vertices* and E is the set of arcs. Each arc is an ordered pair of two distinct vertices (v_i, v_j), usually denoted as just v_iv_j, for $1 \leq i, j \leq n$. A *loop* is an arc from a vertex to itself, v_iv_i. Unless otherwise stated, the digraphs we consider will not have loops. A digraph Γ is *strongly connected* if for all $u, v \in V(G)$, there exists a path from u

to v. Since the study of distance matrices requires it, all digraphs in this paper are assumed to be strongly connected unless otherwise stated.

4.2 The number of digraphs with distance cospectral mates

In this section, we will discuss the number of digraphs with a $\mathcal{D}$-cospectral mate. We begin with the following observation, which explains a large number of instances of $\mathcal{D}$-cospectrality. An example of such $\mathcal{D}$-cospectral mates is shown in Figure 4.1.

Observation 4.2.1. *Let Γ be a strongly regular digraph and let Γ^{T} be the digraph that results from arc reversal of Γ; i.e. $uv \in E(\Gamma^T)$ if and only if $vu \in E(\Gamma)$. Observe $\mathcal{D}(\Gamma^{\mathrm{T}}) = \mathcal{D}(\Gamma)^{\mathrm{T}}$. Therefore, if Γ and Γ^{T} are non-isomorphic, then they are $\mathcal{D}$-cospectral.*

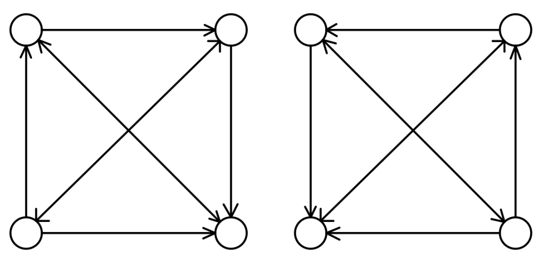

Figure 4.1 Two non-isomorphic digraphs that are $\mathcal{D}$-cospectral by Observation 4.2.1.

Using *Sage*, the number of digraphs with a $\mathcal{D}$-cospectral mate was computed for digraphs on 6 and fewer vertices [13]. Since many instances of cospectrality can be explained by Observation 4.2.1, the number of digraphs with a $\mathcal{D}$-cospectral mate not obtained by arc reversal was also computed. There are no D-cospectral digraphs on 3 or fewer vertices, the results for digraphs on $n = 4, 5, 6$ vertices are summarized in Table 4.1 and are given as percentages of the total number of strongly connected digraphs in Table 4.2.

It is worth noting that the percentage of digraphs with D-cospectral mates not obtained by arc reversal is still quite high. For comparison, the percentage of connected graphs $\mathcal{D}$-cospectral mates is listed in Table 4.3 for graphs on 10 or fewer vertices (there are no D-cospectral graphs on 6 or fewer vertices).

Table 4.1 Number of strongly connected digraphs with a $\mathcal{D}$-cospectral pair

n	# strongly connected digraphs	# with cospec mate	# with cospec mate not by arc reversal
4	83	48	18
5	5,048	4,643	1,943
6	1,047,008	1,037,797	503,399

Table 4.2 Percentage of strongly connected digraphs with a $\mathcal{D}$-cospectral pair

n	% with cospec mate	% with cospec mate not by arc reversal
4	57.8313%	21.6867%
5	91.9770%	38.4905%
6	99.1203%	48.0798%

4.3 Constructions for distance cospectral digraphs using twins

In this section, we will describe a construction that produces sets of $\mathcal{D}$-cospectral digraphs from a digraph containing (in/out) twins with certain properties. To do this, we will first determine the distance characteristic polynomial of a digraph in terms of generalized cycle decompositions (see [3]). For an $n \times n$ matrix $M = [m_{ij}]$,

$$\det(M) = \sum_{\sigma \in S_n} \left(\operatorname{sgn}(\sigma) \prod_{k=1}^{n} m_{k,\sigma(k)} \right).$$

Let Γ_M denoted the digraph such that ij is an arc if and only if $m_{ij} \neq 0$. For a permutation σ that contributes a non-zero value to the above summand, each factor $m_{k,\sigma(k)}$ corresponds to an arc in the digraph and exactly n arcs in the digraph will contribute to each product. This collection of n arcs is a *generalized cycle decomposition* of Γ_M. Each generalized cycle decomposition is a collection of cycles of length 1 (loops), cycles of length 2 (the arcs ij and ji), and cycles of length at least 3, such that each vertex in Γ_M is in exactly one cycle.

We note that in order to determine the characteristic polynomial of $\mathcal{D}(\Gamma)$ for some digraph Γ, we will use generalized cycle decompositions of $\Gamma_{xI_n - \mathcal{D}(\Gamma)}$. Since $M = xI_n - \mathcal{D}(\Gamma)$ has no zero entries, $\Gamma_{xI_n-\mathcal{D}(\Gamma)} = \overleftrightarrow{K}_n^{\ell}$, the complete digraph with loops on every vertex. Furthermore, since $xI_n - \mathcal{D}(\Gamma)$ is not symmetric, there is no need to differentiate between cycles of length 2 and cycles

Table 4.3 Percent of connected graphs that have a $\mathcal{D}$-cospectral mate

n	# connected graphs	# graphs with cospec mate	% of graphs with cospec mate
7	853	22	2.5791%
8	11,117	658	5.91886%
9	261,080	25,058	9.5978%
10	11,716,571	1,389,984	11.8634%

of length at least 3. Thus, from here on, we will consider each generalized cycle decomposition of $\overleftrightarrow{K_n^\ell}$ to be a collection of loops and cycles.

For a particular generalized cycle decomposition of $\overleftrightarrow{K_n^\ell}$, denoted D, let $\ell(D)$ be the number of loops in the decomposition. Let $C(D)$ be the set of arcs contained in cycles in the decomposition and let $e(D)$ be the number of cycles in the decomposition of even length.

Proposition 4.3.1. *For a digraph Γ, the characteristic polynomial of $\mathcal{D}(\Gamma)$ can be written*

$$p_{\mathcal{D}(\Gamma)}(x) = \sum_{D}(-1)^{e(D)+n-\ell(D)}\left(\prod_{ij\in C(D)} \mathrm{d}\,(v_i, v_j)\right) x^{\ell(D)},$$

where the sum runs over all generalized cycle decompositions of the complete digraph with loops $\overleftrightarrow{K_n^\ell}$.

Proof. The distance characteristic polynomial is

$$p_{\mathcal{D}(\Gamma)}(x) = \sum_{\sigma\in S_n}\left(\operatorname{sgn}(\sigma)\prod_{k=1}^{n}(xI_n - \mathcal{D}(\Gamma))_{k,\sigma(k)}\right).$$

Each σ corresponds to a generalized cycle decomposition of $\overleftrightarrow{K_n^\ell}$. For a generalized cycle decomposition D, each loop contributes $(xI_n - \mathcal{D}(\Gamma))_{k,\sigma(k)} = x$ and each arc ij in a cycle contributes $(xI_n - \mathcal{D}(\Gamma))_{i,j} = -\mathrm{d}\,(v_i, v_j)$. Note the total number of arcs in cycles is $n - \ell(D)$. Finally, since cycles of an even length in D correspond to odd permutations, $\operatorname{sgn}(\sigma) = (-1)^{e(D)}$. Thus,

$$\begin{aligned} p_{\mathcal{D}(\Gamma)}(x) &= \sum_{D}(-1)^{e(D)}x^{\ell(D)}\left(\prod_{ij\in C(D)} -\mathrm{d}\,(v_i, v_j)\right) \\ &= \sum_{D}(-1)^{e(D)+n-\ell(D)}\left(\prod_{ij\in C(D)} \mathrm{d}\,(v_i, v_j)\right) x^{\ell(D)} \end{aligned}$$

as desired. $\square$

We now apply Proposition 4.3.1 to prove our main result; constructing cospectral digraphs from a digraph containing twin vertices with certain properties. Theorem 4.3.2 considers digraphs with doubly adjacent out-twins, an example of this construction is shown in Figure 4.2. Note that in the following theorem, Γ_1 need not be strongly connected because the distance matrix of Γ_1 is not considered.

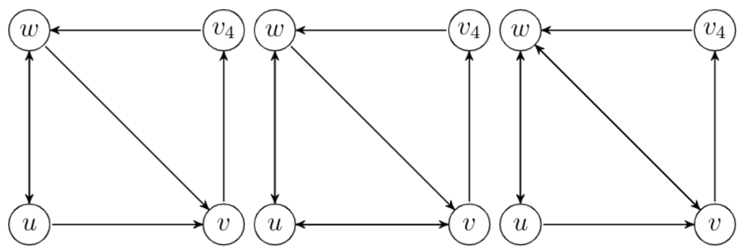

Figure 4.2 From left to right, example of graphs Γ_1, Γ_2, and Γ_3 as in Theorem 4.3.2.

Theorem 4.3.2. *Let Γ_1 be a digraph that has doubly adjacent out-twins u and w and a vertex v such that $vu, vw \notin E(\Gamma_1)$. Let $\Gamma_2 = \Gamma_1 + vu$ and $\Gamma_3 = \Gamma_1 + vw$. If $\mathrm{d}_{\Gamma_1}(v_i, u) = \mathrm{d}_{\Gamma_2}(v_i, u) = \mathrm{d}_{\Gamma_3}(v_i, u)$ and $\mathrm{d}_{\Gamma_1}(v_i, w) = \mathrm{d}_{\Gamma_2}(v_i, w) = \mathrm{d}_{\Gamma_3}(v_i, w)$ for all $v_i \in V(\Gamma_1)$ such that $v_i \neq v, u, w$, and Γ_2 and Γ_3 are strongly connected and non-isomorphic, then Γ_2 and Γ_3 are distance cospectral. Furthermore, if Γ_2^{T} is not isomorphic to Γ_2 and Γ_3, then it is $\mathcal{D}$-cospectral to Γ_2 and Γ_3. If Γ_3^{T} is not isomorphic to Γ_2 and Γ_3, then it is $\mathcal{D}$-cospectral to Γ_2 and Γ_3.*

Proof. Let $|V(\Gamma_1)| = n$ and label the vertices of Γ_1 such that

$$V(\Gamma_1) = \{v = v_1, u = v_2, w = v_3, v_4, \ldots, v_n\}.$$

We will show Γ_2 and Γ_3 are cospectral by showing their characteristic polynomials are the same. But first, we examine distances in Γ_2 and Γ_3.

The only way for a $v_i v_j$ path to be shorter in Γ_2 or Γ_3 than in Γ_1 is if the shorter path includes the new arc, vu or vw. Since u and w are out-twins, a path from v_i to v_j for $j \neq 2, 3$ that contains the arc vu in Γ_2 and such a path that contains vw in Γ_3 have the same length. Therefore

$$\mathrm{d}_{\Gamma_2}(v_i, v_j) = \mathrm{d}_{\Gamma_3}(v_i, v_j) \text{ for all } i \neq 2, 3 \text{ and all } j \neq 2, 3.$$

By assumption,

$$\mathrm{d}_{\Gamma_2}(v_i, v_j) = \mathrm{d}_{\Gamma_3}(v_i, v_j) \text{ for } 4 \leq i \leq n \text{ and } j = 2, 3.$$

Since u and w are doubly adjacent,

$$\mathrm{d}_{\Gamma_2}(v_i, v_j) = \mathrm{d}_{\Gamma_3}(v_i, v_j) = 1 \text{ for } i = 2, 3 \text{ and } j = 2, 3.$$

Because $\Gamma_2 = \Gamma_1 + vu$ and $\Gamma_3 = \Gamma_1 + vw$ and u and w are doubly adjacent,

$$\mathrm{d}_{\Gamma_2}(v_1, v_2) = \mathrm{d}_{\Gamma_3}(v_1, v_3) = 1 \text{ and } \mathrm{d}_{\Gamma_2}(v_1, v_3) = \mathrm{d}_{\Gamma_3}(v_1, v_2) = 2.$$

Finally, note that because u and w are out-twins,

$$\mathrm{d}_{\Gamma_2}(v_2, v_i) = \mathrm{d}_{\Gamma_2}(v_3, v_i) = \mathrm{d}_{\Gamma_3}(v_2, v_i) = \mathrm{d}_{\Gamma_3}(v_3, v_i) \text{ for all } i \neq 2, 3.$$

Thus, we see the distance matrices of Γ_2 and Γ_3 can be written as the following block matrices with vertex partition $\{\{1\}, \{2, 3\}, \{4, \ldots, n\}\}$.

$$\mathcal{D}(\Gamma_2) = \begin{bmatrix} 0 & A_1 & B \\ C_1 & E & F \\ G & H & K \end{bmatrix} \text{ and } \mathcal{D}(\Gamma_3) = \begin{bmatrix} 0 & A_2 & B \\ C_2 & E & F \\ G & H & K \end{bmatrix},$$

where

$$A_1 = \begin{bmatrix} 1 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 2 & 1 \end{bmatrix}, C_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

B has dimension $1 \times n - 3$, F has dimension $2 \times n - 3$, G has dimension $n - 3 \times 1$, H has dimension $n - 3 \times 2$, and K has dimension $n - 3 \times n - 3$.

We now turn our attention to the characteristic polynomials of $\mathcal{D}(\Gamma_2)$ and $\mathcal{D}(\Gamma_3)$. By Proposition 4.3.1, we can show the polynomials are equivalent by showing

$$\sum_D (-1)^{e(D)+n-\ell(D)} \left(\prod_{ij \in C(D)} \mathrm{d}_{\Gamma_2}(v_i, v_j) \right) x^{\ell(D)} = \sum_D (-1)^{e(D)+n-\ell(D)} \left(\prod_{ij \in C(D)} \mathrm{d}_{\Gamma_3}(v_i, v_j) \right) x^{\ell(D)}$$

where the sum runs over all generalized cycle decompositions D of the complete digraph with loops $\overleftrightarrow{K_n^{\ell}}$. First, observe for all D such that v_1v_2 and v_1v_3 are not in $C(D)$, $\mathrm{d}_{\Gamma_2}(v_i, v_j) = \mathrm{d}_{\Gamma_3}(v_i, v_j)$ for all $1 \leq i, j \leq n$ and so

$$(-1)^{e(D)+n-\ell(D)} \left(\prod_{ij \in C(D)} \mathrm{d}_{\Gamma_2}(v_i, v_j) \right) x^{\ell(D)} = (-1)^{e(D)+n-\ell(D)} \left(\prod_{ij \in C(D)} \mathrm{d}_{\Gamma_3}(v_i, v_j) \right) x^{\ell(D)}.$$

So we only need consider D such that v_1v_2 or v_1v_3 are in $C(D)$. We break such decompositions into four cases. For all cases, let D' be any generalized cycle decompositions of the complete digraph with loops on the remaining vertices. We will use $(v_{i_1}\, v_{i_2} \ldots v_{i_k})$ to denote a cycle with arc set $\{v_{i_1}v_{i_2}, v_{i_2}v_{i_3}, \ldots, v_{i_{k-1}}v_{i_k}, v_{i_k}v_{i_1}\}$ for some $2 \leq k \leq n$ and $1 \leq i_1, \ldots, i_k \leq n$ and we will use (v_i) to denote a loop for some $1 \leq i \leq n$.

Case 1: Consider the generalized cycle decompositions $D_1 = \{(v_1\, v_2\, v_3\, v_{i_1}\, v_{i_2} \ldots v_{i_r}) \cup D'\}$ or $D_2 = \{(v_1\, v_3\, v_2\, v_{i_1}\, v_{i_2} \ldots v_{i_r}) \cup D'\}$ for some $0 \leq r \leq n-3$ and for $4 \leq i_1, i_2, \ldots, i_r \leq n$. For this case, since $\mathrm{d}_{\Gamma_2}(v_1, v_2) = \mathrm{d}_{\Gamma_3}(v_1, v_3)$, $\mathrm{d}_{\Gamma_2}(v_2, v_3) = \mathrm{d}_{\Gamma_3}(v_2, v_3) = \mathrm{d}_{\Gamma_3}(v_3, v_2)$, $\mathrm{d}_{\Gamma_2}(v_3, v_i) = \mathrm{d}_{\Gamma_3}(v_3, v_i) = \mathrm{d}_{\Gamma_3}(v_2, v_i)$ for $i \neq 2, 3$, and $\mathrm{d}_{\Gamma_2}(v_i, v_j) = \mathrm{d}_{\Gamma_3}(v_i, v_j)$ for $i \neq 1$ and $j \neq 2, 3$, we have

$$\begin{aligned}
&\prod_{ij \in C(D_1)} \mathrm{d}_{\Gamma_2}(v_i, v_j) \\
= \;& \mathrm{d}_{\Gamma_2}(v_1, v_2)\, \mathrm{d}_{\Gamma_2}(v_2, v_3)\, \mathrm{d}_{\Gamma_2}(v_3, v_{i_1})\, \mathrm{d}_{\Gamma_2}(v_{i_1}, v_{i_2}) \cdots \mathrm{d}_{\Gamma_2}\left(v_{i_{r-1}}, v_{i_r}\right) \mathrm{d}_{\Gamma_2}(v_{i_r}, v_1) \prod_{jk \in C(D')} \mathrm{d}_{\Gamma_2}(v_j, v_k) \\
= \;& \mathrm{d}_{\Gamma_2}(v_1, v_2)\, \mathrm{d}_{\Gamma_3}(v_2, v_3)\, \mathrm{d}_{\Gamma_3}(v_3, v_{i_1})\, \mathrm{d}_{\Gamma_3}(v_{i_1}, v_{i_2}) \cdots \mathrm{d}_{\Gamma_3}\left(v_{i_{r-1}}, v_{i_r}\right) \mathrm{d}_{\Gamma_3}(v_{i_r}, v_1) \prod_{jk \in C(D')} \mathrm{d}_{\Gamma_3}(v_j, v_k) \\
= \;& \mathrm{d}_{\Gamma_3}(v_1, v_3)\, \mathrm{d}_{\Gamma_3}(v_3, v_2)\, \mathrm{d}_{\Gamma_3}(v_2, v_{i_1})\, \mathrm{d}_{\Gamma_3}(v_{i_1}, v_{i_2}) \cdots \mathrm{d}_{\Gamma_3}\left(v_{i_{r-1}}, v_{i_r}\right) \mathrm{d}_{\Gamma_3}(v_{i_r}, v_1) \prod_{jk \in C(D')} \mathrm{d}_{\Gamma_3}(v_j, v_k) \\
= \;& \prod_{ij \in C(D_2)} \mathrm{d}_{\Gamma_3}(v_i, v_j)\,.
\end{aligned}$$

By a similar argument,

$$\prod_{ij \in C(D_2)} \mathrm{d}_{\Gamma_2}(v_i, v_j) = \prod_{ij \in C(D_1)} \mathrm{d}_{\Gamma_3}(v_i, v_j)\,.$$

Furthermore, D_1 and D_2 have the same number of loops and cycles of even length. Thus, D_1 and D_2 together contribute the same quantity to the summand for each digraph Γ_2 and Γ_3.

Case 2: Consider the generalized cycle decompositions $D_1 = \{(v_1\, v_2\, v_{i_1}\, v_{i_2} \ldots v_{i_r}\, v_3) \cup D'\}$, $D_2 = \{(v_1\, v_3\, v_{i_1}\, v_{i_2} \ldots v_{i_r}\, v_2) \cup D'\}$, $D_3 = \{(v_1\, v_2) \cup (v_{i_1}\, v_{i_2} \ldots v_{i_r}\, v_3) \cup D'\}$, or $D_4 = \{(v_1\, v_3) \cup (v_{i_1}\, v_{i_2} \ldots v_{i_r}\, v_2) \cup D'\}$ for some $0 \leq r \leq n-3$ and for $4 \leq i_1, \ldots, i_r \leq n$. For this case, since

$\mathrm{d}_{\Gamma_p}(v_2, v_i) = \mathrm{d}_{\Gamma_p}(v_3, v_i)$ for $i \neq 2, 3$ and $p = 2, 3$ we have,

$$\begin{aligned}
&\prod_{ij \in C(D_1)} \mathrm{d}_{\Gamma_p}(v_i, v_j) \\
= \;& \mathrm{d}_{\Gamma_p}(v_1, v_2)\, \mathrm{d}_{\Gamma_p}(v_2, v_{i_1})\, \mathrm{d}_{\Gamma_p}(v_{i_1}, v_{i_2}) \cdots \mathrm{d}_{\Gamma_p}\left(v_{i_{r-1}}, v_{i_r}\right) \mathrm{d}_{\Gamma_p}(v_{i_r}, v_3)\, \mathrm{d}_{\Gamma_p}(v_3, v_1) \prod_{jk \in C(D')} \mathrm{d}_{\Gamma_p}(v_j, v_k) \\
= \;& \mathrm{d}_{\Gamma_p}(v_1, v_2)\, \mathrm{d}_{\Gamma_p}(v_3, v_{i_1})\, \mathrm{d}_{\Gamma_p}(v_{i_1}, v_{i_2}) \cdots \mathrm{d}_{\Gamma_p}\left(v_{i_{r-1}}, v_{i_r}\right) \mathrm{d}_{\Gamma_p}(v_{i_r}, v_3)\, \mathrm{d}_{\Gamma_p}(v_2, v_1) \prod_{jk \in C(D')} \mathrm{d}_{\Gamma_p}(v_j, v_k) \\
= \;& \mathrm{d}_{\Gamma_p}(v_1, v_2)\, \mathrm{d}_{\Gamma_p}(v_2, v_1)\, \mathrm{d}_{\Gamma_p}(v_{i_1}, v_{i_2}) \cdots \mathrm{d}_{\Gamma_p}\left(v_{i_{r-1}}, v_{i_r}\right) \mathrm{d}_{\Gamma_p}(v_{i_r}, v_3)\, \mathrm{d}_{\Gamma_p}(v_3, v_{i_1}) \prod_{jk \in C(D')} \mathrm{d}_{\Gamma_p}(v_j, v_k) \\
= \;& \prod_{ij \in C(D_3)} \mathrm{d}_{\Gamma_p}(v_i, v_j).
\end{aligned}$$

Note D_1 is the union of a cycle of length $r+3$ with D' and D_3 is the union of cycles of length 2 and $r+1$ with D'. Thus, D_3 has one more even cycle than D_1 and so $(-1)^{e(D_3)+n-\ell(D)} = -(-1)^{e(D_1)+n-\ell(D)}$. Therefore,

$$(-1)^{e(D_1)+n-\ell(D)} \prod_{ij \in C(D_1)} \mathrm{d}_{\Gamma_p}(v_i, v_j) + (-1)^{e(D_3)+n-\ell(D)} \prod_{ij \in C(D_3)} \mathrm{d}_{\Gamma_p}(v_i, v_j) = 0.$$

By a similar argument,

$$(-1)^{e(D_2)+n-\ell(D)} \prod_{ij \in C(D_2)} \mathrm{d}_{\Gamma_p}(v_i, v_j) + (-1)^{e(D_4)+n-\ell(D)} \prod_{ij \in C(D_4)} \mathrm{d}_{\Gamma_p}(v_i, v_j) = 0.$$

Thus, D_1, D_2, D_3, and D_4 together contribute nothing to the summand for each digraph Γ_2 and Γ_3.

Case 3: Consider the generalized cycle decompositions $D_1 = \{(v_1\, v_2\, v_{i_1}\, v_{i_2} \ldots v_{i_r}) \cup (v_3) \cup D'\}$ and $D_2 = \{(v_1\, v_3\, v_{i_1}\, v_{i_2} \ldots v_{i_r}) \cup (v_2) \cup D'\}$ for some $0 \leq r \leq n-2$ and for $4 \leq i_1, i_2, \ldots, i_r \leq n$. For this case, since $\mathrm{d}_{\Gamma_2}(v_1, v_2) = 1 = \mathrm{d}_{\Gamma_3}(v_1, v_3)$, $\mathrm{d}_{\Gamma_2}(v_2, v_i) = \mathrm{d}_{\Gamma_3}(v_2, v_i) = \mathrm{d}_{\Gamma_3}(v_3, v_i)$ for $i \neq 2, 3$, and $\mathrm{d}_{\Gamma_2}(v_i, v_j) = \mathrm{d}_{\Gamma_3}(v_i, v_j)$ for $i \neq 1$ and $j \neq 2, 3$, we have,

$$
\begin{aligned}
&\prod_{ij\in C(D_1)} \mathrm{d}_{\Gamma_2}(v_i,v_j)\\
=\;& \mathrm{d}_{\Gamma_2}(v_1,v_2)\,\mathrm{d}_{\Gamma_2}(v_2,v_{i_1})\,\mathrm{d}_{\Gamma_2}(v_{i_1},v_{i_2})\cdots\mathrm{d}_{\Gamma_2}\left(v_{i_{r-1}},v_{i_r}\right)\mathrm{d}_{\Gamma_2}(v_{i_r},v_1)\prod_{jk\in C(D')}\mathrm{d}_{\Gamma_2}(v_j,v_k)\\
=\;& \mathrm{d}_{\Gamma_2}(v_1,v_2)\,\mathrm{d}_{\Gamma_3}(v_2,v_{i_1})\,\mathrm{d}_{\Gamma_3}(v_{i_1},v_{i_2})\cdots\mathrm{d}_{\Gamma_3}\left(v_{i_{r-1}},v_{i_r}\right)\mathrm{d}_{\Gamma_3}(v_{i_r},v_1)\prod_{jk\in C(D')}\mathrm{d}_{\Gamma_3}(v_j,v_k)\\
=\;& \mathrm{d}_{\Gamma_3}(v_1,v_3)\,\mathrm{d}_{\Gamma_3}(v_3,v_{i_1})\,\mathrm{d}_{\Gamma_3}(v_{i_1},v_{i_2})\cdots\mathrm{d}_{\Gamma_3}\left(v_{i_{r-1}},v_{i_r}\right)\mathrm{d}_{\Gamma_3}(v_{i_r},v_1)\prod_{jk\in C(D')}\mathrm{d}_{\Gamma_3}(v_j,v_k)\\
=\;& \prod_{ij\in C(D_2)} \mathrm{d}_{\Gamma_3}(v_i,v_j)\,.
\end{aligned}
$$

By a similar argument,

$$\prod_{ij\in C(D_2)} \mathrm{d}_{\Gamma_2}(v_i,v_j)=\prod_{ij\in C(D_1)} \mathrm{d}_{\Gamma_3}(v_i,v_j)\,.$$

Furthermore, D_1 and D_2 have the same number of loops and cycles of even length. Thus, D_1 and D_2 together contribute the same quantity to the summand for each digraph Γ_2 and Γ_3.

Case 4: Consider the generalized cycle decompositions $D_1=\{(v_1\,v_2\,v_{i_1}\,v_{i_2}\ldots v_{i_r}\,v_3\,v_{j_1}\,v_{j_2}\ldots v_{j_s})\cup D'\}$, $D_2=\{(v_1\,v_3\,v_{i_1}\,v_{i_2}\ldots v_{i_r}\,v_2\,v_{j_1}\,v_{j_2}\ldots v_{j_s})\cup D'\}$, $D_3=\{(v_1\,v_2\,v_{j_1}\,v_{j_2}\ldots v_{j_s})\cup(v_{i_1}\,v_{i_2}\ldots v_{i_r}\,v_3)\cup D'\}$, $D_4=\{(v_1\,v_3\,v_{j_1}\,v_{j_2}\ldots v_{j_s})\cup(v_{i_1}\,v_{i_2}\ldots v_{i_r}\,v_2)\cup D'\}$ for some $1\le r\le n-3$ and $1\le s\le n-3$ such that $r+s+3\le n$ and for $4\le i_1,\ldots,i_r,j_1,\ldots,j_s\le n$. For this case, since $\mathrm{d}_{\Gamma_p}(v_2,v_i)=\mathrm{d}_{\Gamma_p}(v_3,v_i)$ for $i\ne 2,3$ and $p=2,3$ we have,

$$
\begin{aligned}
&\prod_{ij\in C(D_1)} \mathrm{d}_{\Gamma_p}(v_i,v_j)\\
=\;& \mathrm{d}_{\Gamma_p}(v_1,v_2)\,\mathrm{d}_{\Gamma_p}(v_2,v_{i_1})\,\mathrm{d}_{\Gamma_p}(v_{i_1},v_{i_2})\cdots\mathrm{d}_{\Gamma_p}\left(v_{i_{r-1}},v_{i_r}\right)\mathrm{d}_{\Gamma_p}(v_{i_r},v_3)\\
& \mathrm{d}_{\Gamma_p}(v_3,v_{j_1})\,\mathrm{d}_{\Gamma_p}(v_{j_1},v_{j_2})\cdots\mathrm{d}_{\Gamma_p}\left(v_{j_{s-1}},v_{j_s}\right)\mathrm{d}_{\Gamma_p}(v_{j_s},v_1)\prod_{jk\in C(D')}\mathrm{d}_{\Gamma_p}(v_j,v_k)\\
=\;& \mathrm{d}_{\Gamma_p}(v_1,v_2)\,\mathrm{d}_{\Gamma_p}(v_3,v_{i_1})\,\mathrm{d}_{\Gamma_p}(v_{i_1},v_{i_2})\cdots\mathrm{d}_{\Gamma_p}\left(v_{i_{r-1}},v_{i_r}\right)\mathrm{d}_{\Gamma_p}(v_{i_r},v_3)\\
& \mathrm{d}_{\Gamma_p}(v_2,v_{j_1})\,\mathrm{d}_{\Gamma_p}(v_{j_1},v_{j_2})\cdots\mathrm{d}_{\Gamma_p}\left(v_{j_{s-1}},v_{j_s}\right)\mathrm{d}_{\Gamma_p}(v_{j_s},v_1)\prod_{jk\in C(D')}\mathrm{d}_{\Gamma_p}(v_j,v_k)\\
=\;& \mathrm{d}_{\Gamma_p}(v_1,v_2)\,\mathrm{d}_{\Gamma_p}(v_2,v_{j_1})\,\mathrm{d}_{\Gamma_p}(v_{j_1},v_{j_2})\cdots\mathrm{d}_{\Gamma_p}\left(v_{j_{s-1}},v_{j_s}\right)\mathrm{d}_{\Gamma_p}(v_{j_s},v_1)\\
& \mathrm{d}_{\Gamma_p}(v_{i_1},v_{i_2})\cdots\mathrm{d}_{\Gamma_p}\left(v_{i_{r-1}},v_{i_r}\right)\mathrm{d}_{\Gamma_p}(v_{i_r},v_3)\,\mathrm{d}_{\Gamma_p}(v_3,v_{i_1})\prod_{jk\in C(D')}\mathrm{d}_{\Gamma_p}(v_j,v_k)\\
=\;& \prod_{ij\in C(D_3)} \mathrm{d}_{\Gamma_p}(v_i,v_j)\,.
\end{aligned}
$$

Note D_1 is the union of a cycle of length $r+s+3$ with D' and D_3 is the union of cycles of length $s+2$ and $r+1$ with D'. If r and s are both even or both odd, then $r+s+3$ is odd and exactly one of $s+2$ or $r+1$ is even and the other is odd. If one of r and s are even and the other is odd, then $r+s+3$ is even and either $s+2$ and $r+1$ are both even or both odd. Thus, D_3 has one more or one less even cycle than D_1 and so $(-1)^{e(D_3)+n-\ell(D)} = -(-1)^{e(D_1)+n-\ell(D)}$. Therefore,

$$(-1)^{e(D_1)+n-\ell(D)} \prod_{ij\in C(D_1)} \mathrm{d}_{\Gamma_p}(v_i, v_j) + (-1)^{e(D_3)+n-\ell(D)} \prod_{ij\in C(D_3)} \mathrm{d}_{\Gamma_p}(v_i, v_j) = 0.$$

By a similar argument,

$$(-1)^{e(D_2)+n-\ell(D)} \prod_{ij\in C(D_2)} \mathrm{d}_{\Gamma_p}(v_i, v_j) + (-1)^{e(D_4)+n-\ell(D)} \prod_{ij\in C(D_4)} \mathrm{d}_{\Gamma_p}(v_i, v_j) = 0.$$

Thus, D_1, D_2, D_3, and D_4 together contribute nothing to the summand for each digraph Γ_2 and Γ_3.

Since all D such that v_1v_2 and v_1v_3 are not in $C(D)$ fall into one of these four cases, we have show that $p_{\mathcal{D}}(\Gamma_2) = p_{\mathcal{D}}(\Gamma_3)$. Therefore, Γ_2 and Γ_3 are distance cospectral. The cospectrality of Γ_2^{T} and Γ_3^{T} is see by applying Observation 4.2.1. □

The analogous result holds for digraphs containing doubly adjacent in-twins with certain properties.

Theorem 4.3.3. *Let Γ be a digraph that has doubly adjacent in-twins u and w and a vertex v such that $uv, wv \notin E(\Gamma)$. Let $\Gamma_2 = \Gamma + uv$ and $\Gamma_3 = \Gamma + wv$. If $\mathrm{d}_\Gamma(u, v_i) = \mathrm{d}_{\Gamma_2}(u, v_i) = \mathrm{d}_{\Gamma_3}(u, v_i)$ and $\mathrm{d}_\Gamma(w, v_i) = \mathrm{d}_{\Gamma_2}(w, v_i) = \mathrm{d}_{\Gamma_3}(w, v_i)$ for all $v_i \in V(\Gamma)$ such that $v_i \neq v, u, w$, and Γ_2 and Γ_3 are strongly connected and non-isomorphic, then Γ_2 and Γ_3 are distance cospectral. Furthermore, if Γ_2^{T} is not isomorphic to Γ_2 and Γ_3, then it is $\mathcal{D}$-cospectral to Γ_2 and Γ_3. If Γ_3^{T} is not isomorphic to Γ_2 and Γ_3, then it is $\mathcal{D}$-cospectral to Γ_2 and Γ_3.*

Proof. Proof is analogous to that of Theorem 4.3.2. □

4.4 Concluding remarks

In this paper, we studied distance cospectrality for digraphs. In Section 4.2, we computed the number of digraphs with a $\mathcal{D}$-cospectral mate on 6 and fewer vertices and the number of such digraphs who have a mate not produced by arc reversal. Generalized cycle decompositions were applied in Section 4.3 to produce distance cospectral constructions for digraphs with (in/out)-twin vertices. In the future, it would be interesting to apply the same method of determining the distance characteristic polynomial of a digraph to digraphs with other structural properties.

4.5 References

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CHAPTER 5. GENERAL CONCLUSION

The distance matrix and its variants have been of increasing interest in recent years. Much work has been focused on bounding the spectral radius of the matrices and finding graphs that achieve extremal values. Various aspects of cospectrality, including enumerating the number of graphs with cospectral mates, determining graph parameters that are preserved or not preserved by cospectrality, and producing cospectral constructions, have been of particular interest.

In Chapter 2, we introduced a new variant of the distance matrix, the normalized distance Laplacian and derived bounds on its eigenvalues. In contrast to the normalized Laplacian, we showed that $\partial^{\mathcal{L}} < 2$ for all graphs G on $n \geq 3$ vertices. We conjecture that the maximum $\mathcal{D}^{\mathcal{L}}$ spectral radius tends towards 2 as n becomes large and that this value is achieved by a family of graphs, KPK_{n_1,n_2,n_3}. We also found that the complete graph K_n achieves the minimal spectral radius and conjecture that is the only such graph.

The preservation of parameters by cospectrality was considered in Chapter 3. Several parameters were shown not to be preserved by cospectrality for various matrices. Also, we proved that transmission regular graphs that are distance cospectral must have the same transmission and Wiener index. The preservation or non-preservation by cospectrality has not yet been determined for several parameters and matrices. In particular, it would be interesting to determine if transmission regularity is preserved for $\mathcal{D}$, $\mathcal{D}^{Q}$, and $\mathcal{D}^{\mathcal{L}}$. Since no example exhibiting non-preservation exists on 10 or fewer vertices, it would be useful to find a cospectral construction which produces a transmission regular and non-transmission regular graph.

Finally, in Chapter 4, distance cospectrality was studied for digraphs. The number of digraphs with a distance cospectral mate was found for strongly connected digraphs on 6 and fewer vertices. It was observed that arc reversal produces cospectral digraphs and a cospectral construction was described that uses twin vertices and is not produced by arc reversal. In the future, it would be interesting to apply the method used, computing the distance characteristic polynomial through generalized cycle decompositions, to digraphs with other structures to produce more cospectral constructions.